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The Translog Utility Function and the Tornqvist Quantity Index

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Abstract

We re-visit the proposition: utility-change for the translog utility function equals the corresponding Tornqvist quantity index. We observe that the "linear" terms in the translog function must be zero for the result to obtain. We also report on the role of the assumption of homogeneity for the utility function in allowing the result to obtain.

- key words: translog function; Tornqvist index
- classification: O47, E01

1 Introduction

We establish that the "linear" terms in the translog utility function must be zero in order for the change in utility for the translog function to link precisely to the Tornqvist quantity index (see Diewert (1976; pp. 119-120) for the first statement of the theorem). In addition we find the assumption of homogeneity of the utility function is special and limits the validity of the link between change in utility for the translog function and the Tornvist quantity index. For distinct price and quantity vectors at two dates, the theorem will generally not "hold up". One has to construct "workable" pairs of price and quantity vectors in order for the theorem to "go through". We work with the two commodity case and proceed without referral to Diewert's quadratic equivalence lemma.[1]

The early classic result in this area is Fisher's Ideal quantity index capturing the change in utility levels for the case of a quadratic utility function (Diewert

[1] Diewert's lemma is presented in Diewert (1976; pp. 117-118). A different proof of the lemma is in Hartwick (2020; Chapter 3). This lemma is central to Diewert's analysis of utility-change with the translog function and the Tornqvist quantity index.

(1976; pp. 116-117))[2]. We start below with the general translog utility function for two commodities and indicate the specialization necessary for the utility-change, Tornqvist result to be valid.

2 Translog Utility

The first period is indicated by y and the second by z. We start with the translog utility function for two commodities for periods y and z:

$$\begin{aligned}\ln U^z &= \alpha_0 + \alpha_1 \ln z_1 + \alpha_2 \ln z_2 + \frac{1}{2}\beta_{11} \ln z_1 \ln z_1 + \frac{1}{2}\beta_{12} \ln z_1 \ln z_2 \\ &\quad + \frac{1}{2}\beta_{21} \ln z_2 \ln z_1 + \frac{1}{2}\beta_{22} \ln z_2 \ln z_2 \qquad (1) \\ \ln U^y &= \alpha_0 + \alpha_1 \ln y_1 + \alpha_2 \ln y_2 + \frac{1}{2}\beta_{11} \ln y_1 \ln y_1 + \frac{1}{2}\beta_{12} \ln y_1 \ln y_2 \\ &\quad + \frac{1}{2}\beta_{21} \ln y_2 \ln y_1 + \frac{1}{2}\beta_{22} \ln y_2 \ln y_2.\end{aligned}$$

We obtain derivatives

$$\begin{aligned}\frac{\partial U^z}{\partial z_1} &= [\alpha_1 + \frac{1}{2}\beta_{11} \ln z_1 + \frac{1}{2}\beta_{11} \ln z_1 + \frac{1}{2}\beta_{12} \ln z_2 + \frac{1}{2}\beta_{21} \ln z_2]\frac{U^z}{z_1} \\ \frac{\partial U^z}{\partial z_2} &= [\alpha_2 + \frac{1}{2}\beta_{22} \ln z_2 + \frac{1}{2}\beta_{22} \ln z_2 + \frac{1}{2}\beta_{12} \ln z_1 + \frac{1}{2}\beta_{21} \ln z_1]\frac{U^z}{z_2} \\ \frac{\partial U^y}{\partial y_1} &= [\alpha_1 + \frac{1}{2}\beta_{11} \ln y_1 + \frac{1}{2}\beta_{11} \ln y_1 + \frac{1}{2}\beta_{12} \ln y_2 + \frac{1}{2}\beta_{21} \ln y_2]\frac{U^y}{y_1} \\ \frac{\partial U^y}{\partial y_2} &= [\alpha_2 + \frac{1}{2}\beta_{22} \ln y_2 + \frac{1}{2}\beta_{22} \ln y_2 + \frac{1}{2}\beta_{12} \ln y_1 + \frac{1}{2}\beta_{21} \ln y_1]\frac{U^y}{y_2}.\end{aligned}$$

We are interested in transforming the right side of $\ln U^z - \ln U^y$. We will concentrate on transforming $\ln U^z$ first. $\ln U^y$ works the same. We re-write the first two derivatives to get

$$\begin{aligned}\frac{z_1}{U^z}\frac{\partial U^z}{\partial z_1} \ln z_1 &= [\{\alpha_1 \ln z_1 + \frac{1}{2}\beta_{11} \ln z_1 \ln z_1 + \frac{1}{2}\beta_{12} \ln z_2 \ln z_1 + \frac{1}{2}\beta_{21} \ln z_2 \ln z_1\} + \frac{1}{2}\beta_{11} \ln z_1 \ln z_1] \\ \frac{z_2}{U^z}\frac{\partial U^z}{\partial z_2} \ln z_2 &= [\{\{\alpha_2 \ln z_2 + \frac{1}{2}\beta_{22} \ln z_2 \ln z_2\}\} + \frac{1}{2}\beta_{22} \ln z_2 \ln z_2 + \frac{1}{2}\beta_{12} \ln z_1 \ln z_2 + \frac{1}{2}\beta_{21} \ln z_1 \ln z_2].\end{aligned}$$

The terms in brackets $\{...\}$ and $\{\{...\}\}$ appear exactly in the orginal expression above for $\ln U^z$ (equation 1). We can thus proceed to substitute in $\ln U^z$ to get

$$\ln U^z = \frac{z_1}{U^z}\frac{\partial U^z}{\partial z_1} \ln z_1 + \frac{z_2}{U^z}\frac{\partial U^z}{\partial z_2} \ln z_2 + \alpha_0 - \frac{1}{2}\beta_{11} \ln z_1 \ln z_1 - \frac{1}{2}\beta_{12} \ln z_1 \ln z_2 - \frac{1}{2}\beta_{21} \ln z_2 \ln z_1 - \frac{1}{2}\beta_{22} \ln z_2 \ln z_2.$$

[2] Diewert cites Byushgens (1925) as the first publication on this result with a quadratic utility function.

We move the last four terms to the left side and at the same time, add $\alpha_0 + \alpha_1 \ln z_1 + \alpha_2 \ln z_2$ to both sides to get

$$2 \ln U^z = \frac{z_1}{U^z}\frac{\partial U^z}{\partial z_1}\ln z_1 + \frac{z_2}{U^z}\frac{\partial U^z}{\partial z_2}\ln z_2 + \alpha_0 + [\alpha_0 + \alpha_1 \ln z_1 + \alpha_2 \ln z_2].$$

We redo the above steps for $\ln U^y$ to get

$$2 \ln U^y = \frac{y_1}{U^y}\frac{\partial U^y}{\partial y_1}\ln y_1 + \frac{y_2}{U^y}\frac{\partial U^y}{\partial y_2}\ln y_2 + \alpha_0 + [\alpha_0 + \alpha_1 \ln y_1 + \alpha_2 \ln y_2].$$

We can link these two expressions in $\ln U^z - \ln U^y =$

$$\frac{1}{2}[\frac{z_1}{U^z}\frac{\partial U^z}{\partial z_1}\ln z_1 + \frac{z_2}{U^z}\frac{\partial U^z}{\partial z_2}\ln z_2 - \frac{y_1}{U^y}\frac{\partial U^y}{\partial y_1}\ln y_1 - \frac{y_2}{U^y}\frac{\partial U^y}{\partial y_2}\ln y_2]$$
$$+\frac{1}{2}[\alpha_1 \ln z_1 + \alpha_2 \ln z_2] - \frac{1}{2}[\alpha_1 \ln y_1 + \alpha_2 \ln y_2].$$

We proceed to invoke the assumption of homogeneity: $U^z = z_1\frac{\partial U^z}{\partial z_1} + z_2\frac{\partial U^z}{\partial z_2}$ and $U^y = y_1\frac{\partial U^y}{\partial y_1} + y_2\frac{\partial U^y}{\partial y_2}$.[3] We have then: $\ln U^z - \ln U^y =$

$$\frac{1}{2}\{\frac{z_1\frac{\partial U^z}{\partial z_1}}{z_1\frac{\partial U^z}{\partial z_1} + z_2\frac{\partial U^z}{\partial z_2}}\ln z_1 + \frac{z_2\frac{\partial U^z}{\partial z_2}}{z_1\frac{\partial U^z}{\partial z_1} + z_2\frac{\partial U^z}{\partial z_2}}\ln z_2 - \frac{y_1\frac{\partial U^y}{\partial y_1}}{y_1\frac{\partial U^y}{\partial y_1} + y_2\frac{\partial U^y}{\partial y_2}}\ln y_1$$
$$-\frac{y_2\frac{\partial U^y}{\partial y_2}}{y_1\frac{\partial U^y}{\partial y_1} + y_2\frac{\partial U^y}{\partial y_2}}\ln y_2\} + \frac{1}{2}[\alpha_1 \ln z_1 + \alpha_2 \ln z_2] - \frac{1}{2}[\alpha_1 \ln y_1 + \alpha_2 \ln y_2]\}.$$

Under the assumption that $[\alpha_1 \ln z_1 + \alpha_2 \ln z_2] - [\alpha_1 \ln y_1 + \alpha_2 \ln y_2] = 0$, and prices derive from consumer optimization, we now have $\ln U^z - \ln U^y =$

$$\frac{1}{2}\{\frac{p_1^z z_1}{p_1^z z_1 + p_2^z z_2}\ln z_1 + \frac{p_2^z z_2}{p_1^z z_1 + p_2^z z_2}\ln z_2 - \frac{p_1^y y_1}{p_1^y y_1 + p_2^y y_2}\ln y_1 - \frac{p_2^y y_2}{p_1^y y_1 + p_2^y y_2}\ln y_2\}.$$

(We have introduced appropriate price vectors, (p_1^z, p_2^z) and (p_1^y, p_2^y).) Our expression for $\ln U^z - \ln U^y$ can be written as

$$\frac{1}{2}\{[\frac{p_1^z z_1}{p_1^z z_1 + p_2^z z_2} + \frac{p_2^z z_2}{p_1^z z_1 + p_2^z z_2}]\ln\left(\frac{z_1}{z_2}\right) + [\frac{p_1^y y_1}{p_1^y y_1 + p_2^y y_2} + \frac{p_2^y y_2}{p_1^y y_1 + p_2^y y_2}]\ln\left(\frac{y_1}{y_2}\right)$$

or

$$\ln\left[\frac{U^z}{U^y}\right] = \frac{1}{2}[(s_1^z + s_2^z)\ln\left(\frac{z_1}{z_2}\right) + (s_1^y + s_2^y)]\ln\left(\frac{y_1}{y_2}\right)$$

[3] We return to this assumption of homogeneity below. Homogeneity only appears for special cases, not "arbitrary" initial prices and quantities. Below we construct a numerical example with the appropriate homogeneity.

where $s_i^z = \frac{p_i^z z_i}{p_1^z z_1 + p_2^z z_2}$ and $s_i^y = \frac{p_i^y y_i}{p_1^y y_1 + p_2^y y_2}$ are the shares of the value of commodity i in total value $(i = 1, 2)$. The right side above is the Tornqvist index for the case of two commodities and two prices. The difference in utilities across two consecutive periods, on the left, is precisely linked to the Tornqvist quantity index. Our proof extends in a straightforward fashion to the general case with $n > 2$ commodities.

Our proof turns on the assumption that $[\alpha_1 \ln z_1 + \alpha_2 \ln z_2] - [\alpha_1 \ln y_1 + \alpha_2 \ln y_2] = 0$. This condition is signalling that the theorem is valid only for the case of each translog function defined at the start without the terms $\alpha_1 \ln z_1$, $\alpha_2 \ln z_2$, $\alpha_1 \ln y_1$ and $\alpha_2 \ln y_2$, i.e. with $\alpha_1 = \alpha_2 = 0$. For the left-side, right-side equivalence theorem to hold, one must start with a translog function without the "linear" terms; i.e. $\ln U^y = \frac{1}{2} \sum_i^2 \sum_j^2 \beta_{ij} \ln y_i \ln y_j$ and $\ln U^z = \frac{1}{2} \sum_i^2 \sum_j^2 \beta_{ij} \ln z_i \ln z_j$.The "reduction" in the starting form for the translog function brings the theorem in line with the classic theorem dealing with a quadratic utility function and the Fisher Ideal index, (reference above).

3 The Homogeneity Assumption

The equivalence theorem above requires that the homogeneity assumption ($U^z = z_1 \frac{\partial U^z}{\partial z_1} + z_2 \frac{\partial U^z}{\partial z_2}$ and $U^y = y_1 \frac{\partial U^y}{\partial y_1} + y_2 \frac{\partial U^y}{\partial y_2}$) is satisfied. For "arbitrary" initial price and quantity vectors the homogeneity assumption will not be satisfied. One way to deal with this is to illustrate the theorem with quantity vectors (y_1, y_2) and (z_1, z_2), and price vectors (p_1^y, p_2^y) and (p_1^z, p_2^z) specially constructed *a priori*. We proceed with this approach with an illustrative numerical example, one with two commodities and two prices.[4]

(1) We fix positive numerical values for β_{11}, β_{12}, β_{21} and β_{22}. ($\beta_{11} = 1.4, \beta_{12} = 0.5, \beta_{21} = 0.7$ and $\beta_{22} = 1.2$) We select an arbitrary z_2, ($z_2 = 6$). We then solve for z_1 and U^z in $\ln U^z = \frac{1}{2} \sum_i \sum_j \beta_{ij} \ln z_i \ln z_j$ and $U^z = z_1 \frac{\partial U^z}{\partial z_1} + z_2 \frac{\partial U^z}{\partial z_2}$. We get $z_1 = 0.7823$ and $U^z = 5.4694$, using Matlab.

(2) Given values for z_1 and z_2, we solve for the price ratio p_1^z / p_2^z in $\frac{\partial U^z}{\partial z_1} / \frac{\partial U^z}{\partial z_2} = p_1^z / p_2^z$. The ratio is 2.8008.

[4] Christensen, Jorgenson, and Lau (1975) developed an analysis of the translog utility function as preparatory to considerations of econometrics for such functions.

(3) We repeat (1) and (2) for $\ln U^y = \frac{1}{2}\sum_i \sum_j \beta_{ij} \ln y_i \ln y_j$ and $U^y = y_1 \frac{\partial U^y}{\partial y_1} + y_2 \frac{\partial U^y}{\partial y_2}$. We set $y_2 = 5$. We get $y_1 = 0.5513$ and $U^y = 3.4122$. The price ratio solves to, $p_1^y / p_2^y = 0.7608$.

We have then vectors $(y_1, y_2, p_1^y/p_2^y)$ and $(z_1, z_2, p_1^z/p_2^z)$ with numerical values. We use prices $(p_1^y, p_2^y) = (0.7608, 1)$ and $(p_1^z, p_2^z) = (2.8008, 1)$. We define shares: $s_1^z = \frac{2.8008*0.7823}{(2.8008*0.7823)+6}$ and $s_2^z = \frac{6}{(2.8008*0.7823)+6}$; and $s_1^y = \frac{0.7608*0.5513}{(0.7608*0.5513)+5}$ and $s_2^y = \frac{5}{(0.7608*0.5513)+5}$. These can be inserted into the Tornqvist quantity index number formula:

$$\left[\frac{z_1}{y_1}\right]^{\frac{1}{2}[s_1^z + s_1^y]} * \left[\frac{z_2}{y_2}\right]^{\frac{1}{2}[s_2^z + s_2^y]}$$

The index number comes out as 1.5. We also have numerical values for U^y and U^z. This ratio, U^z/U^y is 5.4674/3.4122, equal to 1.5 also. Hence our numerical example illustrates the equivalence of the Tornqvist quantity index and the ratio of values of the corresponding utilities for the translog form of the utility function. It is clear that starting with two "arbitrary" quantity vectors and two "arbitrary" price vectors the equivalence of the index number and the ratio of utilities will not obtain. One has to build in homogeneity $U^y = y_1 \frac{\partial U^y}{\partial y_1} + y_2 \frac{\partial U^y}{\partial y_2}$ and $U^z = z_1 \frac{\partial U^z}{\partial z_1} + z_2 \frac{\partial U^z}{\partial z_2}$ from the start in order to have the equivalence obtain. Hence the equivalence theorem is valid only for specially constructed price and quantity vectors. We indicated the appropriate construction with our numerical example.

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