Queen's Economics Department Working Paper No. 1225

Finite Sample Accuracy of Integrated Volatility Estimators

Morten Ørregaard Nielsen
Queen's University and CREATES

Per Houmann Frederiksen
Nordea Bank

Department of Economics
Queen's University
94 University Avenue
Kingston, Ontario, Canada
K7L 3N6

1-2005

Finite Sample Accuracy of Integrated Volatility Estimators*

Morten Ørregaard Nielsen† — Cornell University

Per Houmann Frederiksen — Aarhus School of Business

January 19, 2005

Abstract

We consider the properties of three estimation methods for integrated volatility, i.e. realized volatility, the Fourier estimator, and the wavelet estimator, when a typical sample of high-frequency data is observed. We employ several different generating mechanisms for the instantaneous volatility process, e.g. Ornstein-Uhlenbeck, long memory, and jump processes. The possibility of market microstructure contamination is also entertained using a model with bid-ask bounce in which case alternative estimators with theoretical justification under market microstructure noise are also examined. The estimation methods are compared in a simulation study which reveals a general robustness towards persistence or jumps in the latent stochastic volatility process. However, bid-ask bounce effects render realized volatility and especially the wavelet estimator less useful in practice, whereas the Fourier method remains useful and is superior to the other two estimators in that case. More strikingly, even compared to bias correction methods for microstructure noise, the Fourier method is superior with respect to RMSE while having only slightly higher bias.

JEL Classifications: C15, C22.

Keywords: Bid-ask bounce, finite sample bias, integrated volatility, long memory, market microstructure, Monte Carlo simulation, realized volatility, wavelet.

---

*We thank Torben G. Andersen, Esben Høg, and Asger Lunde for valuable comments and suggestions, and Esben Høg and Asger Lunde for making their Gauss codes available to us. The first author is grateful for financial support from the Danish Social Science Research Council (grant no 24-02-0181).

†Corresponding author. Please address correspondence to: Morten Ørregaard Nielsen, Department of Economics, Cornell University, 482 Uris Hall, Ithaca, NY 14853, USA; phone: +1 607 255 6338; fax: +1 607 255 2818; email: mon2@cornell.edu.

1 Introduction

The definition and analysis of realized volatility in financial return series has attracted considerable interest in the literature starting with French, Schwert & Stambaugh (1987), see e.g. Andersen, Bollerslev & Diebold (2004) and the references therein for a review. Essentially, integrated instantaneous volatility is estimated consistently by its sample analogue based on high-frequency return observations. This approach allows gathering much more detailed information on the properties of financial market volatility than previously. However, recently two rival approaches for the estimation of integrated volatility have been introduced. Malliavin & Mancino (2002) suggest estimating integrated volatility by a Fourier transform based method and Høg & Lunde (2003) suggest employing a method based on the wavelet transform. The properties of the three estimation methods for integrated volatility, i.e. realized volatility, the Fourier estimator, and the wavelet estimator, when only a finite sample of the price process (albeit at a high frequency) is observed, have been examined only briefly in the literature.

Previously, Barucci & Reno (2002*a*, 2002*b*) have compared the Fourier method to realized volatility in a Monte Carlo study using the Cox, Ingersoll & Ross (1985) model to generate the latent instantaneous volatility process, and their simulations show that the Fourier method compares favorably with realized volatility. However, Barucci & Reno (2002*a*, 2002*b*) typically contrast a 5 minute realized volatility estimator to a Fourier estimator using all observations (which are measured as often as every 14 seconds on average) creating a very uneven base for comparison, and furthermore they use interpolation between observations rather than the imputation scheme that the literature has settled upon. Within the same framework, i.e. using a Cox et al. (1985) model for the latent instantaneous volatility process, simulations by Høg & Lunde (2003) show that the wavelet method and the Fourier method are virtually indistinguishable with respect to bias and variance, although the wavelet method is computationally much faster than the Fourier method. However, a major drawback of both these studies is that they consider only one generating mechanism for the unobserved instantaneous volatility process, and in particular, their generating mechanism does not allow for any of the recently popularized features of integrated volatility, such as long memory, jumps, and market microstructure effects.

In this paper, we examine all three estimators mentioned above and compare them within the same model setup. Unlike the previous studies, we try to even the playing field compared to Barucci & Reno (2002*a*, 2002*b*) by using the same number of implied intra-daily returns for each estimator. Furthermore, we follow the literature and use imputation rather than interpolation, and we employ several different generating mechanisms for the instantaneous integrated volatility. In particular, we consider logarithmic Ornstein-Uhlenbeck processes, long memory processes (e.g. Comte & Renault (1996, 1998)), and jump processes (e.g. Andersen, Benzoni & Lund (2002), Eraker, Johannes & Polson (2003), and Eraker (2004)). The possibility of market microstructure effects contaminating the data is also entertained in a model that allows for a bid-ask bounce in the spirit of Roll (1984). In the latter case we also consider alternative estimators by Barndorff-Nielsen & Shephard (2003*a*, 2004) and Hansen & Lunde (2004*a*, 2004*b*) with theoretical justification under market microstructure noise.

The estimation methods are compared in a Monte Carlo study which reveals that the theoretical robustness of the estimators towards persistence or jumps in the stochastic process governing the latent volatility carries over to practice. On the other hand, irregularities such as bid-ask bounce effects, which the methods have no theoretical robustness against, in general render the wavelet estimator, and to a lesser degree realized volatility, less useful in practice. However, we find the Fourier method to be superior compared to the other two estimators in the case of market microstructure noise, and indeed this estimator remains very useful even in that case. More strikingly, even when compared to the bias correction methods designed specifically to handle market microstructure effects, the Fourier method is superior with respect to RMSE while having only slightly higher bias.

The remainder of the paper is organized as follows. In the next section we present a typical model for asset returns and integrated volatility, which is the focus of the estimation. Section 3 presents the three estimation methods, realized volatility, the Fourier estimator, and the wavelet estimator. In sections 4 and 5 we describe the setup of the Monte Carlo study for the models without market microstructure (bid-ask bounce) effects and the model including the bid-ask bounce, respectively. Section 5 also introduces three alternative estimators designed for the case with market microstructure contamination. Section 6 presents the simulation results in terms of the finite sample biases and RMSEs of the estimators in sections 3 and 5, and section 7 offers some concluding remarks.

2 Integrated Volatility and Quadratic Variation

Suppose the log-price of an asset, $p(t)$, follows a stochastic volatility model, where the basic Brownian motion is generalized to allow the volatility to vary over time, see e.g. Ghysels, Harvey & Renault (1996) or Barndorff-Nielsen & Shephard (2001) and the references therein for overviews of the vast literature on this topic. In particular, we assume $p(t)$ follows the general stochastic differential equation model

$$dp(t) = \mu(t)\,dt + \sigma(t)\,dw(t), \quad t \geq 0, \tag{1}$$

where the mean process $\mu(\cdot)$ and the instantaneous volatility process $\sigma(\cdot) > 0$ are assumed to be independent of the Brownian motion $w(\cdot)$. Allowing the instantaneous volatility to be random and possibly exhibit serial correlation (which we shall do below, see the examples in section 4), the model (1) will generate returns with unconditional distributions that are fat-tailed and have volatility clustering. This replicates more closely actually observed processes than constant volatility, and e.g. allows the model to overcome some of the shortcomings of the basic Black & Scholes (1973) option pricing model, see Hull & White (1987).

An important feature of the model (1) is that

$$p(t) | \int_0^t \mu(s)\,ds, \sigma^{2*}(t) \sim N\left(\int_0^t \mu(s)\,ds, \sigma^{2*}(t)\right), \tag{2}$$

where

$$\sigma^{2*}(t) = \int_0^t \sigma^2(s)\,ds \tag{3}$$

is called the integrated volatility (or integrated variance) and is the object of interest. For pricing options, this is the relevant volatility measure, see Hull & White (1987), and for the econometrician this is the object to be estimated, see also Andersen & Bollerslev (1998). Thus, it is also the focal point of this paper.

Another important object is the quadratic variation of the process $p(t)$, denoted $[p](t)$, which is defined for any semimartingale (see e.g. Protter (1990)) by

$$[p](t) = p^2(t) - 2\int_0^t p(s-)\, dp(s) \tag{4}$$

or equivalently

$$[p](t) = p\lim \sum_{j=1}^{M} \left(p(s_j) - p(s_{j-1})\right)^2, \tag{5}$$

where $0 = s_0 < s_1 < ... < s_M = t$ and the limit is taken for $\max_j |s_j - s_{j-1}| \to 0$ as $M \to \infty$.

Under some very general regularity conditions, which allow the instantaneous volatility process to exhibit many irregularities, e.g. jumps, long memory, or even nonstationarity, it was shown by Andersen & Bollerslev (1998) and Barndorff-Nielsen & Shephard (2001) that

$$[p](t) = \sigma^{2*}(t) \tag{6}$$

for the model (1). For the purpose of this paper we note that this implies that the object of interest, $\sigma^{2*}(t)$, can be estimated either directly via a parametric model or nonparametrically via the quadratic variation. The latter has been a very popular approach recently.

3 Estimation of Integrated Volatility

We next review briefly three different methods of estimation for the integrated volatility based on a sample of high-frequency return observations. First, we describe the very popular realized volatility approach which utilizes the connection to quadratic variation, and subsequently we describe the Fourier and wavelet estimators which estimate $\sigma^{2*}(t)$ via the Fourier and wavelet transforms, respectively.

To lighten the notation, we make some simplifying assumptions. We assume that only one day (or month or year) of observations is available and denote the intra-daily (or intra-monthly or intra-yearly) observations on the log-price of the asset by p_j. In principle, the time period could be any arbitrary period, but most often in empirical work either intra-daily or intra-monthly observations are considered in order to estimate integrated volatility on a daily or monthly basis. For instance, to obtain estimates of $\sigma^{2*}(t)$, where $t = 1, ..., T$ denotes days, the econometrician employs intra-daily observations on the price process. For the purpose of this section we normalize by setting $T = 1$ and consider the estimation of integrated volatility over one time period.

The intra-daily data may be of the tick-by-tick type, where the price is observed at every trade or quote (tick), or of the fixed interval type, where the price is observed at fixed intervals, e.g. at 5 minute intervals. Both these types of data, commonly denoted high-frequency data, are widely available on many types of assets. For applications to stock return data, see e.g. French et al. (1987), Andersen, Bollerslev, Diebold & Ebens (2001), Eraker et al. (2003), or Eraker (2004), and for introductions to some of the very commonly used Olsen and Associates high-frequency exchange rate data sets, see e.g. Guillaume, Dacorogna, Davé, Müller, Olsen & Pictet (1997), Andersen & Bollerslev (1998), Andersen, Bollerslev, Diebold & Labys (2001), Dacorogna, Gencay, Müller, Pictet & Olsen (2001), or Barndorff-Nielsen & Shephard (2001, 2002).

3.1 Realized Volatility

Suppose n intra-daily observations are available. It is often desirable to have observations that are evenly spaced in time. Suppose M evenly spaced observations are desired based on the n intra-daily and possibly irregularly spaced observations. To avoid the problem of irregularly spaced data in high-frequency data sets, it is common to use the imputation scheme (in contrast to an interpolation scheme, see e.g. Dacorogna et al. (2001) or Barucci & Reno (2002*a*)), i.e. for each of the M evenly spaced observations to use the last observed price. In this way a data set of M evenly spaced intra-daily price observations can be constructed based on an irregularly spaced high-frequency data set (preferably with n much higher than M to avoid using the same observation more than once).

Using these M evenly spaced price observations we denote the continuously compounded intra-daily returns by

$$r_j = p_j - p_{j-1}, \quad j = 1, ..., M. \tag{7}$$

Using (5), quadratic variation can be estimated by

$$\hat{\sigma}^{2*}_{RV,M} = \sum_{j=1}^{M} r_j^2, \tag{8}$$

which is denoted the realized volatility of the process $p(\cdot)$. If several days of intra-daily observations on $p(\cdot)$ (or $r(\cdot)$) were observed, a time series of daily observations on the realized volatility could be obtained, but that is not the issue here so we retain the assumption that only one day of observations is available. Some authors refer to the quantity (8) as the realized variance and reserve the name realized volatility for the square root of (8), but we shall use the more conventional name realized volatility.

Andersen & Bollerslev (1998) and Andersen, Bollerslev, Diebold & Labys (2001) noted that by definition $\hat{\sigma}^{2*}_{RV,M}$ in (8) is a consistent (in probability) estimator of integrated volatility (3), using (5) and (6). The consistency result does not require the observations to be evenly spaced, only that the maximum distance between observations goes to zero in the limit. Barndorff-Nielsen & Shephard (2002) strengthened the consistency result and showed that $\hat{\sigma}^{2*}_{RV,M}$ converges to σ^{2*} in probability at rate $\sqrt{M}$, and furthermore that $\hat{\sigma}^{2*}_{RV,M}$ satisfies

$$\frac{\hat{\sigma}^{2*}_{RV,M} - \sigma^{2*}}{\sqrt{\frac{2}{3}\sum_{j=1}^{M} r_j^4}} \to_d N(0,1) \quad \text{as } M \to \infty. \tag{9}$$

This is a mixed Gaussian asymptotic distribution theory since the denominator is itself random, and hence $\hat{\sigma}^{2*}_{RV,M}$ has fatter tails than the normal distribution. For some simulation evidence on the accuracy of the asymptotic distribution (9), see Barndorff-Nielsen & Shephard (2002) and Barndorff-Nielsen & Shephard (2003*b*).

If there are many intra-daily observations available, the coarseness of the realized volatility estimator is governed by the choice of M. For example, if trading occurs 24 hours per day as in the foreign exchange markets, choosing 5 minute returns in (7) and (8) corresponds to $M = 288$, 15 minute returns corresponds to $M = 96$, and hourly returns corresponds to $M = 24$. Choosing a higher number of intra-daily returns improves the precision of the estimator but at the same time makes it more sensitive towards microstructure effects in the market, e.g. measurement errors, bid-ask bounces, etc., see section 5.

3.2 Fourier Estimator

This estimator was suggested by Malliavin & Mancino (2002) and subsequently applied by Barucci & Reno (2002*a*, 2002*b*). The Fourier method only requires that the quadratic variation (4) or (5) is bounded. The method is based on the Fourier transform,

$$\int_0^{2\pi} \sigma^2(s)\, ds = 2\pi a_0\left(\sigma^2\right), \tag{10}$$

where

$$a_0\left(\sigma^2\right) = \lim_{S\to\infty} \frac{\pi}{2S} \sum_{s=1}^{S} \left(a_s^2(dp) + b_s^2(dp)\right) \tag{11}$$

and the Fourier coefficients are given by

$$a_s(dp) = \frac{1}{\pi} \int_0^{2\pi} \cos(st)\, dp(t), \quad s \geq 1, \tag{12}$$

$$b_s(dp) = \frac{1}{\pi} \int_0^{2\pi} \sin(st)\, dp(t), \quad s \geq 1. \tag{13}$$

Hence, the n intra-daily and possibly irregularly spaced observations at times $t_1, ..., t_n$, i.e. on the interval $[t_1, t_n]$, needs to be normalized into the interval $[0, 2\pi]$, and we denote the renormalized time points by $\tau_j = 2\pi(t_j - t_1)/(t_n - t_1), j = 1, ..., n$. The formal justification for using (10) and (11) is given by Malliavin & Mancino (2002). Barucci & Reno (2002*b*) derived the following approximations that we use to compute estimates of the Fourier coefficients $(a_s(dp), b_s(dp))$ in (12) and (13),

$$\hat{a}_s(dp) = \frac{p(\tau_n) - p(\tau_1)}{\pi} - \frac{1}{\pi} \sum_{j=2}^{n} p(\tau_{j-1})\left(\cos(s\tau_j) - \cos(s\tau_{j-1})\right), \quad s \geq 1, \tag{14}$$

$$\hat{b}_s(dp) = \frac{1}{\pi} \sum_{j=2}^{n} p(\tau_{j-1})\left(\sin(s\tau_j) - \sin(s\tau_{j-1})\right), \quad s \geq 1. \tag{15}$$

To obtain the Fourier estimator of integrated volatility, we thus plug (14) and (15) into (11) and use (10),

$$\hat{\sigma}^{2*}_{F,S} = \frac{\pi^2}{S} \sum_{s=1}^{S} \left(\hat{a}_s^2(dp) + \hat{b}_s^2(dp)\right). \tag{16}$$

The coarseness of the estimator is controlled by the user-chosen number S, i.e. the number of Fourier coefficients to include in the estimation, which is related to M for realized volatility by

$S = M/2$. Higher S thus corresponds to choosing a finer grid (higher M) for realized volatility, e.g. choosing 5 minute returns instead of 15 minute returns. Note that by including only the lowest S frequencies in the Fourier estimator (16), high-frequency noise or short-run noise is ignored by the estimator. Hence, by choosing a smaller number of (low) frequency ordinates to be used for estimation, i.e. by choosing S small, it is in principle possible to render the Fourier estimator invariant to short-run noise introduced by e.g. market microstructure effects.

3.3 Wavelet Estimator

Høg & Lunde (2003) suggest an alternative estimator which is based on the wavelet transform of $dp(\cdot)$, instead of the Fourier transform based estimator above.

A function $y(j) \in \mathbb{L}^2$ with $j = 0, 1, \ldots, 2^K - 1$, where $K \in \mathbb{N}$ and $\mathbb{L}^2$ is the space of square integrable functions, can be expanded into a wavelet series,

$$y(t) = \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} w_{j,k} \psi_{j,k}(t), \tag{17}$$

with wavelet coefficients

$$w_{j,k} = 2^{j/2} \int y(t) \psi_{j,k}(t) dt, \tag{18}$$

where $\psi_{j,k}(t) \equiv 2^{-j/2} \psi\left(2^{-j} t - k\right)$ for $j, k \in \mathbb{N}$, is the collection of dilations (scales), j, and translations, k, of the wavelet function $\psi(t)$.

By design the wavelets strength rests in its ability to simultaneously localize a process in time and scale. At high scales, the wavelet has a small centralized time support enabling it to focus in on short lived time phenomena like a singularity point. At low scales, the wavelet has a large time support allowing it to identify long periodic behavior. By moving from low to high scales, the wavelet zooms in on the behavior of a process at a particular point in time, identifying singularities, jumps, and cusps. Alternatively, the wavelet can zoom out to reveal the long, smooth features of a series. In our implementation we use the Haar wavelet as in Høg & Lunde (2003).

As was the case with the Fourier estimator, the time interval $[t_1, t_n]$ needs to be renormalized. For the wavelet estimator we renormalize into the interval $[0, 1]$, and we denote the renormalized time points by $\tau_j = (t_j - t_1) / (t_n - t_1), j = 1, ..., n$. It is shown by Høg & Lunde (2003) that defining $K(n) = \text{int}(\log_2(n))$ the wavelet estimator can be based on

$$\int_0^1 \sigma^2(s)\,ds = \lim_{n\to\infty} 2^{-K(n)} \sum_{k=0}^{2^{K(n)}-1} w^2_{K(n),k}(dp), \tag{19}$$

where $K(n)$ is the highest power of two below or equal to n.

Given a time series of n possibly irregularly sampled observations $(t_j, p(t_j)), j = 1, ..., n$, we construct the Haar wavelet coefficients

$$w_{j,k}(dp) = 2^{j/2}\left[2p\left(2^{-j}(k+1/2)\right) - p\left(2^{-j}k\right) - p\left(2^{-j}(k+1)\right)\right]. \tag{20}$$

To obtain the wavelet estimator we plug (20) into (19),

$$\hat{\sigma}^{2*}_{W,K} = \sum_{k=0}^{2^K-1}\left[2p\left(2^{-K}(k+1/2)\right) - p\left(2^{-K}k\right) - p\left(2^{-K}(k+1)\right)\right]^2. \tag{21}$$

The coarseness of the wavelet estimator is controlled by the user-chosen number K, i.e. the number of dilations (scales) to include in the estimation, which is related to M by $2^K = M$ or $K = \log_2(M)$. Choosing K high is similar to choosing a high S for the Fourier estimator and a high M for realized volatility.

4 Monte Carlo Setup

In this section we describe the simulation setup used to investigate the biases and root mean squared errors (RMSEs) of the three estimation methods described above. The objective of the simulation exercise is to shed light on which method most accurately estimates the integrated volatility in practical application with realistic sample sizes.

For the Monte Carlo study we let the log-price $p(t)$ be generated by

$$dp(t) = \sigma(t)\,dW_1(t), \tag{22}$$

and assume the instantaneous volatility process $\sigma(t)$ follows

$$\text{Model A}: \qquad d\ln\sigma^2(t) = \alpha\left(\beta - \ln\sigma^2(t)\right)dt + \nu dW_{2d}(t) \tag{23}$$

or

$$\text{Model B :} \qquad d\sigma^2(t) = \alpha\left(\beta - \sigma^2(t)\right)dt + \sigma(t)\,\nu dW_2(t) + \kappa(t)\,dq(t)\,, \tag{24}$$

where $W_1(\cdot)$ and $W_2(\cdot)$ are independent standard Brownian motions. Both volatility models ensure, under regularity conditions, that volatility cannot become negative.

In Model A, $W_{2d}(\cdot)$ is a fractional Brownian motion of order d independent of $W_1(\cdot)$. It may be represented by the Holmgren-Riemann-Liouville fractional integral

$$W_{2d}(t) = \int_0^t \frac{(t-s)^2}{\Gamma(d+1)} dW(s)\,, \quad t > 0, \tag{25}$$

where $W(\cdot)$ is a standard Brownian motion, see e.g. Comte & Renault (1996) and Marinucci & Robinson (1999). Sometimes (25) is denoted a type II fractional Brownian motion, see Marinucci & Robinson (1999) for details on the generation and simulation of this process. The generating mechanism for the instantaneous volatility process in Model A, i.e. (23), is a logarithmic Ornstein-Uhlenbeck process, driven by a possibly fractional Brownian motion. Thus, Model A allows the stochastic process driving volatility to exhibit long memory, which is also empirically well founded, see e.g. Andersen, Bollerslev, Diebold & Ebens (2001), Andersen, Bollerslev, Diebold & Labys (2001, 2003), Andersen et al. (2004), and the references therein. Note that when $d = 0$, (23) is a standard (logarithmic) Ornstein-Uhlenbeck process. For a detailed discussion of the implications of using $d \neq 0$ in Model A, see Comte & Renault (1998).

In Model B the allowance for long memory has been substituted with allowance for jumps, represented by the jump process $dq(t)$. In particular, the process (24) is a Cox et al. (1985) (or CIR), square-root model for the volatility process with the addition of a (positive) jump process. The arrival of jumps is assumed to follow a Poisson process with intensity given by $\lambda_0 + \lambda_1\sigma^2(t)$, i.e. the arrival of jumps is assumed to depend on the volatility through the parameter λ_1. The magnitude of the jumps $\kappa(t)$ is assumed to be exponentially distributed with mean μ. This setup of the jump process follows that in Andersen et al. (2002) (although they have the jump process in the mean and assume $\kappa(t)$ to be log-normally distributed), Eraker et al. (2003), and Eraker (2004). Note that when $\kappa(t) = 0$, i.e. in the absence of jumps, (24) is a standard CIR model.

For each Monte Carlo DGP we simulate (through simple Euler discretization) artificial time series of second-by-second intra-daily data points for $(p(t), \sigma(t)), t = 1, ..., T$, assuming 24 hour trading, i.e. a total of $T = 86,400$ seconds. Then we sample from this log-price "process" by assuming that the time difference between successive observations is exponentially distributed with mean τ. We choose the value $\tau = 14$ (also used by Barucci & Reno (2002a)) corresponding to approximately 6171 observations per day and to the mean time between observations in the DEM-USD exchange rate time series analyzed by e.g. Andersen & Bollerslev (1998). This procedure is repeated for $L = 10,000$ days (replications), which then each have a different number (n) of irregularly spaced observations.

To evaluate the performance of the estimation methods, we calculate, for each method and for each day l, the relative error statistic

$$\pi_l = \frac{\hat{\sigma}_l^{2*} - \sum_{t=1}^{T} (p_l(t) - p_{l-1}(t))^2}{\sum_{t=1}^{T} (p_l(t) - p_{l-1}(t))^2}, \quad l = 1, ..., L, \tag{26}$$

where $\hat{\sigma}_l^{2*}$ denotes any of the estimators above and $\sum_{t=1}^{T} (p_l(t) - p_{l-1}(t))^2$ is the "true" integrated volatility on day l. In the following we focus on the mean and RMSE of π_l, which can be given natural interpretations as (relative) bias and RMSE of the estimators, respectively. We define them in the usual way as

$$\text{Bias} = \bar{\pi} = \frac{1}{L} \sum_{l=1}^{L} \pi_l \tag{27}$$

and

$$\text{RMSE} = \left(\frac{1}{L} \sum_{l=1}^{L} \pi_l^2 \right)^{1/2} = \sqrt{\bar{\pi}^2 + s_\pi^2}, \tag{28}$$

where $s_\pi^2 = L^{-1} \sum_{l=1}^{L} (\pi_l - \bar{\pi})^2$ is the sample variance of π_l. Note that, since we divide by the "true" integrated volatility in (26), the bias $\bar{\pi}$ in (27) is a relative bias and not an absolute bias.

5 Microstructure Effects in Volatility

Finally, to expand further on our Monte Carlo study and perhaps introduce more empirical realism, this section applies the estimation methods on data contaminated by market microstructure effects. Inspired by Roll (1984), we introduce a bid-ask bounce effect, see also

Campbell, Lo & MacKinlay (1997, pp. 100-101). Intuitively, as random buys and sells arrive at the market, prices can bounce back and forth between the ask and the bid prices, thus creating spurious volatility and serial correlation in returns, even if the fundamental value of the asset remains unchanged.

As before, let $p(t)$ denote the simulated log-price at time t when no microstructure effects are present and introduce the order-driven indicator variable $I(t)$, indicating whether, at time t, the observed price is an ask (buyer-initiated, $I(t) = 1$) or a bid (seller-initiated, $I(t) = -1$) price. The new contaminated price $p^*(t)$ becomes

$$p^*(t) = p(t) + \frac{\xi}{2} I(t) \tag{29}$$

so that

$$\begin{aligned} dp^*(t) &= dp(t) + \frac{\xi}{2}\left(I(t) - I(t-1)\right) \\ &= dp(t) + \frac{\xi}{2} dI(t), \end{aligned} \tag{30}$$

where ξ is the percentage spread and the $I(t)$ are independently (across t and from p) and identically distributed with $\Pr(I(t) = 1) = \Pr(I(t) = -1) = 1/2$.

Since $dp(t)$ and $dI(t)$ are independent, at least theoretically, the (instantaneous) variance, covariance, and first-order autocorrelation (the setup does not introduce any higher-order serial correlation) of the contaminated asset return can be easily calculated from (30) and are given as

$$Var(dp^*(t)) = \sigma^2(t) + \frac{\xi^2}{2} \tag{31}$$

$$Cov(dp^*(t), dp^*(t-1)) = -\frac{\xi^2}{4} \tag{32}$$

$$Corr(dp^*(t), dp^*(t-1)) = -\frac{\xi^2/4}{\sigma^2(t) + \xi^2/2} \leq 0. \tag{33}$$

Hence, we notice that $dp^*(t)$ exhibits spurious volatility and negative serial correlation as a result of the bid-ask bounce, and a larger spread, ξ, implies a higher spurious volatility.

Roll (1984) estimates the effective spreads of NYSE and AMEX stocks using returns data from 1963 to 1982, and finds the average effective spread to be 0.298% using daily returns and 1.74% using weekly returns. This corresponds roughly to $\xi = 0.003$ and $\xi = 0.017$ in (29). To ensure that these somewhat outdated estimates are still relevant, we have estimated the effective spreads of six highly traded stocks and three common stock indices using daily returns following the estimation method of Roll (1984). The results are presented in Table 1 for the two periods 1.1.1995–6.30.2004 and 1.1.2001–6.30.2004. The former period is typical for recent empirical applications, where roughly a decade of high-frequency data is employed, and the latter period is after the so-called "decimalization" in 2000 which would presumably have lowered the effective spread. However, it is clear from Table 1 that effective bid-ask spreads are only slightly lower than the earlier results by Roll (1984), and in particular our estimated values correspond to $\xi \in (0.0019, 0.0108)$.

Table 1 about here

It is well known that sampling at the highest possible frequency induces bias due to market microstructure effects, see e.g. Andreou & Ghysels (2002) and Oomen (2002), because intra-day returns become (spuriously) autocorrelated. In applications it is common to use lower-frequency intra-day returns to alleviate the problem, see e.g. Andersen & Bollerslev (1997) and Andersen, Bollerslev, Diebold & Labys (2000), since the (spurious) autocorrelation only lives for a short period of time (Hansen & Lunde (2004*b*)). However, it is only recently that a formal justification for this approach has been established by Bandi & Russell (2003*a*). Unfortunately, the use of lower-frequency observations inevitably implies a loss of information in the sense that fewer data points are applied, resulting in an inefficient measure of volatility. Several bias reduction methods have been suggested in the literature, e.g. the (moving average or autoregressive) filtering techniques by Andersen, Bollerslev, Diebold & Ebens (2001) and Bollen & Inder (2002), the subsampling techniques by Zhang, Mykland & Ait-Sahalia (2003), and the simple autocorrelation correction methods by Hansen & Lunde (2004*a*, 2004*b*).

In the present setup with market microstructure contamination, we extend the investigation of the three above-mentioned estimators to include also the Newey-West correction applied by e.g. French et al. (1987) and Zhou (1996) and the related bias correction procedure suggested by Hansen & Lunde (2004a, 2004b)[1]. Furthermore, since recent literature concerning jump detection in financial markets has focused on bipower variation, see e.g. Andersen, Bollerslev & Diebold (2003), Barndorff-Nielsen & Shephard (2003a, 2004) and Huang & Tauchen (2003), we also briefly explore the performance of (2nd lag) realized bipower variation under market microstructure contamination.

Under the process (29), realized volatility is no longer consistent for integrated volatility, but instead

$$\widehat{\sigma}_{RV,M}^{2*}(t) \to_p [p^*](t) = \sigma^{2*}(t) + \frac{\xi^2}{4}\sum_{j=1}^{M}(dI(j))^2, \tag{34}$$

which diverges as $M \to \infty$. Thus, $\widehat{\sigma}_{RV,M}^{2*}(t)$ estimates the sum of the "true" latent integrated volatility and the summing of an infinite number of (squared) microstructure contributions, i.e. the "true" latent integrated volatility is stochastically dominated by the microstructure component.

As emphasized by Barndorff-Nielsen & Shephard (2003a, 2004), who consider both jump processes and market microstructure noise processes, if the noise (or jump) component is of finite activity, separate non-parametric identification of the two components in (34) is possible using bipower variation measures. In our setup with first order serial correlation, the 2nd lag realized bipower variation measure,

$$\widehat{\sigma}_{BV,M}^{2*}(t) = \frac{\pi}{2}\frac{M}{M-2}\sum_{j=3}^{M}|r_{t,j}|\,|r_{t,j-2}|, \tag{35}$$

would, in the finite activity case, converge in probability (as $M \to \infty$) to $\sigma^{2*}(t)$. Hence, the microstructure noise component (or jump component) could be identified as $\widehat{\sigma}_{RV,M}^{2*} - \widehat{\sigma}_{BV,M}^{2*}$, see Barndorff-Nielsen & Shephard (2003a, 2004), but the identification of jump or noise components is not the focus of the present study. Instead, we wish to examine the impact of the market microstructure effects on the performance of $\widehat{\sigma}_{BV,M}^{2*}$, since Barndorff-Nielsen & Shephard (2003a, p. 29) conjecture that "[i]t does not mean that market microstructure effects have no impact on [2nd lag] BPV. Rather, we take this as meaning that it should be more

[1] Note that if the price observations are not contaminated with noise, applying the bias correction would not be efficient relative to standard (uncorrected) realized volatility.

robust to market microstructure effects than realised variance." To what extent this is true under our particular noise process (29) will be examined in the simulations below.

Hansen & Lunde (2004*b*) consider the estimator

$$\widehat{\sigma}^{2*}_{HL,M}(t) = \sum_{j=1}^{M} r^2_{t,j} + 2\frac{M}{M-1}\sum_{j=2}^{M} r_{t,j}r_{t,j-1}, \tag{36}$$

which is robust to first order serial correlation. More general estimators robust to higher order serial correlation are also considered by Hansen & Lunde (2004*b*), but they argue that the first order correction in (36) is sufficient, at least for the sampling frequencies considered here. Furthermore, Hansen & Lunde (2004*b*) show that $\widehat{\sigma}^{2*}_{HL,M}(t)$ is an unbiased estimator of the "true" integrated volatility $\sigma^{2*}(t)$.

A similar bias correction method is based on the Bartlett kernel, well known from the Newey & West (1987) covariance estimator. The implementation by French et al. (1987) and Zhou (1996) in the context of integrated volatility estimation was

$$\widehat{\sigma}^{2*}_{NW,M}(t) = \sum_{j=1}^{M} r^2_{t,j} + 2\sum_{h=1}^{q_M}\left(1 - \frac{h}{q_M+1}\right)\sum_{j=h+1}^{M} r_{t,j}r_{t,j-h} \tag{37}$$

using $q_M = 1$. In our implementation we use $q_M = \text{int}\left(4(M/100)^{2/9}\right)$, which is typical in the time series literature, e.g. Andrews & Monahan (1992). As mentioned in Hansen & Lunde (2004*b*), this estimator may not be unbiased like $\widehat{\sigma}^{2*}_{HL,M}$ but it may have a smaller asymptotic MSE. Of course, given the nature of the noise that we assume, the Hansen & Lunde (2004*b*) estimator is presumably superior to the Newey & West (1987) correction, which is robust to a more general noise structure - that may or may not be relevant in practice depending on the specific market - involving higher order AR or MA terms.

6 Simulation Results

To introduce as much empirical realism as possible, the parameter values chosen for the simulations of the volatility processes are based on the estimation results of Roll (1984), Andersen et al. (2002), and Eraker (2004).

The parameter values used for the simulations of Model A in (23) are inspired by the analysis of the S&P 500 stock index in Andersen et al. (2002, Table III, p. 1256), who found the estimated parameter values $(\hat{\alpha}, \hat{\beta}, \hat{\nu}) = (0.0062, -1, 0.0374)$ (in our notation) for their model without jumps. In particular, we use the parameter values $\alpha \in \{0, 0.0062, 0.0124\}$, $\beta = -1$, and $\nu \in \{0.0374, 0.1122\}$ in our simulations. Furthermore, since it is empirically well founded that financial volatility time series exhibit long memory (see e.g. Andersen, Bollerslev, Diebold & Ebens (2001), Andersen, Bollerslev, Diebold & Labys (2001, 2003), Andersen et al. (2004), and the references therein), we let the process (23) be governed by a stationary fractional Brownian motion with long memory parameter $d \in \{0, 0.15, 0.30, 0.45\}$. The studies by Andersen, Bollerslev, Diebold & Ebens (2001), Andersen, Bollerslev, Diebold & Labys (2001, 2003), and Andersen et al. (2004) find long memory in volatility with a parameter around $0.3 - 0.45$. The starting values for Model A were chosen as $p(0) = \ln(100)$ and $\sigma(0) = e^{\beta}$. This makes for easy comparisons across days.

In Model B, the square-root jump model (24), we follow the results on jumps in the volatility of the S&P 500 stock index returns by Eraker (2004, Table III, p. 49), who found the estimated parameter estimates $(\hat{\alpha}, \hat{\beta}, \hat{\nu}) = (0.023, 0.943, 0.137)$ which we also employ in our simulations of (24). For the jump process, Eraker (2004) found the estimates $(\hat{\mu}, \hat{\lambda}_0, \hat{\lambda}_1) = (1.530, 0.002, 1.298)$, so we simulate our process using the parameter values $\mu \in \{0.7515, 1.530\}$, $\lambda_0 \in \{0.002, 0.01\}$, and $\lambda_1 \in \{0, 1.298, 2.596\}$. The starting values for Model B were chosen as $p(0) = \ln(100)$ and $\sigma(0) = \beta$.

For the model in (29) including market microstructure distortions, we let the parameter choice be inspired by Roll (1984, Table I, p. 1132-1133) and our own estimates of the more recent effective spreads of some highly traded stocks and stock indices in Table 1. Roll (1984) finds the average effective spread of NYSE and AMEX stocks, using returns data from 1963 to 1982, to be 0.298% and 1.74% for daily and weekly returns, respectively. As shown in Table 1 we estimate effective spreads between 0.19% and 1.08% for six highly traded stocks and three common stock indices using daily returns data and the two sample periods 1.1.1995-6.30.2004 and 1.1.2001-6.30.2004. The former period is typical for recent applications where roughly a decade of high-frequency data is employed and the latter period is after the so-called "decimalization" in 2000 which would presumably have lowered the effective spread. However, according to our results in Table 1, the effective bid-ask spreads are slightly lower than (but still comparable to) the older results by Roll (1984), and in particular we use $\xi \in \{0.003, 0.005, 0.01\}$ in simulating (29). Hence, our simulation is a little conservative compared to Roll (1984), and probably more in line with the current average microstructure frictions such as bid-ask spreads, especially since the so-called "decimalization" in 2000. We simulate with $\alpha = 0.0124$, $\beta = -1$, and $d = 0$ in model (23), and to measure the effect of the bid-ask bounce for different values of the volatility of volatility, we simulate using $\nu \in \{0.05, 0.5\}$. The starting values were chosen as in Model A.

We implement realized volatility using 1 minute returns ($M = 1440$), 5 minute returns ($M = 288$), and 15 minute returns ($M = 96$). The first two values are close to the optimal sampling frequency, in the presence of market microstructure noise, of 1.5 minutes found by Bandi & Russell (2003*a*), which is shorter than the 5 minute frequency used in most empirical work on the subject, see e.g. Andersen, Bollerslev, Diebold & Ebens (2001) and the above references. Previously, Andersen et al. (2000) conjectured that optimal sampling should be based on 15-20 minute returns, corresponding to our third implementation of realized volatility. Note that the results by Bandi & Russell (2003*a*) depend crucially on the very high liquidity of the simulated stocks. Consequently, Bandi & Russell (2003*b*) find that less liquid stocks require lower sampling frequencies leading to optimal sampling schemes that are close to the standard 5 minute frequency. For related (theoretical) results on optimal sampling schemes for maximum likelihood estimation of diffusions in the presence of market microstructure noise, see Ait-Sahalia, Mykland & Zhang (2003).

The Fourier method is implemented using $S = 720$ corresponding to 1 minute returns, $S = 144$ corresponding to 5 minute returns, and $S = 48$ corresponding to 15 minute returns. Finally, the wavelet estimator is implemented using slightly different values since it is limited to powers of 2. In particular, we use $K = 10$ corresponding to $M = 2^{10} = 1024$, which is a little less than for realized volatility and the Fourier estimator, $K = 8$ corresponding to $M = 2^8 = 256$, which is again a little less, and $K = 7$ corresponding to $M = 2^7 = 128$, which is a little more than for realized volatility and the Fourier estimator. That is, our implementations of realized volatility and of the Fourier estimator match with respect to the coarseness of the estimators, whereas the implementation of the wavelet estimator is a little off compared to the other two. The reason for this is two-fold. First, the wavelet estimator must be implemented using a power of 2 which limits the possible choices of coarseness. Second, realized volatility is by far the most applied estimator in empirical studies, the 5 minute estimator being particularly popular, and we thus wish to include that particular implementation here. Hence, when reading our results below, one should take into account this slight discrepancy in the implementation of the wavelet estimator.

In particular, we thus even the playing field between the realized volatility estimator and the Fourier estimator by matching the implied number of intra-daily returns used in the estimation. The Monte Carlo analysis in Barucci & Reno (2002*a*, 2002*b*) typically contrast a 5 minute realized volatility estimator to a Fourier estimator using all observations. Also, again in contrast to Barucci & Reno (2002*a*, 2002*b*) we use the imputation rather than interpolation technique, see also Dacorogna et al. (2001), as this is what the literature has settled upon.

In Tables 2-11 the results of our Monte Carlo study are presented in terms of the relative biases (27) (multiplied by 1,000) and the RMSEs (28) of the estimators. All calculations were made using the computer language Gauss v5.0.

Tables 2-5 display the results for the estimators when instantaneous volatility is governed by a logarithmic fractional Ornstein-Uhlenbeck process, Model A. Tables 6 and 7 display the results for the square-root jump process, Model B, and Tables 8-11 display the results when bid-ask bounce effects are introduced into the price process and the true latent instantaneous volatility is governed by a simple logarithmic Ornstein-Uhlenbeck process, i.e. by Model A with $d = 0$.

We consider first the Monte Carlo results for the three estimation methods in Model A, where the underlying instantaneous volatility follows a logarithmic Ornstein-Uhlenbeck process possible governed by a fractional Brownian motion.

The relative biases are generally very low (less than 1%) as evident from Tables 2 and 4, and seem to be lower for higher sampling frequencies which was to be expected. Intuitively, one could anticipate that the performance of the estimators would be poor when the simulated series exhibit a high degree of persistence in the form of long memory or even non-stationarity, even though all three estimation methods are robust to such persistence in theory. However, the tables reveal that the methods are very robust towards such persistence even in finite samples. When α and d are zero the simulated process is very persistent since the integrated volatility is then governed by a continuous-time random walk. This persistence becomes even more pronounced when d is greater than zero (and $\alpha = 0$), as the order of integration then becomes $1 + d$, but we still do not find any indications that contradict the theoretical robustness.

However, we do find the highest relative biases when the series exhibit slow mean reversion ($\alpha \leq 0.062$) and long memory ($d = 0.45$). In perspective, this is interesting since many empirical studies have concluded that integrated volatility time series exhibit long memory of roughly this magnitude. Furthermore, Table 4 indicates that when increasing the volatility of volatility ($\nu = 0.1122$) we generally encounter slightly higher relative biases.

Tables 2-5 about here

Increasing the volatility of volatility has a very interesting but expected impact on the variability of the estimators. Tables 3 and 5 show a clear pattern when the series exhibit relatively strong long memory ($d \geq 0.3$), where the RMSE (mainly variance since biases are very small) of all the estimators increases noticeably, which is also in accordance with the distribution theory in (9) for realized volatility. On the other hand, when the integrated volatility series are simulated with relatively weak long memory ($d \leq 0.15$), the higher variability of the volatility does not imply less accuracy (higher RMSE), i.e. the RMSEs in Table 5 are nearly identical to those in Table 3 when d is small. Overall, the RMSEs are higher if smaller sampling frequencies are used. Hence, if possible, one should in empirical investigations use a relatively high sampling frequency even if the latent integrated volatility series exhibits long memory or non-stationarity.

However, an important caveat here is the risk of contamination from market microstructure effects which would lead to the selection of a lower sampling frequency. This is discussed in detail below, where indeed our simulations support this important point.

Focusing more explicitly on the three individual estimators, we do not find any noticeable differences between them. However, independent of the size of the mean reversion, α, and the size of the volatility of the volatility, ν, it seems that the Fourier method provides the lowest relative bias and RMSE when looking at the empirically interesting scenario of high d and high sampling frequency, although only marginally so.

An interesting irregularity that has turned up in empirical studies, e.g. Eraker et al. (2003) and Eraker (2004), is the possible presence of jumps in the instantaneous volatility process. Tables 6 and 7 present the results for the square-root jump process, Model B. Since the biases are very low, we again find support of the theoretical robustness mentioned in section 2. Nonetheless, there is a tendency that a larger magnitude of the jump, μ, weakens the performance of the estimators as the (absolute sizes of the) relative biases and RMSEs increase when μ increases. This is generally also the case when increasing the intensity of the arrival of jumps, but only for the part of the intensity that is directly linked to the volatility. That is, there is no clear relationship between the size of the minimum intensity, λ_0, and the performance of the estimators. On the contrary, increasing λ_1 such that the arrival intensity is more sensitive towards the size of the volatility generally implies an increase in the relative biases and as expected an even more pronounced impact is found on the variability of the estimators (the RMSEs). Here we find a noticeable increase when intensity increases.

Tables 6-7 about here

As under the persistence scenario in Tables 2-5, it is recommendable in Model B to use a higher sampling frequency as the relative biases and especially the RMSEs become smaller. Increasing the sampling frequency also seems to imply that the RMSEs of the estimators become relatively insensitive towards the magnitude of the jumps as the RMSEs are almost identical for high frequencies (across μ and λ_0). Again we find that the Fourier method seems to provide better relative bias and RMSE compared to the other estimators, although only marginally so.

Ultimately, the conclusions of the above scenarios are that the estimators, in general, are robust towards irregularities such as long memory, non-stationarity, and jumps. The performance of the estimators depends on the characteristics of the irregularity and especially on the sampling frequency.

We next consider the model in section 5, i.e. Model A with a bid-ask bounce effect. When introducing this market microstructure effect, and thereby introducing spurious volatility and serial correlation, we do not expect the estimators to remain as well behaved as above. This fact is reflected in Tables 8 and 9. Almost all biases are positive, revealing that the methods overestimate the true latent integrated volatility. That is, the bid-ask effect introduces spurious volatility and negative serial correlation which causes the estimators to overestimate the underlying integrated volatility. For example, the relative bias of the wavelet estimator (with $K = 10$) is approximately 0.4 with a 1% spread, see Table 8, meaning that it overestimates the true integrated volatility by as much as 40% in this case.

Tables 8-9 about here

As expected, we generally find that the performance of the estimators declines rapidly as the sampling frequency increases. This is in agreement with the literature where it is widely accepted that increasing the sampling frequency worsens the impact from market microstructure effects. This tendency becomes even more distinct when the spread (in percentage of the underlying asset price) increases. Hence, in general, when conducting an empirical analysis of financial time series contaminated by bid-ask bounce effects (and in general by market microstructure effects), where the spread is known to be significant (even if it is as small as 0.3%), it is extremely important not to use too high sampling frequency. Indeed, the lowest biases in our simulations in Table 8 are found for the very lowest sampling frequencies. Furthermore, the RMSEs in Table 9 indicate the usual trade-off between bias and variance in this model.

In the two scenarios of persistence and jumps in the integrated volatility process, i.e. Models A and B in Tables 2-7, our Monte Carlo study did not reveal any apparent differences between the three estimators. Now, when introducing an irregularity that the methods are not theoretically equipped to handle, we can really unveil the strengths of the three estimators. The wavelet estimator of integrated volatility generally provides the highest biases and RMSEs for all sizes of the spread and for any frequency, thus rendering the wavelet estimator less useful in case of bid-ask bounce effects. The realized volatility estimator is also heavily biased in the presence of the bid-ask bounce, but not quite as badly as the wavelet estimator.

Probably the most striking finding is that the Fourier method is vastly superior in this scenario with market microstructure effects. The Fourier estimator is practically unaffected with respect to both bias and RMSE except in the case with the highest sampling frequency. Even then the bias and RMSE are vastly superior to those of the other methods. The superiority of the Fourier estimator in the presence of market microstructure contamination can be attributed to the decomposition of the integrated variance into components of varying frequencies by the Fourier transform. That is, including only the lowest S frequencies in the Fourier estimator (16) implies that high-frequency noise or short-run noise is ignored by the estimator. Hence, by choosing a smaller number of (low) frequency ordinates to be used for estimation, i.e. by choosing S small, it is in principle possible to render the Fourier estimator invariant to short-run noise introduced by market microstructure effects.

Tables 10-11 about here

In Tables 10-11 we present the relative biases and RMSEs of the three alternative integrated volatility estimators from section 5, i.e. the 2nd lag realized bipower variation (35), the Hansen & Lunde (2004*b*) estimator (36), and the Newey & West (1987) estimator (37). The tables show that the 2nd lag realized bipower variation estimator is greatly affected by the spurious volatility and the negative serial correlation introduced by the bid-ask bounce effect. For small ξ and high M the negative serial correlation causes the 2nd lag realized bipower variation estimator to underestimate the true latent integrated volatility, whereas for higher ξ the 2nd lag realized bipower variation is nearly as biased as realized volatility. Thus, the 2nd lag realized bipower variation seems to be slightly more robust to market microstructure effects, as conjectured by Barndorff-Nielsen & Shephard (2003*a*, pp. 28-29), only in the case of a large spread and high sampling frequency.

Tables 10-11 also demonstrate that the bias correction suggested by Hansen & Lunde (2004*b*) works very well, and in particular outperforms the Newey & West (1987) correction since (at least for higher frequencies) $\widehat{\sigma}^{2*}_{HL,M}$ compares favorably to $\widehat{\sigma}^{2*}_{NW,M}$ both in terms of bias and RMSE. Of course, given the nature of the noise that we assume, the Hansen & Lunde (2004*b*) estimator is presumably superior to the Newey & West (1987) correction, which is robust to a more general noise structure - that may or may not be relevant in practice depending on the specific market - involving higher order AR or MA terms.

More interestingly, the autocorrelation bias correction methods are only noticeably superior to the Fourier method for the very highest sampling frequency ($M = 1440$ and $S = 720$). In fact, the RMSEs for $\widehat{\sigma}^{2*}_{HL,M}$ are generally much higher than the RMSEs for the Fourier method except for the combination of highest sampling frequency and highest spread. This implies that the Fourier method remains a very attractive estimator in the presence of market microstructure effects even in comparison with methods specifically designed to handle such contamination. However, if one wishes to incorporate the full information available in ultra high-frequency data the need for bias correction methods remains.

7 Concluding Remarks

We have considered the properties of three estimation methods for integrated volatility, i.e. realized volatility, the Fourier estimator, and the wavelet estimator, when only a finite (high-frequency) sample of the price process is observed. We considered several different generating mechanisms for the instantaneous integrated volatility: Ornstein-Uhlenbeck, long memory, and jump processes, and the possibility of market microstructure effects contaminating the data is also entertained in a model that allows for a bid-ask bounce effect. In the latter case we also considered alternative estimators with theoretical justification under market microstructure noise.

Our simulation study reveals that the theoretical robustness of the estimators towards persistence or jumps in the stochastic process governing the latent volatility carries over to practice. On the other hand, irregularities such as bid-ask bounce effects, which the methods have no theoretical robustness against, in general render the wavelet estimator, and to a lesser degree realized volatility, less useful in practice. However, we find the Fourier method to be superior compared to the other two estimators in the case of market microstructure noise, and indeed this estimator remains very useful in that case. Even more strikingly, when compared to the bias correction methods designed specifically to handle market microstructure effects, the Fourier method is superior with respect to RMSE while having only slightly higher bias.

References

Ait-Sahalia, Y., Mykland, P. A. & Zhang, L. (2003), 'How often to sample a continuous-time process in the presence of market microstructure noise', *Forthcoming in Review of Financial Studies* .

Andersen, T. G., Benzoni, L. & Lund, J. (2002), 'An empirical investigation of continuous-time equity return models', *Journal of Finance* **57**, 1239–1284.

Andersen, T. G. & Bollerslev, T. (1997), 'Intraday periodicity and volatility persistence in financial markets', *Journal of Empirical Finance* **4**, 115–158.

Andersen, T. G. & Bollerslev, T. (1998), 'Answering the skeptics: Yes, standard volatility models do provide accurate forecasts', *International Economic Review* **39**, 885–905.

Andersen, T. G., Bollerslev, T. & Diebold, F. X. (2003), 'Some like it smooth, and some like it rough: Untangling continuous and jump components in measuring, modeling, and forecasting asset return volatility', *Working Paper, University of Pennsylvania* .

Andersen, T. G., Bollerslev, T. & Diebold, F. X. (2004), Parametric and nonparametric volatility measurement, *in* L. P. Hansen & Y. Ait-Sahalia, eds, 'Handbook of Financial Econometrics (Forthcoming)', North-Holland, Amsterdam.

Andersen, T. G., Bollerslev, T., Diebold, F. X. & Ebens, H. (2001), 'The distribution of realized stock return volatility', *Journal of Financial Economics* **61**, 43–76.

Andersen, T. G., Bollerslev, T., Diebold, F. X. & Labys, P. (2000), 'Great realizations', *Risk* **13**, 105–108.

Andersen, T. G., Bollerslev, T., Diebold, F. X. & Labys, P. (2001), 'The distribution of exchange rate volatility', *Journal of the American Statistical Association* **96**, 42–55.

Andersen, T. G., Bollerslev, T., Diebold, F. X. & Labys, P. (2003), 'Modelling and forecasting realized volatility', *Econometrica* **71**, 579–625.

Andreou, E. & Ghysels, E. (2002), 'Rolling-sample volatility estimators: Some new theoretical, simulation, and empirical results', *Journal of Business and Economic Statistics* **20**, 363–376.

Andrews, D. W. K. & Monahan, J. C. (1992), 'An improved heteroskedasticity and autocorrelation consistent covariance matrix estimator', *Econometrica* **60**, 953–966.

Bandi, F. M. & Russell, J. R. (2003*a*), 'Microstructure noise, realized volatility, and optimal sampling', *Working paper, University of Chicago Graduate School of Business* .

Bandi, F. M. & Russell, J. R. (2003*b*), 'Separating market microstructure noise from volatility', *Working paper, University of Chicago Graduate School of Business* .

Barndorff-Nielsen, O. E. & Shephard, N. (2001), 'Non-Gaussian Ornstein-Uhlenbeck-based models and some of their uses in financial economics (with discussion)', *Journal of the Royal Statistical Society Series B* **63**, 167–241.

Barndorff-Nielsen, O. E. & Shephard, N. (2002), 'Econometric analysis of realized volatility and its use in estimating stochastic volatility models', *Journal of the Royal Statistical Society Series B* **64**, 253–280.

Barndorff-Nielsen, O. E. & Shephard, N. (2003*a*), 'Econometrics of testing for jumps in financial economics using bipower variation', *Working Paper, Oxford University* .

Barndorff-Nielsen, O. E. & Shephard, N. (2003*b*), How accurate is the asymptotic approximation to the distribution of realized variance?, *in* D. W. K. Andrews, J. L. Powell, P. A. Rudd & J. H. Stock, eds, 'Identification and Inference for Econometric Models. A Festschrift in Honour of Thomas J. Rothenberg (Forthcoming)', Econometric Society Monograph, Cambridge University Press, Cambridge.

Barndorff-Nielsen, O. E. & Shephard, N. (2004), 'Power and bipower variation with stochastic volatility and jumps (with discussion)', *Journal of Financial Econometrics* **2**, 1–48.

Barucci, E. & Reno, R. (2002*a*), 'On measuring volatility and the GARCH forecasting performance', *Journal of International Financial Markets, Institutions and Money* **12**, 183–200.

Barucci, E. & Reno, R. (2002*b*), 'On measuring volatility of diffusion processes with high frequency data', *Economics Letters* **74**, 371–378.

Black, F. & Scholes, M. (1973), 'The pricing of options and corporate liabilities', *Journal of Political Economy* **81**, 637–654.

Bollen, B. & Inder, B. (2002), 'Estimating daily volatility in financial markets utilizing intraday data', *Journal of Empirical Finance* **9**, 551–562.

Campbell, J. Y., Lo, A. W. & MacKinlay, A. C. (1997), *The Econometrics of Financial Markets*, Princeton University Press, Princeton.

Comte, F. & Renault, E. (1996), 'Long-memory continuous-time models', *Journal of Econometrics* **73**, 101–149.

Comte, F. & Renault, E. (1998), 'Long-memory in continuous-time stochastic volatility models', *Mathematical Finance* **8**, 291–323.

Cox, J. C., Ingersoll, J. E. & Ross, S. A. (1985), 'A theory of the term structure of interest rates', *Econometrica* **53**, 385–408.

Dacorogna, M. M., Gencay, R., Müller, U. A., Pictet, O. V. & Olsen, R. B. (2001), *An Introduction to High-Frequency Finance*, Academic Press, San Diego.

Eraker, B. (2004), 'Do stock prices and volatility jump? Reconciling evidence from spot and option prices', *Journal of Finance* **59**, 1367–1403.

Eraker, B., Johannes, M. & Polson, N. (2003), 'The impact of jumps in volatility and returns', *Journal of Finance* **58**, 1269–1300.

French, K. R., Schwert, G. W. & Stambaugh, R. F. (1987), 'Expected stock returns and volatility', *Journal of Financial Economics* **19**, 3–30.

Ghysels, E., Harvey, A. C. & Renault, E. (1996), Stochastic volatility, *in* C. R. Rao & G. S. Maddala, eds, 'Statistical Methods in Finance', North-Holland, Amsterdam, pp. 119–191.

Guillaume, D. M., Dacorogna, M. M., Davé, R. R., Müller, U. A., Olsen, R. B. & Pictet, O. V. (1997), 'From the bird's eye to the microscope: A survey of new stylized facts of the intra-daily foreign exchange markets', *Finance and Stochastics* **1**, 95–129.

Hansen, P. R. & Lunde, A. (2004*a*), 'Realized variance and IID market microstructure noise', *Working Paper, Brown University* .

Hansen, P. R. & Lunde, A. (2004*b*), 'An unbiased measure of realized variance', *Working Paper, Brown University* .

Høg, E. & Lunde, A. (2003), 'Wavelet estimation of integrated volatility', *Working Paper, Aarhus School of Business* .

Huang, X. & Tauchen, G. (2003), 'The relative contribution of jumps to total price variance', *Working Paper, Duke University* .

Hull, J. C. & White, A. (1987), 'The pricing of options on assets with stochastic volatilities', *Journal of Finance* **42**, 281–300.

Malliavin, P. & Mancino, M. E. (2002), 'Fourier series method for measurement of multivariate volatilities', *Finance and Stochastics* **6**, 49–61.

Marinucci, D. & Robinson, P. M. (1999), 'Alternative forms of fractional Brownian motion', *Journal of Statistical Planning and Inference* **80**, 111–122.

Newey, W. K. & West, K. D. (1987), 'A simple positive semi-definite, heteroskedasticity and autocorrelation consistent covariance matrix', *Econometrica* **55**, 703–708.

Oomen, R. A. C. (2002), 'Modelling realized variance when returns are serially correlated', *Working Paper, Warwick Business School* .

Protter, P. (1990), *Stochastic Integration and Differential Equations: A New Approach*, Springer-Verlag, New York.

Roll, R. (1984), 'A simple implicit measure of the effective bid-ask spread in an efficient market', *Journal of Finance* **39**, 1127–1139.

Zhang, L., Mykland, P. A. & Ait-Sahalia, Y. (2003), 'A tale of two time scales: Determining integrated volatility with noisy high-frequency data', *Forthcoming in Journal of the American Statistical Association* .

Zhou, B. (1996), 'High-frequency data and volatility in foreign-exchange rates', *Journal of Business and Economic Statistics* **14**, 45–52.

Table 1: Estimates of the Bid-Ask Spread

Sample period	AA	GE	IBM	IP	JNJ	JPM	DJIA	S&P100	S&P 500
1.1.1995-6.30.2004	0.62%	0.66%	1.08%	0.56%	0.68%	0.53%	0.19%	0.50%	0.27%
1.1.2001-6.30.2004	0.94%	0.38%	0.36%	0.41%	0.50%	0.80%	0.47%	0.49%	0.41%

Note: The estimated percentage spreads are $2\sqrt{-Cov(dp^*(t), dp^*(t-1))}$ following Roll (1984).

Table 2: Simulation Results for Model A I: Relative bias (x1,000)

		Realized Volatility			Fourier			Wavelet		
α	d	15 min	5 min	1 min	$S = 48$	$S = 144$	$S = 720$	$K = 7$	$K = 8$	$K = 10$
0.0000	0.00	-0.7055	-0.5915	-0.4304	-1.0062	-0.8192	-0.6201	0.4496	-1.7567	-0.4530
	0.15	0.4583	0.0927	0.2498	1.4775	-0.1334	-0.5432	-0.0119	-0.9714	-0.1007
	0.30	0.4863	-1.3222	-0.8911	-0.6410	-0.9654	-0.7419	-0.1815	-1.4819	-0.7038
	0.45	-2.7630	-3.0976	-1.4876	-3.6948	-2.3962	-1.0222	-2.3314	-2.7952	-1.0631
0.0062	0.00	0.0755	0.5924	-0.1730	-1.6686	-0.8413	-0.2669	-0.6830	-0.6474	-0.5130
	0.15	2.0025	0.4435	0.1936	1.4967	0.1044	-0.2624	1.5461	-1.5653	0.0488
	0.30	0.4254	1.3451	-0.1448	2.1558	0.7306	-0.3160	-1.9598	-0.9732	-0.1840
	0.45	-5.3383	-3.3919	-2.0422	-4.7310	-2.9846	-1.8246	-1.2312	-1.9397	-2.7308
0.0124	0.00	-1.7707	-1.4212	-0.2095	-2.2512	-1.1830	-0.4279	-0.1238	-0.2991	-0.4353
	0.15	-0.8734	-0.5482	-0.9901	-0.8381	-1.0543	-1.0814	1.4513	-2.6114	-0.8863
	0.30	-0.6384	-0.0582	-0.3482	0.7188	0.1207	-0.2306	-0.3850	0.1175	-0.2032
	0.45	0.1016	-0.6692	-0.3883	-1.6604	-0.8492	-0.3214	-1.6346	-0.1256	-1.6033

Note: $\beta = -1, \nu = 0.0374$.

Table 3: Simulation Results for Model A I: RMSE

		Realized Volatility			Fourier			Wavelet		
α	d	15 min	5 min	1 min	$S = 48$	$S = 144$	$S = 720$	$K = 7$	$K = 8$	$K = 10$
0.0000	0.00	0.1435	0.0829	0.0394	0.1436	0.0829	0.0395	0.1261	0.0876	0.0454
	0.15	0.1449	0.0835	0.0395	0.1451	0.0841	0.0394	0.1266	0.0883	0.0454
	0.30	0.1522	0.0875	0.0417	0.1511	0.0881	0.0414	0.1316	0.0930	0.0481
	0.45	0.2366	0.1344	0.0635	0.2299	0.1328	0.0630	0.2004	0.1420	0.0740
0.0062	0.00	0.1443	0.0831	0.0394	0.1453	0.0825	0.0390	0.1259	0.0880	0.0455
	0.15	0.1451	0.0834	0.0394	0.1447	0.0841	0.0392	0.1267	0.0889	0.0457
	0.30	0.1540	0.0880	0.0407	0.1539	0.0885	0.0405	0.1323	0.0939	0.0471
	0.45	0.2324	0.1348	0.0625	0.2283	0.1332	0.0619	0.2010	0.1430	0.0733
0.0124	0.00	0.1444	0.0831	0.0391	0.1444	0.0821	0.0389	0.1253	0.0878	0.0447
	0.15	0.1454	0.0831	0.0392	0.1458	0.0833	0.0391	0.1250	0.0882	0.0453
	0.30	0.1524	0.0880	0.0417	0.1521	0.0880	0.0413	0.1329	0.0928	0.0482
	0.45	0.2332	0.1368	0.0638	0.2284	0.1339	0.0625	0.2040	0.1430	0.0747

Note: $\beta = -1, \nu = 0.0374$.

Table 4: Simulation Results for Model A II: Relative bias (x1,000)

		Realized Volatility			Fourier			Wavelet		
α	d	15 min	5 min	1 min	$S = 48$	$S = 144$	$S = 720$	$K = 7$	$K = 8$	$K = 10$
0.0000	0.00	1.2748	-0.3355	-0.7811	0.7110	-0.2561	-0.9026	0.0917	-2.0880	-0.7733
	0.15	-1.4604	-1.1342	-1.0978	-1.5769	-0.9251	-0.6643	0.8388	-1.9007	-0.3863
	0.30	-0.5957	-1.6722	-0.3615	-1.5510	-1.3847	-0.7577	-0.7989	-1.1150	-0.6636
	0.45	3.6261	-2.4180	-2.0503	1.3231	-0.9371	-1.1779	-2.8642	-2.8783	-1.8356
0.0062	0.00	0.2415	0.3961	-0.9228	-1.3742	-1.1969	-0.8255	-1.8643	-1.3787	-0.4259
	0.15	-0.7589	-0.4210	0.5303	-2.2068	-0.3985	0.2660	-0.7397	0.1676	0.3289
	0.30	-2.2294	-2.1213	-0.7497	-1.0979	-0.8559	-0.7321	-0.7992	-2.6153	-1.1359
	0.45	4.0332	-2.5570	-2.8106	0.5308	-2.0275	-0.7129	-8.6316	4.1822	-2.0436
0.0124	0.00	0.8175	1.0354	-0.3810	1.5470	-0.2110	-0.6553	-0.4869	-1.4444	-0.4940
	0.15	-0.1336	-1.0675	-0.9770	-1.4175	-1.4680	-0.9212	-2.2219	-1.4821	-1.0009
	0.30	-0.6238	-2.9059	-1.3050	-3.4262	-2.0246	-1.2529	-0.3346	-0.1858	-2.0064
	0.45	0.0824	-2.5836	-1.8345	0.8803	-1.9273	-0.9515	-3.0580	1.5135	-3.1962

Note: $\beta = -1, \nu = 0.1122$.

Table 5: Simulation Results for Model A II: RMSE

		Realized Volatility			Fourier			Wavelet		
α	d	15 min	5 min	1 min	$S = 48$	$S = 144$	$S = 720$	$K = 7$	$K = 8$	$K = 10$
0.0000	0.00	0.1444	0.0834	0.0392	0.1441	0.0843	0.0391	0.1252	0.0878	0.0454
	0.15	0.1475	0.0853	0.0399	0.1470	0.0844	0.0399	0.1270	0.0901	0.0464
	0.30	0.1928	0.1126	0.0529	0.1876	0.1108	0.0523	0.1687	0.1193	0.0604
	0.45	0.3972	0.2241	0.1080	0.3802	0.2201	0.1060	0.3331	0.2425	0.1244
0.0062	0.00	0.1427	0.0830	0.0392	0.1444	0.0823	0.0391	0.1260	0.0889	0.0452
	0.15	0.1473	0.0850	0.0403	0.1476	0.0847	0.0400	0.1289	0.0904	0.0462
	0.30	0.1942	0.1106	0.0518	0.1906	0.1112	0.0515	0.1666	0.1187	0.0611
	0.45	0.4040	0.2298	0.1073	0.3801	0.2268	0.1059	0.3363	0.2437	0.1229
0.0124	0.00	0.1447	0.0823	0.0392	0.1453	0.0832	0.0388	0.1244	0.0865	0.0454
	0.15	0.1463	0.0849	0.0401	0.1474	0.0847	0.0404	0.1266	0.0893	0.0460
	0.30	0.1929	0.1106	0.0525	0.1910	0.1103	0.0519	0.1679	0.1199	0.0607
	0.45	0.3852	0.2204	0.1065	0.3707	0.2172	0.1055	0.3291	0.2399	0.1237

Note: $\beta = -1, \nu = 0.1122$.

Table 6: Simulation Results for Model B: Relative bias (x1,000)

			Realized Volatility			Fourier			Wavelet		
μ	λ_0	λ_1	15 min	5 min	1 min	$S = 48$	$S = 144$	$S = 720$	$K = 7$	$K = 8$	$K = 10$
0.7515	0.0002	0.000	-0.4096	-0.1961	-0.0101	-0.1788	-1.0763	-0.5357	-0.7135	-1.7050	-0.5489
		1.298	0.7465	0.1119	-0.5220	-0.1551	0.2411	-0.3065	0.3924	-0.6000	-0.1430
		2.596	0.0689	0.1546	-1.1297	0.7316	-0.0242	-0.9588	-0.0525	-2.1016	-1.4776
	0.0100	0.000	0.7938	-0.6632	-0.0514	-0.3697	-1.1959	-0.5438	-2.0806	-0.6514	-1.1300
		1.298	-1.2241	0.5597	0.1855	-1.5922	-0.0911	0.0822	-0.2367	0.7899	-0.0018
		2.596	-0.8487	-0.6688	-0.0896	0.0389	-0.1353	-0.2071	0.4876	-0.8325	0.2767
1.5300	0.0002	0.000	-0.8915	0.1760	0.2460	-0.1027	-0.2596	-0.1420	-1.8248	0.9044	-0.2842
		1.298	-0.1182	-1.6774	-1.6449	-0.8492	-1.3033	-2.0398	-2.6302	-3.7401	-2.2957
		2.596	-2.8403	-2.2980	-1.0465	-0.5911	-2.1993	-0.6012	-2.9168	-2.5319	-0.9122
	0.0100	0.000	-0.3969	-1.0942	-0.0919	-0.5675	-0.7069	-0.2713	0.6571	-1.5714	-0.1078
		1.298	-1.3928	-1.3416	-0.7572	-1.1821	-0.5097	-0.7751	-3.0634	-1.8643	-0.3814
		2.596	-3.3209	-2.2607	-1.1896	-2.2697	-1.2317	-1.2272	-3.1931	-2.2495	-1.2734

Note: $\alpha = 0.023, \beta = 0.943, \nu = 0.137$.

Table 7: Simulation Results for Model B: RMSE

μ	λ_0	λ_1	Realized Volatility			Fourier			Wavelet		
			15 min	5 min	1 min	$S=48$	$S=144$	$S=720$	$K=7$	$K=8$	$K=10$
0.7515	0.0002	0.000	0.1451	0.0838	0.0395	0.1456	0.0843	0.0396	0.1245	0.0891	0.0462
		1.298	0.1546	0.0888	0.0410	0.1509	0.0876	0.0413	0.1312	0.0941	0.0479
		2.596	0.1624	0.0933	0.0445	0.1597	0.0931	0.0439	0.1410	0.0995	0.0511
	0.0100	0.000	0.1453	0.0832	0.0392	0.1450	0.0839	0.0394	0.1255	0.0893	0.0456
		1.298	0.1531	0.0890	0.0412	0.1527	0.0887	0.0413	0.1315	0.0936	0.0480
		2.596	0.1628	0.0944	0.0442	0.1620	0.0942	0.0438	0.1415	0.0996	0.0505
1.5300	0.0002	0.000	0.1449	0.0826	0.0394	0.1450	0.0830	0.0392	0.1255	0.0873	0.0457
		1.298	0.1609	0.0936	0.0440	0.1606	0.0947	0.0440	0.1409	0.0976	0.0512
		2.596	0.1893	0.1105	0.0518	0.1904	0.1096	0.0514	0.1661	0.1169	0.0600
	0.0100	0.000	0.1460	0.0836	0.0396	0.1453	0.0841	0.0392	0.1275	0.0899	0.0455
		1.298	0.1627	0.0931	0.0437	0.1610	0.0937	0.0434	0.1405	0.0985	0.0509
		2.596	0.1885	0.1096	0.0512	0.1877	0.1087	0.0509	0.1631	0.1164	0.0600

Note: $\alpha = 0.023, \beta = 0.943, \nu = 0.137$.

Table 8: Simulation Results for Model A with Bid-Ask Bounce I: Relative bias (x1,000)

		Realized Volatility			Fourier			Wavelet		
ξ	ν	15 min	5 min	1 min	$S = 48$	$S = 144$	$S = 720$	$K = 7$	$K = 8$	$K = 10$
0.003	0.05	0.3586	2.0200	16.7259	-0.5529	-0.8312	9.7284	2.8724	9.7648	34.9593
	0.50	-1.2278	3.6067	17.5627	-0.7294	0.3012	10.7018	7.5200	9.8177	35.4649
0.005	0.05	3.8316	10.1418	48.0333	0.6336	1.6777	28.3677	12.7942	25.2158	97.3825
	0.50	1.7986	8.5590	48.8193	-1.3183	0.0210	28.9011	13.5345	25.8629	99.6641
0.01	0.05	12.2471	40.2888	192.4841	0.4416	6.6724	114.7427	52.6355	103.6826	390.1397
	0.50	14.0446	39.6586	195.3937	1.1314	6.1340	116.5520	52.9863	104.7563	396.4723

Note: $\alpha = 0.0124, \beta = -1, d = 0$.

Table 9: Simulation Results for Model A with Bid-Ask Bounce I: RMSE

		Realized Volatility			Fourier			Wavelet		
ξ	ν	15 min	5 min	1 min	$S = 48$	$S = 144$	$S = 720$	$K = 7$	$K = 8$	$K = 10$
0.003	0.05	0.1438	0.0842	0.0434	0.1435	0.0837	0.0409	0.1272	0.0904	0.0586
	0.50	0.1462	0.0854	0.0444	0.1467	0.0851	0.0419	0.1292	0.0902	0.0599
0.005	0.05	0.1458	0.0850	0.0629	0.1454	0.0837	0.0491	0.1272	0.0941	0.1094
	0.50	0.1466	0.0862	0.0659	0.1465	0.0853	0.0507	0.1305	0.0960	0.1153
0.01	0.05	0.1458	0.0962	0.1981	0.1445	0.0841	0.1229	0.1413	0.1421	0.3951
	0.50	0.1492	0.0974	0.2084	0.1470	0.0852	0.1290	0.1448	0.1479	0.4173

Note: $\alpha = 0.0124, \beta = -1, d = 0$.

Table 10: Simulation Results for Model A with Bid-Ask Bounce II: Relative bias (x1,000)

		2nd Lag Realized Bipower Variation			Realized Volatility HL			Realized Volatility NW		
ξ	ν	15 min	5 min	1 min	15 min	5 min	1 min	15 min	5 min	1 min
0.003	0.05	0.6040	1.4069	-21.1331	-3.8274	-1.3772	-1.0926	-4.0726	-2.0171	0.7270
	0.50	-2.0761	2.6984	-20.5927	-2.9800	-1.6996	-0.3671	-3.0280	-1.5966	1.2103
0.005	0.05	3.9707	9.5815	11.1772	-0.3028	1.5037	0.3467	-1.1657	1.3116	6.7039
	0.50	0.0750	7.6059	11.7717	-3.6051	-1.8898	-0.4615	-1.6500	-0.7138	4.7324
0.01	0.05	12.9514	40.3629	159.7021	-1.5864	-0.0097	2.9440	1.4349	5.7893	24.9240
	0.50	14.6658	38.6138	162.7062	-0.7714	1.3298	2.3916	4.0392	7.3932	25.1500

Note: $\alpha = 0.0124, \beta = -1, d = 0$.

Table 11: Simulation Results for Model A with Bid-Ask Bounce II: RMSE

		2nd Lag Realized Bipower Variation			Realized Volatility HL			Realized Volatility NW		
ξ	ν	15 min	5 min	1 min	15 min	5 min	1 min	15 min	5 min	1 min
0.003	0.05	0.1653	0.0971	0.0480	0.2466	0.1432	0.0657	0.2357	0.1665	0.0859
	0.50	0.1685	0.0979	0.0488	0.2488	0.1474	0.0656	0.2381	0.1688	0.0882
0.005	0.05	0.1662	0.0967	0.0453	0.2496	0.1452	0.0660	0.2389	0.1686	0.0871
	0.50	0.1679	0.0983	0.0495	0.2521	0.1472	0.0668	0.2417	0.1702	0.0884
0.01	0.05	0.1675	0.1070	0.1677	0.2490	0.1460	0.0694	0.2381	0.1679	0.0904
	0.50	0.1711	0.1077	0.1803	0.2527	0.1489	0.0703	0.2420	0.1699	0.0920

Note: $\alpha = 0.0124, \beta = -1, d = 0$.