Applicability of control function separability as a condition for feasible control function estimation of simultaneous equations models

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Abstract

An existing condition, called control function separability, characterizing the feasibility of control function estimation of nonparametric systems of simultaneous equations with scalar disturbances, developed by Blundell and Matzkin, 2014, is shown to be equivalent to a triangular representation condition. An alternative characterization of control function separability in terms of model structural derivatives is given. Necessary and almost necessary conditions are developed, showing that control funciton separability is restrictive to the point of effectively requiring two additional structural monotonicity conditions, symmetry in the inverse structural equations in the main dependent variable, and a control function that is linear in the structural disturbances. It is further shown how these conditions can be used easily to rule out control function separability, and thus to rule out the feasibility of the control function approach.

Contents

1 Introduction

Identification of parameters and structural features in the presence of endogeneity in the disturbances is a well-known problem in econometrics. In settings where models are linear in the parameters, there are relatively straightforward and well-established strategies for dealing with this endogeneity, including the method of instrumental variables (IV) and the generalized method of moments (GMM).

In nonparametric settings where linearity assumptions are relaxed, however, and especially when the model disturbances are nonadditive, identification in the presence of endogeneity becomes much more of a challenge. In nonparametric models, generally, identification and estimation of structural functions is not always feasible, particularly in smaller datasets (Matzkin, 2006). In some cases, IV and GMM or other approaches remain applicable. Frequently, however, the most efficient and feasible approach is that of control variables, or control functions (Wooldridge, 2010).

While the control function approach is generally efficient and relatively easy to implement, it is known to lack robustness across various sets of assumptions, and frequently requires restrictive distributional conditions (Imbens & Wooldridge, 2007). Characterizing the kinds of econonometric models suitable for control function estimation, therefore, is important for applied work. Control function methods have proven generally versatile[1], but their application to nonlinear systems of simultaneous equations with nonparametric disturbances in particular has been restricted to

[1]For recent examples, see Petrin & Train, 2010 and Rau, 2013.

triangular systems of equations.

A previous finding in the literature, by Blundell and Matzkin, 2014, however, shows that a condition they name "control function separability" completely characterizes the (nonempty) set of *nontriangular* systems of simultaneous equations with scalar disturbances that can be estimated using the control function approach. In this paper, I show that showing that this condition holds is equivalent to showing that the system can be written as a triangular system by reparametrization of the disturbances. This result both gives an alternative method of verification of control function separability and permits a better understanding of its restrictive nature, which is later discussed.

I also give an alternative characterization of this condition in terms of a ratio of partial derivatives of the structural functions, which is simpler and more accessible than a similar condition given by Blundell and Matzkin.

Furthermore, I argue that, in practice, control function separability in nonparametric systems of simultaneous equations requires two additional monotonicity conditions on the structural equations, as well as linearity in the control function itself and certain formal similarity between the two inverse structural functions. The restrictiveness of the separability condition was noted briefly by Blundell and Matzkin, but the specific restrictions have not been previously spelled out. The necessary conditions developed in this paper can be used quickly to rule out control function separability without attempting to verify it explicitly.

Because control function separability is a recently developed condition which has not been explored in the literature beyond preliminary exposition, I make reference in this analysis mainly to the paper by Blundell and Matzkin (2014), where the condition was first presented.

The remainder of this paper is structured as follows. Section 2 gives the theoretical background for control functions and control function separability by way of referring to the existing literature, and defines the structural model used by Blundell and Matzkin that will be referred to throughout the paper. Section 3 presents two alternative characterizations of control function separability, including the triangular representation condition, with illustrative examples. Section 4 presents necessary and almost necessary conditions for the separability condition, with applications. Section 5 concludes.

2 Background

2.1 Control functions

Control functions are observed variables, or functions of variables, which have the effect of "purging" the explanatory variables in a model of their endogeneity, so that the parameters or non-parametric features of interest can be identified. Typically, a control function is an estimator of an unobserved variable, such as a reduced form disturbance. The control function approach, which requires the presence of an excluded exogenous variable, is conceptually similar to that of instrumental variables, and in the ordinary linear case amounts to the same thing (see Imbens & Wooldridge, 2007). In nonlinear settings, the control function method is more parsimonious, and likely more efficient, than alternative methods (Wooldridge, 2010).

The earliest complete demonstration of what is now called control function estimation is Heckman's two-step method for estimation in the presence of sample selection effects (Heckman, 1976), although the conceptual framework for the approach can be seen in Telser, 1964. The method was formalized in a later paper (Heckman, 1985). Blundell and Powell, 2003, were the first to show how the control function method can be applied to semiparametric and nonparametric models.

The control function approach involves writing down an equation for a given endogenous variable in a structural model. When this equation is in reduced form, it can be estimated to yield estimates of its disturbances, which under certain conditions can be used as a control function, to be included in the estimation of a corresponding structural equation so as to remove simultaneity. A structural equation combined with a reduced form equation for an endogenous variable corresponds in form to a "triangular" system of simultaneous equations, where the second equation does not depend on the left-hand side of the first.

For illustration, a very general nonparametric, nonseparable example of such a system from Imbens and Newey (2009) can be considered. The structural equation is as follows:

$$Y = g(X, \epsilon),$$

where X are observed explanatory variables, ϵ is a vector of disturbances and the unknown function g is the econometric object of interest. The reduced form for a unique endogenous variable X_2 is given:

$$X_2 = h(Z, \eta),$$

where Z is a vector of exogenous covariates and η is a scalar disturbance in which the unknown function $h(Z, \eta)$ is monotonically increasing.

In this framework, a control variable V is any observable variable such that X and ϵ, the explanatory variables and the disturbances, are independent conditional on V. In practice, an estimate $\hat{\eta}$ of the reduced form disturbance η, or a one-to-one function of η, can often serve as the control function. In Imbens and Newey (2009), this control function was the conditional CDF of the reduced form disturbance η. Once a control variable is available, the structural relationship between Y and X can be identified from the distribution of Y conditional on X and V.

It should be noted that nonparametric estimation is known to face the problem of dimensionality. Large sample sizes are required to identify and estimate an unknown structural function which is not specified up to a vector of parameters. As a result, researchers have turned to estimating various averages and derivatives of structural functions (Matzkin, 2006). These are lower-dimensional, but still useful, objects which can be feasibly estimated using smaller datasets. For example, Imbens and Newey, 2009, showed how to identify and estimate certain quantile, average and policy effects.

Imbens and Newey examined triangular systems, remarking that the general model, described above, is not compatible with general simultaneous systems, such as supply-demand systems, where the reduced form for each equation would contain two disturbances. Similarly, Wooldridge (2010), surveying the literature, shows how a control function approach can be used in nonlinear triangular systems with nonadditive disturbances, but notes that triangularity is a restrictive assumption.

In some cases, however, such as in the supply-demand system case, a simultaneous relationship of economic nature between two or more variables is known or assumed, where each dependent or left-hand variable is a function of at least one of the other dependent variables. Here, estimation cannot proceed by straightforward estimation of a reduced-form equation, as each equation in the system is subject to the simultaneity problem. As a result, it is at first unclear in what manner or under what conditions control functions can be used in nontriangular systems of nonlinear equations, such as systems of the following form,

$$
\begin{aligned}
Y &= g(X, \epsilon), \\
X_2 &= h(Z, Y, \eta),
\end{aligned}
$$

where X_2 is included in X and Z is exogenous. Blundell and Matzkin (2014), on that question, proved an important result, showing that a condition they refer to as "control function separability" was a necessary and sufficient condition for the observational equivalence of certain non-triangular simultaneous systems with triangular systems, and thus for the feasibility of control function estimation.

Their result can be seen as an application of a result by Kasy (2010), regarding the one-dimensionality of the reduced form of the endogenous variable, to structural systems of simultaneous equations.[2] Blundell and Matzkin's contribution is important in addition to Kasy's because it applies specifically to simultaneous systems. In some cases, the reduced form equation for the endogenous variable is obscured by the form of the corresponding simultaneous system given from economic theory.

Observational equivalence with a triangular system, established by control function separability, allows for the identification and estimation of certain structural quantities in nonlinear systems of simultaneous equations with nonadditive, nonparametric disturbances using the control function technique. Blundell and Matzkin show that structural derivatives of the function, the average structural function as defined by Blundell and Powell (2003), the local average response function as defined by Altonji and Matzkin (2005) and the quantile structural function as defined by Imbens and Newey (2009) can be identified in this way.

While the control function separability condition gives a conclusive answer to the question of when control functions can be used in the nonparametric setting examined

[2] Blundell and Matzkin (2014) give an alternative characterization of control function separability. If the reduced form equation for y_2, the endogenous variable, can be expressed as a function of the exogenous variable x and a function of the disturbances vector $(\varepsilon_1, \varepsilon_2)$, then the condition is met. This is equivalent to the one-dimensionality of the equation in x given the control function. See Propositions 2 and 3, Kasy (2010).

by Blundell and Matzkin, it is a restrictive condition outside of linear models. In some settings, it may also be difficult to show that the condition does *not* hold for a given model, because establishing the condition amounts to showing the existence of functions it may simply be difficult to construct.

Before showing my own result regarding the equivalence of control function separability and the existence of a triangular system with scalar disturbances corresponding to the structural system by a reparametrization of the disturbances, as well as further results, I briefly summarize Blundell and Matzkin's main result below, using their notation.

2.2 Control function separability

Consider a triangular system whose form is given by **Model (T)** below, and which satisfies several nonparametric assumptions. There is first, Assumption T.1, that m^1 and s are continuously differentiable on the supports of their arguments. The second, Assumption T.2, is that (ε_1, η) is independent of scalar x (x is exogenous). Third, Assumption T.3 states that conditional on any value of x on its support, the densities of the scalar disturbances (ε_1, η) and the scalar dependent variables (y_1, y_2) are continuous and have convex support. And fourth, Assumption T.4 requires that functions m^1 and s are strictly increasing[3] in their disturbances given any values of

[3]The results of this paper can be shown to hold under general monotonicity, but I assume positive monotonicity for simplicity.

their respective covariates.

$$\textbf{Model (T)} \qquad y_1 = m^1(y_2, \varepsilon_1),$$
$$y_2 = s(x, \eta).$$

Given a system of this form and under these assumptions, control function estimation is known to be relatively straightforward. The second equation, free of simultaneity, can be estimated in order to yield an estimate $\hat{\eta}$ of η, which can be conditioned upon in estimation of the first equation in order to yield consistent estimates of various structural quantities relating to m^1.

While triangular systems are amenable to control function estimation under reasonable assumptions, triangularity is a significant restriction on the variety of models the structural features of which can be identified in this way. Blundell and Matzkin ask whether systems of the form given by **Model (S)**, shown below, are ever such that they are *observationally equivalent* to a triangular system of form **(T)**, so that the alternative triangular system can be estimated and yield the same estimates of structural quantities as the non-triangular system. As with the triangular system, the authors restrict their analysis to systems of continuous scalar variables.

$$\textbf{Model (S)} \qquad y_1 = m^1(y_2, \varepsilon_1),$$
$$y_2 = m^2(y_1, x, \varepsilon_2).$$

We can consider **Model (S)** in the context of similar assumptions as those described for **Model (T)**. S.1 through S.4 correspond closely to assumptions T.1 through T.4. Under these assumptions, **(S)** can always be written in an inverse form, with disturbances as functions of observables, as a system of the form of **Model (I)** below. There is also an additional assumption, S.5, which is necessary to show the equivalence of control function separability and the feasibility of the control function approach. S.5 is a crossing assumption stating that the partial cross-derivatives of the two structural functions with respect to the endogenous variables, when multiplied together, are less than one.

$$\textbf{Model (I)} \qquad \begin{aligned} \varepsilon_1 &= r^1(y_1, y_2), \\ \varepsilon_2 &= r^2(y_1, y_2, x). \end{aligned}$$

A structural inverse system of equations given by **Model (I)** above is said to satisfy control function separability if, on the supports of the observable variables, there exist functions $v : \mathbb{R}^2 \to \mathbb{R}$ and $q : \mathbb{R}^2 \to \mathbb{R}$ such that

a) $r^2(y_1, y_2, x) = v(q(y_2, x), r^1(y_1, y_2))$,

b) q is strictly increasing in its first argument,

c) v is strictly increasing in its first argument.

Blundell and Matzkin prove that if, and only if, a simultaneous system in the form of **(I)**, whether triangular or not, satisfies control function separability, then it is observationally equivalent to some triangular system **(T)**.

Observational equivalence means that the distributions of the observable variables (y_1, y_2), conditional on x for all values of x on its support, in each of these models are identical. Thus, a simultaneous system's eligibility for estimation with the control function approach for triangular systems described above is characterized completely by the control function separability condition.[4]

3 Characterizing separable systems

3.1 Triangular representation

It can be shown that simultaneous systems satisfying control function separability and meeting a set of assumptions given below can be rewritten in triangular form by a change of variables, and, conversely, that triangular systems can be written as simultaneous systems which satisfy control function separability. In other words, the existence of a triangular representation of a formally non-triangular simultaneous system is necessary and sufficient for control function separability. The observational equivalence between a system of form **(S)** that satisfies the condition and a triangular system of form **(T)** corresponds, as intuition suggests, to their formal equivalence.

For illustration, consider Model **(S)** in the semiparametric linear case (where

[4] See Blundell & Matzkin, 2014, for a more complete explanation of observational equivalence as it relates to the feasibility of the control function approach.

control function separability always holds), as follows,

$$y_1 = \alpha y_2 + \varepsilon_1, \tag{3.1}$$

$$y_2 = \beta y_1 + \gamma x + \varepsilon_2, \tag{3.2}$$

where identification conditions requiring that $\alpha\beta < 1$ and $\gamma \neq 0$ are met and assumptions S.1 through S.5 are met more generally. The system, rewritten in terms of the inverse equations, can be seen to satisfy the separability condition,

$$\varepsilon_1 = y_1 - \alpha y_2, \tag{3.3}$$

$$\varepsilon_2 = (1 - \alpha\beta) y_2 - \gamma x - \beta(\varepsilon_1) \tag{3.4}$$

where $v(\eta, \varepsilon_1) = \eta - \beta\varepsilon_1$ and $q(y_2, x) = (1 - \alpha\beta)y_2 - \gamma x$ are each increasing in their first arguments. Equivalently, the system can also be written in triangular form,

$$y_1 = \alpha y_2 + \varepsilon_1, \tag{3.5}$$

$$y_2 = \frac{\gamma}{1 - \alpha\beta} x + \eta, \tag{3.6}$$

where $\eta = \varepsilon_2 + \beta\varepsilon_1$. It can be seen intuitively that control function separability allows y_1 to "disappear" from the second structural equation, allowing a kind of triangularity.

A formal statement of the equivalence of control function separability and triangular representation under general (nonlinear) conditions, given the assumptions in Blundell and Matzkin (2013), and the corresponding proof, follow, using similar notation to Blundell and Matzkin.

Proposition 1. *For a structural system of form **(S)** for which assumptions S.1 to S.5 hold, control function separability holds if and only if there exists a system of form **(T)**, where assumptions T.1 to T.4 hold, for which the structural disturbances* $(\varepsilon_1, \varepsilon_2)$ *of **(S)** can be reparameterized as* (ε_1, η)*, where* η *is an increasing function of* ε_2 *given* ε_1*, so that the inverse equations of **(S)** are equivalent to the inverse equations of **(T)**.*

PROOF Consider a simultaneous system of equations of inverse form **(I)**, corresponding to a system of form **(S)**, that satisfies the condition of control function separability as defined above. The system can be written

$$\begin{aligned} \varepsilon_1 &= r^1(y_1, y_2), \\ \varepsilon_2 &= r^2(y_1, y_2, x), \\ &= v(q(y_2, x), r^1(y_1, y_2)), \\ &= v(q(y_2, x), \varepsilon_1), \end{aligned}$$

where, as assumed, $(\varepsilon_1, \varepsilon_2)$ is distributed independently of x. By the definition of control function separability, v is strictly monotonically increasing in $q(y_2, x)$, so we can write

$$q(y_2, x) = v^{-1}(\varepsilon_2, \varepsilon_1), \tag{3.7}$$

where v^{-1}, the inverse function of v with respect to its first argument, is strictly increasing in ε_2 given any value of ε_1 by assumption. Because q is strictly increasing in its first argument we can also invert it and write the following:

$$y_2 = q^{-1}(x, v^{-1}(\varepsilon_2, \varepsilon_1)).$$

Now we define $s \equiv q^{-1}$. Since $(\varepsilon_2, \varepsilon_1)$ is independent of x, so should be $(v^{-1}(\varepsilon_2, \varepsilon_1), \varepsilon_1)$, which we can call (η, ε_1). Also note that s is strictly increasing in η.

Since **(I)** is a special case of **(S)**, we have that $y_1 = m^1(y_2, \varepsilon_1)$. We can now write the observationally equivalent system, which is known to exist, as

$$\textbf{Model (T)} \qquad \begin{aligned} y_1 &= m^1(y_2, \varepsilon_1), \\ y_2 &= s(x, \eta), \end{aligned}$$

where y_1 is strictly increasing in ε_1 by assumption, s is strictly increasing in η by the monotonicity of q and thus of q^{-1}, and (η, ε_1) is independent of x as discussed. The continuous differentiability of m^1 is assumed, and the continuous differentiability of s follows from the implicit function theorem and the continuous differentiability of r^2. The conditional density of ε_1 is continuous and has convex support by assumption, while the continuity and convex support of the conditional density of η follow from the continuity and monotonicity of v^{-1}, so we have a triangular system of the form of **Model (T)**. It can be easily seen that the two inverse equations for **(T)** take the forms, respectively, of the first inverse equation in **(I)**, and (3.7), rewritten in terms of η, so that the equivalence between the two inverse systems is satisfied.

The converse can also be proven. Consider a triangular system of form **(T)** with disturbances (ε_1, η) which has been obtained through a system of form **(S)** by remapping the disturbances from $(\varepsilon_1, \varepsilon_2)$ to $(\varepsilon_1, \eta) = (\varepsilon_1, u^{-1}(\varepsilon_2, \varepsilon_1))$ where u is continuous, differentiable and increasing in η. By monotonicity, we can write the inverse system,

$$\varepsilon_1 = r^1(y_1, y_2),$$
$$\eta = s^{-1}(y_2, x).$$

Redefining $s^{-1} \equiv q$, we have

$$\varepsilon_1 = r^1(y_1, y_2),$$
$$\eta = q(y_2, x).$$

We can define an arbitrary[5] continuously differentiable function $v : \mathbb{R}^2 \to \mathbb{R}$ that is monotonically increasing in its first argument for all values of its second argument and further define

$$\begin{aligned} \varepsilon_2 &\equiv v(\eta, \varepsilon_1), \\ &= v(q(y_2, x), r^1(y_1, y_2)), \\ &\equiv r^2(y_1, y_2, x). \end{aligned}$$

Note that v and q are each increasing in their first arguments as required for control function separability, and that r^2 is separable as required. v and q can always be

[5]The form of the system that can be shown to satisfy control function separability is only unique up to the relationship between η and ε_2.

defined such that r^2 is strictly increasing in y_2 and thus invertible in y_2.

To see this, note that we require

$$\frac{\partial r^2(y_1, y_2, x)}{\partial y_2} = \frac{\partial v(\varepsilon_1, \eta)}{\partial \eta}\frac{\partial q(y_2, x)}{\partial y_2} + \frac{\partial v(\varepsilon_1, \eta)}{\partial \varepsilon_1}\frac{\partial r^1(y_1, y_2)}{\partial y_2} > 0 \tag{3.8}$$

for all values of the arguments in the domain. This condition follows from taking the partial derivatives of both sides of the equation

$$r^2(y_1, y_2, x) = v(q(y_2, x), r^1(y_1, y_2)), \tag{3.9}$$

which is a condition of control function separability. Monotonicity in the first argument of each of the structural functions requires the first term of (3.8) to be positive. Since v is chosen, we can construct it with respect to ε_1 such that $\partial r^2/\partial y_2 > 0$ by making $(\partial v/\partial \eta)(\partial q/\partial y_2)$ sufficiently large that when $\partial v/\partial \varepsilon_1$ cannot be made the same sign as $\partial r^1/\partial y_2$ (for non-monotonic r_1 in y_2) the inequality still holds.

This is a more general result than required, but u is included within the set of possible functions v that can be defined, since it corresponds to a system **(S)** with its monotonicity assumptions, so the specific result holds.

Now that we have the inverse system, we can invert and write

$$y_1 = m^1(y_2, \varepsilon_1),$$
$$y_2 = m^2(y_1, x, \varepsilon_2),$$

where m^2 is increasing in ε_2 by the monotonicity of s in its second argument and m^1 increases in its disturbance by assumption. m^1 is continuously differentiable by assumption and m^2 is as well by construction of r^2. As discussed, $(\varepsilon_1, \varepsilon_2)$ is independent of x. The conditional density of ε_1 is continuous and has convex support by assumption, while the continuity and convex support of the conditional density of ε_2 follow from the continuity and monotonicity of v and T.3.

It can be shown, by the way v and q were constructed, that the crossing assumption S.5 is met and the equilibrium is unique, which is a requirement for the result. By construction, and indeed for any system satisfying control function separability, it follows by (3.8) in more compact notation, that

$$r^2_{y_2} = v_\eta q_{y_2} + v_{\varepsilon_1} r^1_{y_2}, \tag{3.10}$$

where subscripts indicate the variable with respect to which the partial derivative was taken. We also have, by taking the partial derivative of (3.9) with respect to y_1 of the inverse function r^2:

$$r^2_{y_1} = v_{\varepsilon_1} r^1_{y_1}. \tag{3.11}$$

And substituting, we have

$$r^2_{y_2} = v_\eta q_{y_2} + \frac{r^1_{y_2} r^2_{y_1}}{r^1_{y_1}}.$$

It is shown in the Appendix that $r^1_{y_2} r^2_{y_1} / r^1_{y_1} r^2_{y_2} = m^1_{y_2} m^2_{y_1}$, so that we can write the following:

$$1 = \frac{\upsilon_\eta q_{y_2}}{r_{y_2}^2} + m_{y_2}^1 m_{y_1}^2,$$

$$m_{y_2}^1 m_{y_1}^2 = 1 - \frac{\upsilon_\eta q_{y_2}}{r_{y_2}^2} < 1,$$

where the second equality holds by the assumed positivity of υ_η, q_{y_2} and $r_{y_2}^2$. The crossing assumption is thus met.

So we have a system of form **(S)** for which the conditions for control function separability holds, as required. □

While the control function separability condition does characterize systems where the control function approach described earlier can be used, there remains a question whether this condition is more or less practically useful or easily verified than the triangular representation condition.

3.1.1 Triangularity in previous examples of separability

Here I re-examine two previous examples in the literature of simultaneous systems that satisfy control function separability, in light of the result above. Blundell and Matzkin (2014) include an example of a demand system with latent taste variables in their paper to illustrate the applicability of control function separability. I discuss it here, omitting the exposition. The first order conditions of the demand system yield a nonparametric, nonlinear inverse simultaneous system that satisfies control function separability, as follows:

$$\begin{aligned}
\varepsilon_1 &= (y_1 - u(y_2))x_1, \\
\varepsilon_2 &= \left(\frac{x_2}{u'(y_2)} + x_1\right) - (y_1 - u(y_2))x_1, \\
&= \left(\frac{x_2}{u'(y_2)} + x_1\right) - \varepsilon_1,
\end{aligned}$$

where x_1 and x_2 are exogenous prices. In this system, $v(\eta, \varepsilon_1) = \eta - \varepsilon_1$ and $q(y_2, x) = x_2/u'(y_2) + x_1$. However, the system can be easily rewritten as a triangular system since u is strictly concave, in the following form,

$$\begin{aligned}
y_1 &= u(y_2) + \frac{\varepsilon_1}{x_1}, \\
y_2 &= (u')^{-1}\left(\frac{x_2}{\varepsilon_1 + \varepsilon_2 - x_1}\right), \\
&= (u')^{-1}\left(\frac{x_2}{\eta - x_1}\right),
\end{aligned}$$

where y_1 is increasing in ε_1 and, by the concavity of u, y_2 is increasing in $\eta = \varepsilon_1 + \varepsilon_2$. The equation for y_2 is in reduced form, free of any simultaneity in y_1. While a non-triangular simultaneous system in terms of ε_1 and ε_2 does exist, it cannot generally be written down analytically, and seems to exist as an abstraction away from the underlying triangular structure.[6] Since control function separability is a property of simultaneous systems (not triangular systems), it is not ultimately helpful for this example, and can be verified in any case by establishing triangularity as shown.

[6] The authors in fact wrote the original utility functions in terms of latent variables ε_1 and $\varepsilon_1 + \varepsilon_2$.

As a second example, in a related paper, a nonlinear, nonadditive system given in terms of the inverse equations is shown to satisfy control function separability (Blundell, Kristen & Matzkin, 2013). The system of structural inverse equations can be written as follows:[7]

$$\begin{aligned}\varepsilon_1 &= \Lambda_1^{-1}(y_2) + \Lambda_2^{-1}(y_1),\\ \varepsilon_2 &= \left(\frac{1}{b_2(x)} - c\right)\Lambda_1^{-1}(y_2) - c\Lambda_2^{-1}(y_1) - \frac{g_2(x)}{b_2(x)},\\ &= \left(\frac{1}{b_2(x)}\right)\Lambda_1^{-1}(y_2) - \frac{g_2(x)}{b_2(x)} - c\varepsilon_1,\end{aligned}$$

where c is a constant, x is exogenous, b_2, Λ_1, and Λ_2 are strictly increasing functions, and $1/b_2(x) - c$ is positive on the support of x. Once again, this inverse system can be easily rearranged, yielding a triangular system, as follows:

$$\begin{aligned}y_1 &= \Lambda_2(\Lambda_1^{-1}(y_2) + \varepsilon_1),\\ y_2 &= \Lambda_1(g_2(x) + b_2(x)(\varepsilon_2 + c\varepsilon_1)),\\ &= \Lambda_1(g_2(x) + b_2(x)\eta),\end{aligned}$$

where $\eta = \varepsilon_2 + c\varepsilon_1$.

These examples are illustrative of the implied triangularity of systems that satisfy control function separability. While both could be represented as non-triangular simultaneous systems in terms of ε_1 and ε_2, a simple linear change of variable permits

[7] In the original paper, the inverse equation for ε_2 was written down incorrectly, as is implicitly made clear later in the paper.

the triangular representation. The non-triangular representation of the system above is as follows:

$$y_1 = m^1(y_2, \varepsilon_1) = \Lambda_2(\Lambda_1^{-1}(y_2) + \varepsilon_1),$$
$$y_2 = m^2(y_1, x, \varepsilon_2) = \Lambda_1\left(\frac{c\Lambda_2^{-1}(y_1) + g_2(x)/b_2(x) + \varepsilon_2}{1/b_2(x) - c}\right).$$

This unwieldy system is unlikely to arise from economic theory. It is also restrictive in a particular way.

To the first point, the system would be more likely to arise in terms of its inverse system, in which case its triangular representation would be easily seen. To the second, this simultaneous system can be seen to require the monotonicity of m^1 in y_2 and of m^2 in y_1. It also requires that y_1 enter both inverse equations in the same way, up to a linear transformation, and that the control function $\eta = v^{-1}(\varepsilon_2, \varepsilon_1) = \varepsilon_2 + c\varepsilon_1$ is linear. These are not incidental features of the model, but necessary in general, in practice, as will be shown later in this paper.

In both cases, furthermore, control function separability does not appear to be more easily verified or mathematically accessible than the equivalent condition of admitting triangular representation; both are arrived at by straightforward manipulation of the first order conditions or inverse equations.

3.2 Additional characterizations

An equivalent characterization of control function separability that follows from the proof of Proposition 1 is the existence of functions q and w, both monotonically increasing in their first arguments, such that with respect to a system of form **(I)**, the following holds:

$$q(y_2, x) = w(r^2(y_1, y_2, x), r^1(y_1, y_2)). \tag{3.12}$$

In other words, control function separability can also be thought of as the condition where a one-to-one function of $\varepsilon_2 = r^2(y_1, y_2, x)$, given $\varepsilon_1 = r^1(y_1, y_2)$, can be constructed that does not depend on y_1.

An additional new condition characterizing control function separability can be shown. It is derived from one given by Blundell & Matzkin in terms of partial derivatives of the structural and inverse functions, which follows [8]. Given a system of form **(I)** and another of form **(T)** not known to be equivalent, if and only if the following condition holds for all x, y_1 and y_2,

$$\frac{r_x^2}{r_{y_1}^2 m_{y_2}^1 + r_{y_2}^2} = \frac{s_x^{-1}}{s_{y_2}^{-1}}, \tag{3.13}$$

then the control function separability condition is met. In (3.13), subscripts indicate the variable with respect to which the partial derivative is taken and s^{-1} refers to the inverse function of s with respect to its second argument. By Blundell and Matzkin's

[8] In Blundell and Matzkin's paper, the function s^{-1} is erroneously denoted by s for the condition (3.13).

theorem, if and only if this condition is met, the two systems are observationally equivalent.

Verifying this condition, however, requires computing partial derivatives of the inverse functions. In some cases of interest, the inverse system may be unwieldy. In the Appendix, I derive a similar, equivalent condition of somewhat simpler form which depends only on the partial derivatives of the structural functions, for cases in which the structural functions are more easily accessible. If and only if the equation,

$$s_x = \frac{m_x^2}{1 - m_{y_1}^2 m_{y_2}^1}, \tag{3.14}$$

always holds, then the control function separability condition is met, by my argument and by Theorem 2 in Blundell & Matzkin. This alternative characterization is free of reference to the inverse functions, and can be used to verify observational equivalence of two systems of form **(S)** and **(T)**. Intuitively, it can be quickly seen that in a system already written in triangular form, $m_{y_1}^2 = 0$ and $m_x^2 = s_x$, yielding the equality.

This condition also illustrates some of the significance of the crossing assumption S.5, which states that for all values of $(y_1, y_2, x, \varepsilon_1, \varepsilon_2)$ in their domain, $m_{y_2}^1 m_{y_1}^2 < 1$. We can rearrange the condition,

$$\frac{m_x^2}{s_x} = 1 - m_{y_1}^2 m_{y_2}^1,$$

and see that the right-hand side will, by the crossing assumption, always be positive. We can see that a necessary (but not sufficient) condition for control function separability in a system that meets assumptions S.1-S.5 and a possibly equivalent triangular system is that the partial derivatives of $m^2(y_1, x, \varepsilon_2)$ and $s(x, \eta)$ with respect to x are of the same sign for all values of $(y_1, y_2, x, \varepsilon_1, \varepsilon_2)$ on their supports.

4 Applicability of control function separability

4.1 Restrictiveness of the condition

What makes the control function separability worthy or unworthy of study is its usefulness for identification and estimation of quantities relevant to economic theory. Its practical restrictiveness has not been investigated in the literature other than in some preliminary observations made by Blundell and Matzkin. Here I develop some necessary derivative conditions for control function separability and argue that, in practice, the condition is restrictive to the point that it is equivalent to requiring two additional monotonicity conditions, as well as additional functional form restrictions including linearity in the control function.

To examine the necessary derivative conditions, recall that the condition (3.8) must hold, as follows,

$$\frac{\partial r^2(y_1, y_2, x)}{\partial y_2} = \frac{\partial v(\eta, \varepsilon_1)}{\partial \eta} \frac{\partial q(y_2, x)}{\partial y_2} + \frac{\partial v(\eta, \varepsilon_1)}{\partial \varepsilon_1} \frac{\partial r^1(y_1, y_2)}{\partial y_2} > 0,$$

for all values of the variables on their supports. For a given simultaneous system of nonparametric equations arising from economic theory, of the form **(I)**, and for a corresponding control function $\eta = v^{-1}(\varepsilon_2, \varepsilon_1)$, assuming it exists, the equality and the inequality above are necessary, respectively, for control function separability to be satisfied and for the monotonicity of the structural equations in their disturbances to hold.

The condition in this form can be seen to be highly restrictive. Because it must hold for all admissible values of the variables, it is specifically the case that $(\partial v(\eta, \varepsilon_1)/\partial \varepsilon_1)(\partial r^1(y_1, y_2)/\partial y_2)$ is finitely bounded below by the changing (negative) value of $-(\partial v(\eta, \varepsilon_1)/\partial \eta)(\partial q(y_2, x)/\partial y_2)$. For a structural simultaneous system, there is no guarantee on the sign of $\partial r^1(y_1, y_2)/\partial y_2$, despite the crossing assumption, which requires the quantity to be of the opposite sign to $\partial r^2(y_1, y_2, x)/\partial y_1$, or else of the same sign and of sufficiently small magnitude.

In the permissible case that r^1 is not monotonic in the endogenous variable y_2, the partial derivative of v with respect to its second argument is required to either remain very small or else balance on a knife edge, changing sign approximately when $\partial r^1(y_1, y_2)/\partial y_2$ does.

In practice, there is no reason to expect these knife-edge scenarios, and control function separability is perhaps only realistically applicable when r^1 is monotonic in y_2 (or equivalently when m^1 is monotonic in y_2), so that simply by v being monotonic in the same direction, in its second argument, $(\partial v(\eta, \varepsilon_1)/\partial \varepsilon_1)(\partial r^1(y_1, y_2)/\partial y_2)$ is positive and the condition is met. Indeed, Blundell and Matzkin's demand example, where control function separability is met, is a case where the structural function $m^1(y_2, \varepsilon_1)$ is monotonic in y_2.

The inequality above, as well as an earlier result that $r^1_{y_2} r^2_{y_1} / r^1_{y_1} r^2_{y_2} = m^1_{y_2} m^2_{y_1}$, which is proved in the Appendix, can be combined with the fact that $r^2_{y_2} = 1/m^2_{\varepsilon_2}$ to show that the following is necessary for a system that satisfies assumption S.5 and control function separability,

$$-\frac{\partial v(\eta, \varepsilon_1)}{\partial \eta} \frac{\partial q(y_2, x)}{\partial y_2} \frac{\partial m^2(y_1, \varepsilon_2, x)}{\partial \varepsilon_2} < \frac{\partial m^1(y_2, \varepsilon_1)}{\partial y_2} \frac{\partial m^2(y_1, \varepsilon_2, x)}{\partial y_1} < 1,$$

where the left hand side is always negative by the positive monotonicity of v, q and m^2 in their second arguments. Written this way, control function separability and the crossing assumption jointly hem in the magnitude of the product of the structural cross-derivatives in both the positive and negative directions.

It can also be shown that m^2 must always, in practice, be monotonic in y_1. Consider the equation (3.11), which is a necessary condition for the separability condition:

$$r^2_{y_1} = v_{\varepsilon_1} r^1_{y_1}.$$

It can be rearranged and written explicitly as follows:

$$\frac{\partial r^2(y_1, y_2, x)/\partial y_1}{\partial r^1(y_1, y_2)/\partial y_1} = \frac{\partial v(\eta, \varepsilon_1)}{\partial \varepsilon_1}. \tag{4.1}$$

This is certainly a restrictive condition on the functional form of the inverse structural equations r^1 and r^2. Without parametric restrictions (since this is a nonparametric setting), it can first be observed that maintaining even the same sign on each side of the equation, without strict monotonicity, is a highly nongeneric outcome. In practice, it cannot be expected to hold under ordinary conditions unless, at the very least, r^2 is monotonic in y_1 and v is monotonic in the same direction in ε_1 (r^2 is monotonic by assumption). As can be easily shown using prior results, $r^2_{y_1} > 0$ if and only if $m^2_{y_1} < 0$, assuming S.4 holds. That is, m^2 must be monotonic in y_1 in the opposite direction of m^1 in y_2. In the examples discussed in Blundell & Matzkin (2014) and Blundell, Kristensen & Matzkin (2013), the structural equation m^2 is indeed strictly decreasing in y_1.

The condition (4.1), however, further requires the quotient of two functions in y_1, y_2, and x (for r^2) to yield a function in η and ε_1. (In particular, this requires $r^2_{y_1}$ to be free of x.) In Blundell and Matzkin's demand system example, this condition is met by the quotient being equal to -1. In other words, y_1 enters each of the inverse equations r^1 and r^2 in the same way, up to a linear transformation.[9] For the condition to be met in a less restrictive way would require, again, very nongeneric cases or still-restrictive parametric restrictions.

It is also shown in the Appendix that

$$\frac{\partial r^2(y_1, y_2, x)/\partial y_1}{\partial r^1(y_1, y_2)/\partial y_1} = -\frac{(\partial m^2(y_1, \varepsilon_2, x)/\partial y_1)(\partial m^1(y_2, \varepsilon_1)/\partial \varepsilon_1)}{\partial m^2(y_1, \varepsilon_2, x)/\partial \varepsilon_2}, \tag{4.2}$$

[9] Such a linear transformation must, of course, have constants as coefficients.

which implies that, equivalently, each of the disturbances ε_1 and ε_2 must enter the structural equation m^2 in the same way (indirectly through y_1 in the case of ε_1), up to a linear transformation, in practice.

In practice, then, control function separability will thus usually only hold in cases where the function $v(\eta, \varepsilon_1)$ is linear in ε_1. In such a case, v would also generally need to be linear in η so as to satisfy (3.9). Accordingly, the control function $\eta = v^{-1}(\varepsilon_1, \varepsilon_2)$ would also be linear in ε_1 and ε_2.

In summary, a system **(S)**, except in unusual nongeneric circumstances, will satisfy control function separability only when m^1 is strictly monotonic in y_2, m^2 is strictly monotonic in y_1 in the opposite direction, y_1 enters each of the inverse equations in the same way, up to a linear transformation, and the control function $\eta = v^{-1}(\varepsilon_1, \varepsilon_2)$ is linear in each of its arguments. These conditions hold for the demand system example in Blundell & Matzkin, 2014, as well as the nonlinear system example in a related paper (Blundell, Kristensen & Matzkin, 2013), which are the only examples in the literature in which control function separability is explicitly investigated and satisfied. The conditions are also, of course, satisfied for linear simultaneous equations systems with additive scalar disturbances.

4.2 Ruling out control function separability

Given an arbitrary nonlinear simultaneous system where assumptions S.1 through S.5 are thought to hold, it may be difficult to conclude with certainty that control function separability does not hold in systems where that is the case. The practical inability to construct functions v and q satisfying the condition does not itself imply their nonexistence.

The alternative characterization of the separability condition, (3.14), given earlier can be used to rule it out. Necessary conditions for control function separability, when shown not to hold, also, of course, amount to sufficient conditions for the nonexistence of v and q. Two examples follow.

4.2.1 Example

A simultaneous system **(S)** could take the following functional form with additive nonparametric disturbances,

$$y_1 = c(y_2) + \varepsilon_1, \tag{4.3}$$

$$y_2 = y_1 + y_1 x + \varepsilon_2, \tag{4.4}$$

where unknown function $c(.)$ is the econometric object of interest and x is exogenous. We assume that c is differentiable, strictly increasing and strictly concave. We also assume that assumptions S.1 to S.4 are met, and that c is such that the crossing assumption required for a unique equilibrium is met.

We know that a necessary (and sufficient) condition for the applicability of the control function approach, (3.14), is that

$$s_x = \frac{m_x^2}{1 - m_{y_1}^2 m_{y_2}^1}. \tag{4.5}$$

While this condition is not very useful as a sufficient condition, since the function s is usually unknown at the outset, it can be easily used to rule out control function separability. For the system described above, it can be shown that the condition requires that, for all values of y_1, x and y_2,

$$s_x = \frac{y_1}{1 - (1 + x)(c'(y_2))},$$

which is an impossibility, since s cannot be a function of y_1 by the definition of triangularity and y_1 will not cancel out. Therefore, the control function approach cannot be used to identify and estimate this system.

Observational equivalence can also be ruled out in a weaker sense by reference to the almost-necessary conditions described in Section 4.1. The inverse system corresponding to equations (4.3) and (4.4) is written as follows:

$$\varepsilon_1 = y_1 - c(y_2), \tag{4.6}$$

$$\varepsilon_2 = -y_1(1 + x) + y_2. \tag{4.7}$$

It is quickly observed that y_1 does not enter each of the equations in the same way up to a linear transformation. This rules out, in all likelihood, the possibility of an observationally equivalent triangular system.

4.2.2 Example

Another example can be considered from the literature on property taxation and house valuation. In a book by Bloom, Borsch-Supan, Ladd and Yinger (1988), a simultaneous system of equations is developed, modelling house valuation and the effective tax rate between two periods. The model's structural form is simplified for the purposes of presentation in this paper, as follows:

$$\frac{y_1 - x_1}{x_1} = \frac{-\beta(y_2 - x_2)}{0.03 + \beta y_2} + \varepsilon_1, \tag{4.8}$$

$$y_2 = x_3 \frac{1}{y_1} + \varepsilon_2, \tag{4.9}$$

where all observable variables are strictly positive and exogenous and β is assumed nonnegative. Assumptions S.1 through S.5 are assumed to hold. In terms of economic theory, y_1 represents the value of a house in the second period, and x_1 its value in the first, so that the left hand side of (4.8) represents the growth in house valuation between the two periods. y_2 is the effective property tax rate in the second period, and x_2 the rate in the first. x_3 is an observable, exogenous variable representing housing and neighborhood characteristics.

It can be observed that (4.8) (the left-hand side of which is a positive monotonic function of y_1) is strictly decreasing in y_2, and that (4.9) is strictly decreasing in y_1. In other words, the two structural cross-derivatives have the same sign. While this is permissible under weak support and parametric conditions for identification, by the crossing assumption, it almost certainly is not permissible for control function separability, by the conditions developed earlier.

Alternatively, the inverse system can be written down as follows:

$$\varepsilon_1 = \frac{y_1 - x_1}{x_1} + \frac{\beta(y_2 - x_2)}{0.03 + \beta y_2}, \tag{4.10}$$

$$\varepsilon_2 = y_2 - x_3 \frac{1}{y_1}, \tag{4.11}$$

by which we see that y_1 does not enter each inverse equation in the same way. Accordingly, we can almost certainly rule out control function separability, without attempting to find functions v and q.

Finally, we can conclusively confirm, by (4.1), that the above system is not observationally equivalent to a triangular system by noting that the partial derivative $r^2_{y_1}$ is a function of x_3.

5 Conclusion

In this paper, I have investigated the applicability of the control function separability condition introduced by Blundell and Matzkin, 2014, which characterizes structural systems of simultaneous equations in scalar variables where the method of control functions can be used to estimate the main structural equation.

First, in an effort to discover whether the condition allows for the application of the control function method to a wide class of nontriangular models, I have shown that control function separability holds if and only if a system of simultaneous equations can be represented, by reparameterizing the model disturbances, as a triangular system. In applying this result to the previous examples of systems satisfying control function separability, I illustrate the unlikelihood of developing, from economic theory, nontriangular structural systems which satisfy the condition and are not already easily seen to be triangular.

Second, a new characterization of control function separability is given in terms of the structural derivatives, and is used to rule out control function separability in a nonparametric model.

Finally, conditions are developed which are variously necessary or almost necessary for control function separability, with almost-necessity referring to conditions such that only models with highly nongeneric properties would satisfy control function separability without the conditions holding. These include monotonicity conditions in the endogenous variables in the structural functions, the condition that the main dependent variable of interest enters both inverse equations in the same way up to a linear transformation, and the condition that the control function is linear in the structural disturbances. Making reference to two examples, including a system of simultaneous equations from the literature on house valuation and taxation, I use these necessary conditions to rule out control function separability.

The contribution of this paper to the literature is to explain the practical limitations of the control function separability condition, in part by recharacterizing it as a triangular representation condition. Future applied work with nonlinear, nonparametric models with simultaneity may benefit from additional characterizations of the condition and from a simple way to rule out the possibility of using the control function method for nonparametric estimation of nonlinear systems of simultaneous equations.

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A Appendix

Here I show that $r^1_{y_2} r^2_{y_1} / r^1_{y_1} r^2_{y_2} = m^1_{y_2} m^2_{y_1}$. For a system **(S)** meeting assumptions S.1 through S.4, we have the identities

$$y_1 = m^1(y_2, r^1(y_1, y_2)),$$
$$y_2 = m^2(y_1, x, r^2(y_1, y_2, x)).$$

Differentiating the first equation with respect to y_1 and y_2 yields

$$1 = m^1_{\varepsilon_1} r^1_{y_1}, \tag{A.1}$$
$$0 = m^1_{y_2} + m^1_{\varepsilon_1} r^1_{y_2}, \tag{A.2}$$

which together yield $m^1_{y_2} = -r^1_{y_2}/r^1_{y_1}$. A similar process with respect to the second equation yields $m^2_{y_1} = -r^2_{y_1}/r^2_{y_2}$. Multiplying both expressions together, and noting that $r^1_{y_1}$ and $r^2_{y_2}$ are nonzero by monotonicity assumptions, we have our result.

I now prove the alternative characterization. Consider the following identity for a simultaneous system **(S)** which meets assumptions S.1-S.5:

$$y_2 = m^2(y_1, x, r^2(y_1, y_2, x)). \tag{A.3}$$

Partial derivatives can be taken, so that

$$1 = m^2_{\varepsilon_2} r^2_{y_2}, \tag{A.4}$$

$$0 = m^2_{y_1} + m^2_{\varepsilon_2} r^2_{y_1}, \tag{A.5}$$

$$0 = m^2_x + m^2_{\varepsilon_2} r^2_x, \tag{A.6}$$

where partial derivatives are written compactly as before. Similarly, by the following identity for a triangular system **(T)**,

$$y_2 = s(x, s^{-1}(y_2, x)), \tag{A.7}$$

and its partial derivatives,

$$1 = s_\eta s^{-1}_{y_2},$$

$$0 = s_x + s_\eta s^{-1}_x,$$

and solving for $r^2_{y_2}$, $r^2_{y_1}$, r^2_x, $s^{-1}_{y_2}$ and s^{-1}_x, we can substitute into Blundell and Matzkin's condition, whereby it follows that

$$\frac{-m^2_x/m^2_{\varepsilon_2}}{(-m^2_{y_1}/m^2_{\varepsilon_2})m^1_{y_2} + 1/m^2_{\varepsilon_2}} = \frac{-s_x/s_\eta}{1/s_\eta}, \tag{A.8}$$

since $m^2_{\varepsilon_2}$ is nonzero everywhere by the strict monotonicity of m^2 in ε_2, and since s_η is nonzero everywhere by the strict monotonicity of s in η. After simplifying, this yields the following condition:

$$\frac{m_x^2}{m_{y_1}^2 m_{y_2}^1 - 1} = -s_x, \tag{A.9}$$

I also show that

$$\frac{\partial r^2(y_1, y_2, x)/\partial y_1}{\partial r^1(y_1, y_2)/\partial y_1} = -\frac{(\partial m^2(y_1, \varepsilon_2, x)/\partial y_1)(\partial m^1(y_2, \varepsilon_1)/\partial \varepsilon_1)}{\partial m^2(y_1, \varepsilon_2, x)/\partial \varepsilon_2}. \tag{A.10}$$

We have, by (A.5) and (A.1), that $r_{y_1}^2 = -m_{y_1}^2/m_{\varepsilon_2}^2$ and $r_{y_1}^1 = 1/m_{\varepsilon_1}^1$. We then have (A.10) by substitution.