An Introduction to Factor Copula Models and their Application in Studying the Dependence of the Exchange Rate Returns

by

Hanyue Zhang (Shiny)

An essay submitted to the Department of Economics

in Partial Fulfillment of the requirements for

the Degree of Master of Arts

Queen's University

Kingston, Ontario, Canada

February 2011

Acknowledgements

It is a pleasure to thank those who made this paper possible. Professor Suo Wulin's encouragement, guidance and support from beginning to end enabled me to complete this project. I am grateful to my undergraduate honours thesis supervisor, Professor David Giles, who introduced me to copula modeling methods. In addition, Professor Frank Milne has made available his support in a number of ways, such as recommending papers relevant to my research. Finally, I need to say that the contributions of my best friends, Jiao Feng and Tyler Cadigan, were of inestimable importance. They painstakingly read every draft and were helpful in every phase of this undertaking.

Abstract

This paper applies multivariate factor copula modeling methods in order to study the dependence relationships of daily returns of four exchange rates: the Canadian Dollar, the British Pound, the Japanese Yen, and the Euro. Conditional on the principal components or common factors identified, we estimated the dependence parameters and their corresponding rank correlations for the Clayton, Gumbel, and Gaussian copulas. We found that the dependence among the chosen currencies is strongly asymmetric, and the unconditional Gumbel copula is preferable. In contrast, conditional on the common factors, the dependence among the chosen currencies is weakly asymmetric, and the two-factor Gaussian copula modeling hypothesis is more appropriate.

Contents

1 Introduction

The global financial crisis that began in 2007 has led to extreme volatility in financial markets. Uncertainty and panic have flooded the world's financial markets during the global recession. As a result, the foreign exchange market experienced a very volatile period. Regardless of whether the Canadian dollar rises or falls during this crisis period, its movement has a great impact on the Canadian people's living standards and domestic economy. The Canadian government needs to predict the position of the Canadian dollar in global markets quickly in order to stabilize the economy and minimize the risk of holding foreign exchange reserves. Corporations need to know the same information in order to minimize the risk of holding certain currencies when they are conducting international business. Individual investors also increasingly need to know the same information to plan household budgets and change consumption habits to accommodate the rising and falling currency. It is therefore necessary, and important, to investigate the relationships among the Canadian dollar and other currencies, especially during volatile periods. This research will examine these relationships during economic booms and recessions.

Currently, many researchers are concerned with the causes and potential damages of this past financial crisis. Most investment decisions are based on the trade-off between risk and returns. Generally speaking, in asset allocation and risk management, the more independent the selected assets are, the more diversification the portfolio has, and therefore, less risk is involved. As a result, it is very important for people, especially risk practitioners, policy makers, and regulators to identify and model dependent relationships among currencies, which are important to consider for a diversified portfolio and a good risk management strategy.

One of the simplest and most commonly used approaches to study dependence among the exchange rates is to calculate the linear correlation coefficients between the values of their returns with the modelling assumption that they have normal or Gaussian distributions. The smaller the correlation coefficients, the less these markets depend on each other. Unfortunately, this approach is problematic. First, it is a widely recognized that the return data usually exhibit typical asymmetric distributions instead of a symmetric normal distribution (i.e., the data does not exhibit equally likelihood of falling and rising), which is an underlying assumption of the linear correlation calculation. In addition, a correlation of zero does not indicate independence of the data series. For example, let us assume that y and x have a dependence relationship where $y = x^2$. The linear correlation between x and y is zero. Therefore, the linear correlation may not provide an accurate measure of the dependence degree (i.e., how much the markets depend on each other). Second, the linear correlation cannot measure the dependence structure (i.e., how the markets depend on each other). The exchange rate returns' correlations may increase more when the economy is in recession than when the economy is booming. The linear correlation coefficient may underestimate the dependence of the financial exchange markets during the crisis since it does not capture the excess co-movements when the markets move downwards (Patton, 2006). Consequently, the linear correlation coefficients are inappropriate. Fortunately, an alternative approach, copula modelling, can overcome the limitations of the linear correlation coefficients technique. This paper applies copulas to study both dependence degree and dependence structure among the exchange rate returns.

A copula is a cumulative distribution function connecting multivariate marginal distributions in a specified form. Usually, researchers can obtain information about the distributions of the marginal functions easily, but it is difficult to accurately specify joint distribution. Copula modelling offers an important improvement in this respect: for example, in a bivariate case, when both marginals are Gaussian (i.e., normally distributed) and the copula has a Gaussian distribution, the joint distribution generated will be Gaussian as well. However, even if the marginals are not Gaussian, a copula approach can still be used to generate a joint distribution (Rockinger & Jondeau, 2001).

Besides modelling the joint distribution with copulas, this paper applies factor analysis to identify the common factors and construct two-factor copula models to study the dependence among the exchange rate returns. The common factors play an important role in determining the joint dependence among the returns. The common factors can be oil price, global consumer confidence index, or the combination of these factors. When the values on these factors change, all exchange rate returns will be affected to different degrees depending on how much their representative economies are tied with these factors, and this contributes to the asymmetric joint distributions of the chosen returns. To focus solely on the interactions of the dependence among the currencies themselves, the copula models conditional on the common factors (also named as factor copula models in this paper) are more preferable. The common factors, also known as principal components in this study, can be found by using a principal component analysis technique, further details of which will be in the methodology section.

The dataset in this study consists of four exchange rates. We chose the daily exchange rates of the Canadian Dollar, the British Pound, the Japanese Yen, and the Euro expressed in the US dollar. The sample period starts right after the Euro was introduced on

January 2nd, 2002 and ends on October 29th, 2010. In contrast to the existing literature that focuses mainly on bivariate studies, this paper focuses on Gaussian, Clayton, and Gumbel copula modelling in the multivariate case. The aim of this paper is to apply factor copula models to study dependence relationships among daily returns of these chosen exchange rates. In addition, this paper confirms if there is asymmetric dependence among the chosen exchange rate returns or not (i.e., this paper confirms if these returns tend more likely to go up or down together), and provides suggestions on identifying the appropriate copula for the chosen exchange rates.

Despite the fact that the copula approach has flourished in the statistical and actuarial literature, this approach is a newly introduced concept in the social science fields of finance and economics, but since 1998, a few studies have utilized copula modelling. Yet, to the best of our knowledge, the existing literature on international financial exchange markets has not paid much attention to directly investigating the tail dependencies among currencies and using factor copulas on the exchange rate returns. Therefore, one of the important contributions of this paper is to formally introduce the factor copulas and their applications on studying the dependence among the currencies.

The structure of the remainder of this paper is organized as follows: Section 2 presents a review of relevant literature; Section 3 reviews the history, basic concepts, special forms and properties of copulas selected for this study, and includes a brief introduction to principal component analysis; Section 4 describes the data and the estimation procedures used for empirical investigation; and finally, the last section contains suggestions for future research and concludes the study.

2 Literature Review

Abandoning traditional tools such as extreme value theory and using copula tools to study the extreme fluctuations in the exchange rate returns are not new concepts. The actuaries and statisticians were among the first to apply copula techniques in the finance field. Many of these scholars enjoyed the challenges of being the first few to be exposed to such tools. Genest and MacKay (1986) even published a paper with a name ”The Joy of Copulas: Bivariate Distributions with Uniform Marginals” to express their enjoyment of this tool to model non-Gaussian financial series. Researchers with actuarial or statistical backgrounds focused more on applying the tool to the insurance world, and their technical training allow them to explore further the innovative ways of applying this tool to better assist risk management. Their research can be easily extended to the finance world.

Building on the research of the first few scholars such as Frey, McNeil and Nyfeler (2001) who combined factor and copula tools, Laurent and Gregory (2003) thoroughly introduced one-factor Gaussian copulas, one-factor mean variance Gaussian mixtures, and one-factor Archimedean copulas. The factor approach helped achieve the goal of data reduction by locating the common factor variables affecting the selected data series. Conditional on these factors, the copulas can better assist in identifying the dependence relationships and model the chosen financial data. Furthermore, Anderson and Sidenius (2004) extended the standard Gaussian copula model to two new models to study the portfolio default loss. In one of their extended models, they randomized the systematic factor loadings, which allow default correlations to be higher in bear markets than in bull markets. They said that this model can induce a strong correlation skew similar to that observed in the credit market, making it easy to parameterize and efficient for obtaining numerical results. The existing studies focus mainly on the one-factor copulas. In contrast, this research focuses on using two-factor copulas, and the selected copula models are Gaussian, Gumbel, and Clayton copulas.

Many papers have been published on the topic of exchange rates. The attention on exchange rates has gradually shifted towards using copula tools to verify if the distributions of these exchange rates are asymmetric and to model their dependence. These studies will in turn assist the risk management of the international investment portfolio that can be significantly affected by the fluctuations in currencies. For example, Hurd, Salmon, and Schleicher (2007) applied copulas to construct bivariate foreign exchange distributions with a focus on the application of the Sterling Exchange Rate Index. Built on their study, Patton (2006) extended the copulas conditional on variables or common factors found through the factor approach, and constructed flexible models of the conditional dependence structure of the mark/U.S. dollar and yen/U.S. dollar exchange rates. Notably, the research that has been done focuses on applying bivariate copulas only for the dependence between two exchange rates. This research contributes to the existing literature by focusing on multivariate factor copula models to study four exchange rates: CAN/U.S., pound/U.S., yen/U.S., and euro/U.S. dollar.

3 Methodology

This section is organized as follows: First, subsection 3.1 formally introduces the basic definitions and properties of copula functions. Subsection 3.2 presents some important examples of copula functions, and their main characteristics; more specifically, it covers the three types of multivariate copula functions applied in my empirical studies. In subsection 3.3, the characteristics and measure of dependence of copulas are described. The estimation methods are introduced in subsection 3.4. Subsection 3.5 introduces factor structure in copula modeling and depicts how the factors play roles in the chosen types of copula functions. Finally, the subsection 3.6 describes the chosen data.

3.1 Definitions and Properties of Copula Functions

In this subsection, we first concentrate on using copulas to restore the joint distribution for two marginal variables (i.e., cumulative distribution functions or CDFs). We then extend to a multi-dimensional framework. Before we proceed to the foundational theorem for copulas according to Sklar (1959), we begin with the formal definition of "copula" given by Rockinger and Jondeau (2001):

Definition 1 : A two-dimensional copula is a function $C : [0,1]^2 \to [0,1]$ which has the following properties:

(1) $C(u, v)$ is increasing in u and v;

(2) $C(0, v) = C(u, 0) = 0, C(1, v) = v, C(u, 1) = u$;

(3) For every $u_1, u_2, v_1, v_2 \in [0, 1]$ such that $u_1 < u_2$ and $v_1 < v_2$, we have $C(u_2, v_2) - C(u_2, v_1) - C(u_1, v_2) + C(u_1, v_1) \geq 0$.

The first property indicates that the joint distribution function increases when allowing one variable to increase while keeping the other one constant. The second property ensures that the copula function is zero when the probability of one variable is zero. Additionally, the joint probability is determined by the marginal probability that is not equal to one. The third property is equivalent to saying that

$$\int_{v_1}^{v_2} \int_{u_1}^{u_2} \frac{\partial^2 c}{\partial u \partial v} \, dudv \geq 0$$

for all $u_1 < u_2$ and $v_1 < v_2$ in the range. This implies that if both u and v increase, the joint probability increases.

To further introduce the formal definition of copulas and show how copulas are used to restore joint distributions for marginals, now we state Sklars Theorem.

Sklar's Theorem in n-dimensions (1959). Let H be an n-dimensional distribution function with marginals $F_i(\cdot)$ with $i = 1, \ldots, n$. Then, there exists a copula C such that for random variables X_i, such that

$$H(x_1, x_2, \ldots, x_n) = C(F_1(x_1), F_2(x_2), \ldots, F_n(x_n)).$$

If $F_i(\cdot)$ are continuous for all $i = 1, \ldots, n$, then C is unique. Conversely, if $F_i(\cdot)$ are marginals or CDFs and C is a copula with a range of $[0, 1]^n$, then the function H is a joint distribution function with one-dimensional marginals $F_i(\cdot)$. Notably, when H is continuous, the unique C will be

$$C(u_1, u_2, \ldots, u_n) = H({F_1}^{-1}(u_1), {F_2}^{-1}(u_2), \ldots, {F_n}^{-1}(u_n)).$$

Note that, even though in our study the exchange rates are discrete; we can apply the kernel method to approximately smooth the data and obtain continuous CDFs for the exchange rate returns.

Applying a copula function to restore the multivariate distribution has several advantages. First, it provides flexibility in the model specification by separating the specifications of the marginals from those of the copula. In this way, it is possible to construct a complex non-Gaussian joint distribution. Second, a copula is a powerful technique because it directly models the dependence between the marginal distribution functions.

3.2 Some Common Copulas

After giving a general definition for copulas, in this section, we present several important copulas and their properties. We first define the product copula, which is the simplest copula function. Then we define the Gaussian copula, which is the basic and most commonly used copula. After this, we present an important class of copula functions: Archimedean copulas. The copulas that fall in this class can be stated directly and usually have a simple closed form expression. In addition, these copulas are popular because they can be easily derived and can capture a wide range of dependence structures.

3.2.1 Product Copula

If we set $u_1 = F(x_1)$ and $u_2 = G(x_2)$, then $C(F(x_1), G(x_2))$ describes the joint distribution of X_1 and X_2. If u and v are independent, then the product copula has the form

$$C(u_1, u_2) = u_1 u_2.$$

3.2.2 Gaussian Copula

As Schmidt (2006) outlined in his work, Gaussian copulas are an extension from the multivariate normal distribution. Let us assume that X_1 and X_2 are normally distributed and they are also jointly normal. Then we can use a linear correlation to fully describe their dependence structure and their correlation is

$$\text{Corr}(X_1, X_2) = \frac{\text{Cov}(X_1, X_2)}{\sqrt{\text{Var}(X_1)\text{Var}(X_2)}}.$$

The two-dimensional Gaussian copula is

$$\begin{aligned} C^G(u_1, u_2; \theta) &= \Phi_G(\Phi^{-1}(u_1), \Phi^{-1}(u_2); \theta) \\ &= \int_{-\infty}^{\Phi^{-1}(u_1)} \int_{-\infty}^{\Phi^{-1}(u_2)} \frac{1}{\sqrt{2\pi(1-\theta^2)}} e^{\frac{-x_1^2 - 2\theta x_1 x_2 + x_2^2}{2(1-\theta^2)}} dx_1 dx_2. \end{aligned}$$

Instead of applying the straightforward bivariate Gaussian copula, this paper presents a multivariate Gaussian copula to study dependence among more than two variables. The multivariate Gaussian copula for a correlation matrix Σ is given by

$$C_\Sigma^G(\mathbf{u}) = \Phi_\Sigma(\Phi^{-1}(u_1), \Phi^{-1}(u_2); \Sigma). \tag{1}$$

Note that the correlation matrix is the covariance matrix that scales each component of the variable to have a variance of one. The multivariate Gaussian copula is one special case of the elliptical copula family. Linear correlation is a good and efficient measure of the dependence relationships in this case. The equivalent way of expressing Gaussian (i.e., normal) and elliptical distributions' independence is zero correlation. For example, in the bivariate Gaussian copula, if θ=0, the bivariate Gaussian copula is a Product copula. The positive (or negative) sign of θ reveals a positive (or negative) linear dependence relationship among the variables. In other words, the bivariate Gaussian copula allows us to use one simple parameter, the correlation coefficient (θ) to capture the three fundamental dependence structures: independent, positive, and negative dependence structures. In the multivariate Gaussian copula, the correlation matrix is used to capture these dependence structures.

3.2.3 Clayton Copula

As mentioned by Trivedi and Zimmer (2005), with $\theta \in [-1, \infty) \setminus \{0\}$, the Clayton Copula takes the form:

$$C(u_1, u_2; \theta) = \max[(u_1^{-\theta} + u_2^{-\theta} - 1)^{-1/\theta}, \theta].$$

When $\theta > 0$, we simplify the above equation as follows:

$$C(u_1, u_2; \theta) = (u_1^{-\theta} + u_2^{-\theta} - 1)^{-1/\theta}.$$

Applying similar idea, we can have a family of n dimensions Clayton copulas for $\theta > 0$ and $n \geq 2$:

$$C^n(\mathbf{u}; \theta) = (u_1^{-\theta} + u_2^{-\theta} - 1 + \cdots + u_n^{-\theta} - n + 1)^{-1/\theta}. \tag{2}$$

3.2.4 Gumbel Copula

As mentioned by Trivedi and Zimmer (2005), with $\theta \geq 1$, the Gumbel copula takes the form:

$$C(u_1, u_2; \theta) = \exp\{-[(-\ln u_1)^\theta + (-\ln u_2)^\theta]^{1/\theta}\}.$$

To generalize the Gumbel family of bivariate copulas to a family of n dimensions copulas for $\theta \geq 1$ and any $n \geq 2$, we have

$$C_n^\theta(\mathbf{u}) = \exp\{-[(-\ln u_1)^\theta + (-\ln u_2)^\theta + \cdots + (-\ln u_n)^\theta]^{1/\theta}\}. \tag{3}$$

3.3 Measuring Dependence

In a financial context, the measures of dependence among random variables have drawn a lot of attention recently. In this section, we introduce three important measures of dependence: linear correlation, rank correlation and tail dependence using bivariate examples, which can be easily extended to multivariate ones.

3.3.1 Linear Correlation

In statistics and economics literature, the most familiar concept in studying dependence is the correlation coefficient between two random variables. The linear correlation coefficient is a traditional dependence measure. The correlation coefficient between two random variables x and y is defined as follows: $\text{Corr}(X_1, X_2) = \frac{\text{Cov}(X_1,X_2)}{\sqrt{\text{Var}(X_1)\text{Var}(X_2)}}$ or $\rho_{X_1,X_2} = \frac{\text{Cov}(X_1,X_2)}{\sigma_{X_1}\sigma_{X_2}}$. Note that σ_{X_1} and σ_{X_2} represent the standard deviations of X_1 and X_2, $\text{Cov}(X_1, X_2) = \text{E}[X_1X_2] - \text{E}[X_1]E[X_2], \sigma_{X_1}\sigma_{X_2} > 0$.

Several properties of the correlation coefficient are worth mentioning here. First, the correlation coefficient, ρ_{X_1,X_2}, is a well studied measure of linear dependence and it is symmetric. Second, $-1 \leq \rho_{X_1,X_2} \leq 1$ and its lower and upper bounds measure perfect negative and positive linear dependence. Third, the correlation measure is invariant under linear transformations of the variables, but it does not hold for general transformations. For example, if $G(\cdot)$ is a strictly increasing nonlinear transformation of the random variables, then we have the formula $\rho_{G(X_1),G(X_2)} \neq \rho_{X_1,X_2}$. Finally, $\rho_{X_1,X_2} = 0$ implies independence for bivariate normal distributed random variables, but it does not hold in general.

3.3.2 Rank Correlation

The limitations of the measure of linear dependence provide us the motivation to consider rank correlation, which is an alternative measure of nonlinear dependence relationships among variables with non-Gaussian marginals. As implied by its name, rank correlation concentrates on modeling the rankings of given observed data rather than on the actual values of the data themselves. Given by Trivedi and Zimmer (2005), there are two well-established measures of rank correlation, *Spearman's rho* and *Kendall's tau*, which provide a way to fit copulas to data.

Both Spearman's rho, $\rho_s(x, y)$, and Kendall's tau, $\rho_\tau(x, y)$, have the following four properties: first, they are symmetric; second, they are bounded by (-1, 1), and their lower and upper bounds on this inequality measure perfect negative and positive linear dependence; third, they are equal to zero when the random variables are independent from each other; finally, they are co- and counter- monotonic. The expressions in terms of copulas for the Rank correlations are as follows:

$$\rho_s(x,y) = 12\int_0^1\int_0^1 \left(C(u_1,u_2) - u_1u_2\right) du_1du_2$$

and

$$\rho_\tau(x,y) = 4\int_0^1\int_0^1 C(u_1,u_2)dC(u_1,u_2) - 1.$$

According to Nelson (1999), Kendall's tau and Spearman's rho are equivalent with the same underlying assumptions, but they usually have similar but different magnitudes. These two methods can verify the changes in the dependence relationships in different subsamples since the directions of the changes are usually the same under both methods. In this paper, our results include the estimates of the Kendall's tau, the Spearman's rho, as well as the tail indices (i.e., tail dependence), which is introduced in the following section.

3.3.3 Tail Dependence

In this subsection, we introduce the concept of tail dependence, which is applied to measure the dependence between the extreme values of random variables, for copula models. We refer to extreme co-movement relationships as concepts of concordance and discordance. Basically, according to Trivedi and Zimmer (2005), concordance means that there is a dependent relationship between large values of two random variables, and discordance means that there is a dependent relationship between large values of one random variable with small values of another.

Consider two continuous uniform random variables X_1 and X_2 with marginal distribution functions F_1 and F_2. Tail dependence measures the conditional probability that X_1 exceeds a given value k given that X_2 exceeds the same value. Intuitively, upper tail dependence means that large values of X_2 are expected with large values of X_1 (Schmidt, 2006). If this is the case, we say that the random variables do not exhibit tail dependence. Specifically, given the limits $\lambda_U \in [0,1]$ and $\lambda_L \in [0,1]$, we can define λ_U and λ_L while the measures of upper tail dependence and that of lower tail dependence are given in the following equations, respectively:

$$\lambda_U = \lim_{m \to 1^-} P(X_2 > F_2^{-1}(m) | X_1 > F_1^{-1}(m))$$

and

$$\lambda_L = \lim_{m \to 0^+} P(X_2 \leq F_2^{-1}(m) | X_1 \leq F_1^{-1}(m)).$$

If $\lambda_U \in (0,1]$, then X_1 and X_2 are said to have upper tail dependence. If $\lambda_U = 0$, then X_1 and X_2 are said to be asymptotically independent in the upper tail. Similarly, if $\lambda_L \in (0,1]$, then X_1 and X_2 are said to have lower tail dependence. If $\lambda_L = 0$, then X_1 and X_2 are said to be asymptotically independent in the lower tail.

To find a relationship between the above tail dependence equations with our copula functions, the following calculations give the relationship:

$$\lambda_U = \lim_{m \to 1^-} P(X_2 > F_2^{-1}(m) | X_1 > F_1^{-1}(m)) = \lim_{m \to 1^-} \frac{C^s(m,m)}{1-m} \tag{4}$$

and

$$\lambda_L = \lim_{m \to 0^+} P(X_2 \leq F_2^{-1}(m) | X_1 \leq F_1^{-1}(m)) = \lim_{m \to 0^+} \frac{C^s(m,m)}{m}. \tag{5}$$

Then, by substituting the Clayton Copula and Gumbel Copula formulas into the measurement function of tail dependence, we can obtain the coefficients of tail dependence for these copulas.

In particular, the relationship between the dependence parameter (θ) and Kedall's tau (τ) of the Clayton copula in (2) is $\tau = \theta/(\theta+2)$. Using the formulas in (4) and (5), we can show that a Clayton copula has positive lower tail dependence and we have $\lambda_L = 2^{1/\theta}$ and $\lambda_U = 0$. The relationship between the dependence parameter (θ) and Kedall's tau of the Gumbel copula in (3) is $1/\theta = 1 - \tau$. Using the formulas in (4) and (5), we can show that a Gumbel copula has positive upper tail dependence and we have $\lambda_L = 0$ and $\lambda_U = 2 - 2^{1/\theta}$. Similar relationship can be identified between the dependence parameter and Spearman's rho.

The left and right tail dependence of the Gaussian Copula is equally likely to happen (i.e., the number of values that are less than a mean is the same as the number of values greater than the mean). The Gaussian Copula is a good choice for modeling between two variables when there is no strong tail dependence. Or, in the context of exchange rate returns, when two rates are not strongly correlated at low (or high) values but less correlated at high (or low) values, the Gaussian copula is an appropriate modeling choice.

In formula (2), θ is the dependence parameter that has a restricted region of $(0, \infty)$. As θ approaches zero, the marginals will be independent of each other. The Clayton Copula has strong positive left tail dependence and relatively weak right tail dependence. In other words, it models extreme negative co-movements. The Clayton Copula is a good choice for modeling between two variables when their left tail dependence is strong. Or, in the context of exchange rate returns, when two returns are strongly correlated at low values but less correlated at high values, the Clayton copula is an appropriate modeling choice.

The Gumbel copula has strong positive right tail dependence and relatively weak left tail dependence. In other words, it does not tolerate extreme negative co-movements. In the context of exchange rate returns, when correlation between two returns is strong in the right tail of the joint distribution, the Gumbel copula is an appropriate choice.

Therefore, if we can estimate the dependence parameter in each copula function, we can easily calculate out the values of rank correlations, which include both the Kendall's tau and the Spearman's rho, and the tail dependence structure for that copula.

3.4 Estimation Methods

This paper's main interest is to estimate the dependence parameters in copula functions. Usually, according to Trivedi and Zimmer (2005), there are three approaches to estimate the parameters in a copula function. The first and most direct estimation method is a maximum likelihood approach, which is used to estimate the copula and the marginal distributions simultaneously. To apply this method, we need to specify the marginal distributions and any mistake in such specifications will affect the estimation results. A second approach is a generalized method of moments (GMM), which is used to estimate the parameters after deriving the moment functions. A third approach is a two-stage maximum likelihood estimation method: in the first stage, the marginal distribution functions are estimated with the assumption of independence between the two random variables; in the second stage, the estimated marginal distributions are substituted into the copula function and the dependence parameter of this copula function is estimated. The marginal distributions and the dependence structure are independently estimated. Using the two-stage maximum likelihood method, we don't need to make any assumption on the marginal distributions and we can use the estimated marginal distributions, which means that the estimated distributions are free of specification error. Therefore, this paper focuses on the two-stage maximum likelihood method.

3.5 Factor Copula Structure

In this paper, we introduce an alternative way of approaching the problem of finding the dependence relationships between fluctuations on exchange rates by using copula models conditional on the common factors found through the factor analysis.

Factor analysis is based on the fundamental assumption that some underlying factors, which are smaller in number than the number of observed variables, are responsible for the co-variation among the observed variables. This analysis method is mainly used for data reduction purposes or as a natural start point to find the common factors that describe the underlying variables' relationships. There are a few forms of factor analysis, including principal component analysis. This paper focuses on the principal component analysis method, which is the most common form of factor analysis. This method is particularly appropriate since we can use the obtained principal components (which are also referred as "common factors for simplification in explanation) as criterion variables in subsequent copula analyses. These principal components or common factors are able to account for most of the variance in the observed exchange rate returns.

The correlation matrix of returns on exchange rates is calibrated by developing factor models for exchange rate returns, where the underlying factors could be interpreted as a small set of economic or financial factors. Here we first focus on a one-factor model and it can be easily extended to a two-factor model.

Instead of directly applying the marginal distributions of the exchange rate returns into our copula function, we can use their marginals conditional on the common factors. The correlation coefficients of these returns depend on the estimated common factors. The ways the returns are interrelated with each other depend largely on the common factors. To focus solely on the interactions among the returns, the factor copula correlations conditional on the estimated common factors can more accurately describe the dependence relationships among multiple variables. Specifically, conditional on the common factors, the exchange rate returns will only be dependent on the joint distributions of $Z_1, Z_2, \cdots, Z_n$, which are the unique parts for their respective exchange rate returns.

3.5.1 Two factor Gaussian copulas

We define $r_i (i = 1, 2, \ldots, n)$ as the returns on the exchange rates. If these returns were normally distributed, the joint distribution of them may be multivariate normal. As is well-known in the academic world, the probability distribution of financial series tends not be normal. To apply a Gaussian copula to model our data, we followed the suggestions of Hull (2009) and first transformed the returns into new variables $x_i (i = 1, 2, \ldots, n)$ using

$$x_i = N^{-1}[Q_i(r_i)], i = 1, 2, \ldots, n.$$

where N^{-1} is the inverse of the cumulative normal distribution and $Q_i (i = 1, 2, \ldots, n)$ are the cumulative distribution functions for respective exchange rate returns, $r_i (i = 1, 2, \ldots, n)$. In this transformation, the new variables, x_i, are constructed to have a standard normal distribution with mean equals to zero and standard deviation equals to one. This transformation is percentile to percentile so that the correlations among the returns can be measured by the ones among the new variables. Then introducing a Gaussian copula, we can study the copula correlations or dependence relationships among financial returns that do not have normal distribution, and separate the estimations for unconditional marginal distributions and the joint distribution.

To avoid defining a different correlation between x_i and x_j for each pair of exchange rate returns i and j in the Gaussian copula models or a different copula correlation between the distribution function or the copula models, a one-factor model is often used. The assumption is that

$$x_i = a_i F + \sqrt{1 - a_i} Z_i. \tag{6}$$

In this equation, F is a common factor affecting all exchange rate returns and Z_i have independent standard normal distributions. The $a_i (i = 1, 2, \ldots, n)$ are constant parameters between -1 and +1. The correlation between x_i and x_j is $a_i a_j$.

Suppose that the probability that exchange rate i will be below a threshold of m is $Q_i(m)$. Under the Gaussian copula model, such low returns happen when $N(x_i) = Q_i(m)$ or $x_i = N^{-1}[Q_i(m)]$. From equation (6), this condition is

$$a_i F + \sqrt{1 - a_i} Z_i = N^{-1}[Q_i(m)]$$

or

$$Z_i = \frac{N^{-1}[Q_i(m)] - a_i F}{\sqrt{1 - a_i}}.$$

Conditional on the value of the factor F, the probability of having a return lower than m is therefore

$$Q_i(m|F) = N\left(\frac{N^{-1}[Q_i(m)] - a_i F}{\sqrt{1-a_i}}\right)$$

as given in the book written by Hull (2009). Therefore, setting a threshold of m, we can find out the probability of having such disappointing returns. In this paper, we extended the above model and used a two-factor model for our data analysis. In the two-factor model,

$$x_i = \alpha_i F_1 + \beta_i F_2 + \sqrt{1-\alpha_i-\beta_i} Z_i. \tag{7}$$

In this equation, F_1 and F_2 are two common factors affecting defaults for all companies and Z_i have independent standard normal distributions. The α_i and β_i are constant parameters between -1 and +1. The correlation between x_i and x_j is $\alpha_i\alpha_j + \beta_i\beta_j$. Suppose that the probability that exchange rate i will be below a threshold of m is $Q_i(m)$. Under the Gaussian copula model, such low returns happen when $N(x_i) = Q_i(m)$ or $x_i = N^{-1}[Q_i(m)]$. From equation (7), this condition is

$$\alpha_i F_1 + \beta_i F_2 + \sqrt{1-\alpha_i-\beta_i} Z_i = N^{-1}[Q_i(m)]$$

or

$$Z_i = \frac{N^{-1}[Q_i(m)] - \alpha_i F_1 - \beta_i F_2}{\sqrt{1-\alpha_i-\beta_i}}.$$

Conditional on the value of the factors F_1 and F_2, the probability of having a return lower than m is therefore

$$Q_i(m|F) = N\left(\frac{N^{-1}[Q_i(m)] - \alpha_i F_1 - \beta_i F_2}{\sqrt{1-\alpha_i-\beta_i}}\right). \tag{8}$$

Therefore, setting a threshold of m, we can find out the probability of having such disappointing returns.

3.5.2 Two-Factor Archimedean Copulas

In addition, by using the factors we found, we can extend the copula models for conditional variables. Besides one factor Gaussian copulas, Laurent & Gregory (2003) outlined the one-factor Clayton copula model in their paper. Based on their work, we extended their model and constructed two-factor Clayton and Gumbel copulas. The first step is to identify the common factors. The common factors are extracted from the returns without transforming them to normal variables. Instead of having a normal distribution, the common factors follow a Gamma distribution with parameter $1/\theta$, where $\theta > 0$, and with a scale parameter equal to one. More precisely, the factor follows a Gamma distribution with the probability density

$$f(x) = \frac{1}{\Gamma(1/\theta)} e^{-x} x^{\frac{1-\theta}{\theta}}.$$

Then we define:

$$X_i = \left(\frac{1 - \ln(U_i)}{F} \right)^{1/\theta}$$

where $U_i (i = 1, \ldots, n)$ are independent uniform random variables and they are independent from the common factors, F.

Conditionally on the common factors, F, for Clayton copula, the probability of having a return lower than m is therefore

$$Q_i^{\text{Clayton}}(m|F) = \exp\left(V(1 - Q_i(t)^{-\theta}) \right) \tag{9}$$

and using similar logic for Gumbel copula, we have

$$Q_i^{\text{Gumbel}}(m|F) = \exp\left(V(\log(Q_i(t))^{\theta})\right). \tag{10}$$

Using above conditional CDFs, we were able to find more accurate copula correlations or dependence relationships among the exchange rate returns excluding the common factors.

3.6 Data

The data used in this study were downloaded from Yahoo!Finance. Our research interest is to investigate the dependence among foreign exchange rates. The data of interest therefore were the daily returns of exchange rates in four different countries/regions. These selected currencies are the Canadian Dollar, British Pound, Japanese Yen, and Euro. The abbreviations used are CAD (Canadian Dollar), GBP (British Pound), JPY (Japanese Yen), and EUR (Euro). The exchange markets would have been closed for the countries which celebrate different specific holidays. Therefore, it was very important to filter the sample data by removing observations corresponding to these holidays.

The chosen four countries/regions have differences in the development of their financial markets. As a result of this, their exchange rates have different start dates. We chose the start date of launching the Euro, which is January 2nd, 2002, as the cut-off point, and eliminated any observations before this. The sample data ends with October 29th, 2010. This reduces the sample to 1,730 observations. In addition, the current financial crisis would influence the empirical results we obtained on a large scale. It is interesting to compare the pre- and post- crisis empirical results of sample observations. To achieve this goal, we used the same filtered sample data but we further separated the sample by the cut-off date of July 1st, 2007 (note: this is a rough estimation of the start date for our current crisis). The first subsample has 1,037 observations between January 2nd, 2002 and June 30th, 2007, while the second subsample has 693 observations between July 2nd, 2007 and October 29th, 2010.

4 Empirical Results

After obtaining the sample datasets, we estimated the dependence parameters and their corresponding rank correlations in the selected copula functions. We also conducted normality tests on all data series and calculated the linear correlation coefficients for all pairs of exchange rate returns. To further our analysis, we conducted a null hypothesis of independence on both the linear correlation coefficients and the rank correlations. At the end of the empirical results analysis, we obtained the tail dependence parameters, which give a clear idea of the dependence structure among the exchange rate returns. Based on the estimation results obtained, we should be able to compare and find the appropriate copula models for the chosen data.

4.1 Test of Normality and Linear Dependence

Linear correlation coefficients cannot capture the extreme fluctuations and nonlinear dependence relationships in the sample data especially for non-Gaussian financial data. Therefore, it is important for us to find an approach such as copulas to study the nonlinear dependent relationships of financial data (i.e., exchange rate returns) that are not necessarily normal. To confirm that the exchange rates have nonlinear dependence and are not normally distributed, we conducted normality tests and zero linear dependence tests. We rejected normality by obtaining p-value that equals zero from the Jarque-Bera test for all data series. This is consistent to the findings of existing literature.

To verify if the data series are dependent on each other or not, we ran a simple linear regression between the different data series. We obtained positive values of linear correlations for all pairs of exchange rates (Tables 1-3). In other words, all data series exhibit dependence relationships with each other. Moreover, after testing a simple null hypothesis of zero linear independence, we rejected the linear independence relationship for all of the data series as exhibited in Tables 1-3. As mentioned earlier, the exchange rate returns may exhibit nonlinear dependence, and the linear correlation coefficients may fail to capture these dependence relationships. From the results exhibited in Table 4, we found that after separating the full sample into two samples by picking a start date for the crisis and naming the two samples as pre- and post-crisis samples, the correlations significantly changed after the crisis occurred. In other words, the linear correlations are not consistent at different time periods. We can argue that the data series follow a nonlinear dependence relationship and the linear correlations are unlikely able to accurately describe the dependence relationships. Therefore, we need to use models such as copula models that are able to capture the nonlinear dependence relationships.

Table 1: Linear Correlation Coefficients and Test of Independence for Full Sample

	t-Statistic	p-value	Correlation Coefficients
Rho (rc, rj)	-4.92676	9.16E-07	-0.1177
Rho (rc, rg)	12.18942	0.00E+00	0.281384
Rho (rc, re)	13.9246	0.00E+00	0.317627
Rho (rg, rj)	3.264054	1.12E-03	0.07828
Rho (re, rj)	8.554479	0.00E+00	0.201565
Rho (re, rg)	38.37558	0.00E+00	0.678318

*Note: rc, re, rj, and rg represent the returns of CAD/US dollar, Euro/US dollar, Japanese Yen/US dollar, and Pound/US dollar. Rho $(\cdot, \cdot)$ represents the linear correlation between two different returns.

Table 2: Linear Correlation Coefficients and Test of Independence for Pre-Crisis Sample

	t-Statistic	p-value	Correlation Coefficients
Rho (rc, rj)	0.228344	8.19E-01	0.007098
Rho (rc, rg)	2.533704	1.14E-02	0.078513
Rho (rc, re)	5.060373	4.95E-07	0.155384
Rho (rg, rj)	11.27572	0.00E+00	0.330761
Rho (re, rj)	12.13028	0.00E+00	0.352806
Rho (re, rg)	35.87575	0.00E+00	0.744498

*Note: rc, re, rj, and rg represent the returns of CAD/US dollar, Euro/US dollar, Japanese Yen/US dollar, and Pound/US dollar. Rho $(\cdot, \cdot)$ represents the linear correlation between two different returns.

4.2 Copula Model Selection

Patton (2006) found that the univariate distribution of daily exchange rate returns resembles the Student t's distribution. Therefore, a multivariate t distribution seems to be an appropriate assumption for the true underlying model. However, as Patton (2006) and Bollerslev (1987) mentioned in their paper, the multivariate Student's t distribution can be problematic since different exchange rates are likely to have different degrees of freedom parameters and cannot capture the possibility of an asymmetric dependence structure. Therefore, an alternative solution for this problem for nonlinearly dependent data series

Table 3: Linear Correlation Coefficients and Test of Independence for Post-Crisis Sample

	t-Statistic	p-value	Correlation Coefficients
Rho (rc, rj)	10.77742	0.00E+00	0.379347
Rho (rc, re)	11.95872	0.00E+00	0.414094
Rho (rg, rj)	-1.60283	1.09E-01	-0.06086
Rho (re, rj)	2.732024	6.46E-03	0.103374
Rho (re, rg)	21.91728	0.00E+00	0.640383

*Note: rc, re, rj, and rg represent the returns of CAD/US dollar, Euro/US dollar, Japanese Yen/US dollar, and Pound/US dollar. Rho (·, ·) represents the linear correlation between two different returns.

Table 4: Compare the Linear Correlation Coefficients for Different Samples

	Full Sample	Pre-Crisis	Post-Crisis
Rho (rc, rj)	-0.1177	0.007098	-0.1804
Rho (rc, rg)	-0.28138	-0.07851	-0.37935
Rho (rc, re)	-0.31763	-0.15538	-0.41409
Rho (rg, rj)	-0.07828	-0.33076	0.060861
Rho (re, rj)	-0.20157	-0.35281	-0.10337
Rho (re, rg)	0.678318	0.744498	0.640383

*Note: rc, re, rj, and rg represent the returns of CAD/US dollar, Euro/US dollar, Japanese Yen/US dollar, and Pound/US dollar. Rho (·, ·) represents the linear correlation between two different returns.

is to use copula models, which decompose the multivariate distribution into the marginal distributions and connect these distributions using a copula relationship. This allows us to implement different copulas that capture the asymmetric dependence structure and avoid the misspecification by assuming multivariate distributions. In addition, this paper introduces an innovative way of combining factor structure into copula models. To select the most fit copula models, we compared the conditional Gaussian, Gumbel, and Clayton copula models.

From our principal component analysis results shown in tables 6 and 10, we found that the first factor or principal component can explain at least 47 percent of the original returns data, and the first two factors or principal components can explain around 75 percent for full, pre, and post crisis samples. In other words, these results suggest that the number of common factors or components is two. Therefore, the conditional models studied are two-factor copula models.

4.3 Estimation of the Copula Models

For the two-factor Gaussian copula models, we transformed the non-Gaussian data series into new variables that are normally distributed. Then we applied the factor analysis to identify the common factors that are assumed to be normal. Conditional on these factors, we applied the copula to the marginals to obtain the copula correlations. For the two-factor Gumbel and Clayton copula models, we used the common factors that follow standard gamma distributions as conditional variables in the copula models to obtain the estimates on dependence.

In this paper, we adopted the estimation methods chosen by Hu (2006) and used a two stage maximum likelihood semi-parametric method. In other words, the marginal distributions or empirical CDFs (i.e., cumulative distribution functions) are estimated non-parametrically, and then we substituted the marginal distributions or empirical CDFs into the factor copula and estimated the dependence parameter in this copula. Similar to what Hu (2006) has done in her paper, we assumed the observation data series $(x_1, x_2, \ldots, x_n)$ obtained are independent with the univariate marginal distribution or the empirical CDF of X_i, $F_X(\cdot)$, where $\hat{F}_X(x) = \frac{1}{2}\sum_{t=1}^{2} \mathbf{1}\{Z_t \leq z\}$. The univariate marginal distribution or the empirical CDFs of $X_i(i = 1, \ldots, n)$ are expressed as $\hat{F}_X(x_i)$, then we restore the joint distribution function as follows:

$$\hat{H}(x_1, x_2 \dots, x_n; \theta) = C(\hat{F}_{X_1}(x_1), \dots, \hat{F}_{X_n}(x_n); \theta).$$

There are several advantages to using this approach. The main advantage is that the estimation results are free of specification errors and robust. The results are reported in the next section.

4.4 Results Analysis

After applying the two stage estimation methods, we obtained and compared the full sample, pre and post crisis estimates for the dependence parameters (θ), rank correlation coefficients, and tail dependence for Clayton and Gumbel copulas. We also presented the copula correlations for the Gaussian copula in all three samples.

4.4.1 Gaussian Copula with Factor Loadings

After transforming the non-Gaussian exchange rate returns into the normal variables, we found the common factors of these variables with the factor loadings shown in Table 5 and the percentages that the principal components or common factors can explain the underlying data exhibited in Table 6.

Since the transformation is a percentile-to-percentile transformation (note: the details are described under the methodology section), the correlations among the exchange rate returns can be measured as the correlations among these transformed normal variables, and these calculated correlations are referred to as the copula correlations (Hull, 2008). The

Table 5: Compare the Linear Correlation Coefficients for Different Samples

	Full Sample		Pre-Crisis Sample		Post-Crisis Sample	
	Factor 1	Factor 2	Factor 1	Factor 2	Factor 1	Factor 2
rcad	0.46	-0.66	0.03	0.97	0.66	-0.35
reur	0.89	0.04	0.88	0.19	0.86	0.14
rjpy	0.86	0.03	0.88	0.11	0.84	0.01
rgdp	0.33	0.79	0.66	-0.19	0.02	0.96

*Note: rcad, reur, rjpy, and rgdp represent the returns of CAD/US dollar, Euro/US dollar, Japanese Yen/US dollar, and Pound/US dollar.

Table 6: Proportions Explained by the Common Factors of the Transformed Variables

	Proportion of Variance		
Proportion Explained by	Full Sample	Pre-Crisis Sample	Post-crisis sample
Factor 1	0.46	0.49	0.47
Factor 2	0.27	0.26	0.26
Cumulative Proportions	0.73	0.75	0.73

copula correlations in both the unconditional and two-factor Gaussian Copula models are reported in Tables 7 and 8.

Comparing the Tables 7 and 8 with Table 4, we noticed that without using copula models that can capture nonlinear relationships, there are no significant linear correlations of the CAN/US returns with the JPY/US as shown in Table 4, while using copula models, we identified significant copula correlations as shown in both Tables 7 and 8. As we can see from both Tables 7 and 8, the copula correlations tend to exhibit a mixed result in terms of increased or decreased dependence relationships. All copula correlations between the exchange rate returns decreased significantly after the crisis occurred except the correlations of the CAN/US returns with the JPY/US and UK/US returns for the unconditional Gaussian copula models. For the two-factor copula, all copula correlations decreased significantly after the crisis occurred except the correlations of the CAN/US returns with the JPY/US and EUR/US returns. This can be explained by the relative stability of the economy and banking system in Canada compared to the ones in Japan, UK, and European.

Table 7: Unconditional Gaussian Copula Results

	Full Sample	Pre-Crisis	Post-Crisis
Rho (rc, rj)	0.266391	0.157107	0.398597
Rho (rc, rg)	0.212978	0.083562	0.354494
Rho (rc, re)	-0.06569	0.003878	-0.14168
Rho (rg, rj)	0.683859	0.761216	0.625544
Rho (re, rj)	0.223964	0.371452	0.072583
Rho (re, rg)	0.165226	0.363254	-0.03366

*Note: rc, re, rj, and rg represent the returns of CAD/US dollar, Euro/US dollar, Japanese Yen/US dollar, and Pound/US dollar. Rho $(\cdot, \cdot)$ represents the copula correlation between two different returns.

Table 8: Factor Gaussian Copula Results

	Full Sample	Pre-Crisis	Post-Crisis
Rho (rc, rj)	0.751957	0.264077	0.952702
Rho (rc, rg)	-0.73271	-0.42395	-0.61651
Rho (rc, re)	0.470419	0.227673	0.714544
Rho (rg, rj)	-0.12428	0.652986	-0.46903
Rho (re, rj)	0.910232	0.913827	0.866368
Rho (re, rg)	0.153417	0.713231	-0.04502

*Note: rc, re, rj, and rg represent the returns of CAD/US dollar, Euro/US dollar, Japanese Yen/US dollar, and Pound/US dollar. Rho $(\cdot, \cdot)$ represents the copula correlation between two different returns.

4.4.2 Clayton and Gumbel Copula with Factor Loadings

For Clayton and Gumbel Copulas, the estimates of the common factor loadings for the exchange rate returns are exhibited in Table 9 and the percentages that the principal

Table 9: Factor Loadings of the Exchange Rate Returns

	Full Sample		Pre-Crisis Sample		Post-Crisis Sample	
	Factor 1	Factor 2	Factor 1	Factor 2	Factor 1	Factor 2
rcad	0.56	-0.55	0.04	0.97	0.67	-0.39
reur	0.89	0.14	0.88	0.2	0.88	0.18
rjpy	0.19	0.89	0.65	-0.19	0.02	0.96
rgdp	0.86	0.03	0.88	0.11	0.85	-0.02

*Note: rcad, reur, rjpy, and rgdp represent the returns of CAD/US dollar, Euro/US dollar, Japanese Yen/US dollar, and Pound/US dollar.

Table 10: Proportion Explained by the Common Factors of the Exchange Rate Returns

	Proportion of Variance		
Proportion Explained by	Full Sample	Pre-Crisis Sample	Post-crisis sample
Factor 1	0.48	0.49	0.49
Factor 2	0.28	0.26	0.27
Cumulative Proportions	0.75	0.75	0.76

components or common factors can explain the underlying data are exhibited in Table 10. Conditional on the common factors, the estimates for the dependence parameters, the rank correlations, and tail indices for both the unconditional and two-factor Clayton and Gumbel copulas are shown in Table 11.

Table 11 shows that the dependence parameters tend to decrease after the crisis occurred. In other words, after the crisis occurred, the returns on the currencies tend to be less dependent on each other, and the dependence parameters of unconditional Gumbel copula tend to be more than double the estimations of unconditional Clayton copula. This means that the dependence among the returns is asymmetric, and these returns are more negatively dependent on each other. Therefore, the models such as Gumbel copula that can capture the extreme negative dependence would be a better fit to the chosen currencies.

Table 11: Unconditional and Factor Clayton & Gumbel Copula Estimation Results

Copula		Full Sample	Pre-Crisis Sample	Post-Crisis Sample
Unconditional Clayton Copula	Dependence Parameter	0.5182035	0.6735244	0.4390656
	Kendall's Tau	0.205783	0.2519238	0.1800138
	Spearman's Rho	0.3038211	0.368643	0.2669235
	Tail Index	[0.2624758, 0]	[0.3573161, 0]	[0.2062457, 0]
Unconditional Gumbel Copula	Dependence Parameter	1.266502	1.337486	1.237634
	Kendall's Tau	0.2104238	0.2523284	0.1920068
	Spearman's Rho	0.3104698	0.3683127	0.2841702
	Tail Index	[0, 0.2714333]	[0, 0.3209192]	[0, 0.2492256]
Factor Clayton Copula	Dependence Parameter	4.197289	3.872766	3.712977
	Kendall's Tau	0.6770352	0.6588136	0.6505415
	Spearman's Rho	0.8551584	0.8399345	0.8328992
	Tail Index	[0.8476179, 0]	[0.8357017, 0]	[0.8301302, 0]
Factor Gumbel Copula	Dependence Parameter	3.595267	3.685626	3.289485
	Kendall's Tau	0.7218565	0.7286757	0.696001
	Spearman's Rho	0.8923946	0.8973188	0.8730768
	Tail Index	[0, 0.7873666]	[0, 0.7930848]	[0, 0.7654383]

Examining the null hypothesis that the dependence among the returns of the spot exchange rates can be modelled by the Gaussian, Gumbel, or Clayton copulas, the goodness-of-fit tests confirmed that the Gumbel copula would be a better fit among the three copulas to model the unconditional data series.

Using one of the common factor analysis techniques, the principal component analysis, two common factors were identified for the chosen currencies for all three different samples (i.e., the sample before the crisis occurred, the sample since the crisis occurred, and the full sample that includes at least one full business cycle). Conditional on these common factors, the estimated dependence parameters of the Clayton and Gumbel copulas are found to be similar to each other as shown in Table 11. In other words, the chosen exchange rates exhibited no clear asymmetric dependence, and thus, both the Clayton and Gumbel copulas are not appropriate tools for modelling the joint dependence of the chosen exchange rate returns. These patterns are also evident when comparing the estimated results of the Kendall' Tau, Spearman's Rho, and Tail dependence between the unconditional and two-factor copula models. Furthermore, the goodness-of-fit tests showed that the two-factor Gaussian copula is a good fit for modelling the joint distribution of the chosen currencies. Therefore, the copula correlations obtained from the conditional Gaussian copula are reliable estimates.

Notably, without being conditional on the common factors into the copula modelling analysis, the joint dependence among the returns is asymmetric, and thus the unconditional Gumbel copula is a more appropriate tool to model these returns' joint dependence. However, conditional on the common factors, the joint dependence is symmetric, and thus the two-factor Gaussian copula is a better tool. These findings are particularly interesting since they confirm that the common factors play an important role in determining the joint dependence among the exchange rate returns. The common factors can be oil price, global consumer confidence index, or the combination of these factors. When the returns on these factors change, all exchange rate returns will be affected to different degrees depending on how much their representative economies are tied with these factors, and this contributes to the asymmetric joint distributions of the chosen returns. To focus solely on the interactions of the dependence among the currencies themselves, the two-factor Gaussian copula models conditional on the common factors are preferable.

Summing up, the Clayton copula would fit best if negative changes in the chosen exchange rate returns are more highly correlated than positive changes; the Gumbel copula would fit best in the opposite situation. The Gaussian copula fits best if the dependence among the data series is symmetric. The above results analysis indicates that the estimated dependence results are similar under both the conditional Clayton and Gumbel copulas. Combining with the goodness-of-fit tests, the results analysis leads to the conclusion that the two-factor Gaussian copula is a good fit for these four exchange rate returns. Malevergne and Sornette (2003) failed to reject the Gaussian copula hypothesis at the 95% confidence level for more than 50% of the pairs of currencies over the five-year time interval. In this research, the study is extended from the pairs of currencies to the joint distributions of the chosen four currencies, and found that the symmetric joint dependence among the returns of these currencies can be appropriately modeled with the Gaussian copula hypothesis.

5 Conclusions

This paper applies multivariate copula modeling methods in order to study the dependence relationships of daily returns of the four exchange rates: the Canadian Dollar, British Pound, Japanese Yen, and Euro. Conditional on the common factors identified, the copulas are used to estimate the dependence parameters (θ) and their corresponding rank correlation (e.g., Spearman's rho) for two different copulas: Clayton and Gumbel copulas. These two copulas capture the left and right tail dependence, respectively. For the Gaussian copula, we obtained the linear correlation parameters instead of rank correlations to capture the dependence relationships. In the two-step estimation approach, we first obtained empirical CDFs to model the marginal distribution functions, and then we used these CDFs to estimate the dependence parameter in the maximum likelihood function for each copula. For exchange rate returns in our case, the copula modeling method gives more accurate results on the dependence relationships since we are able to utilize this method to capture the nonlinear dependence relationships between non-Gaussian daily returns data of exchange rates. Notably, since the common factors may affect the chosen exchange rates to different degrees and this may contribute to the asymmetric dependence measures, to focus solely on the interactions among the exchange rates, we used the copula models conditional on the common factors, which are estimated from principal component analysis.

A few researchers such as Longin and Solnik (2001) found that the stock market exhibits greater left tail dependence, and in other words, the stock markets tend more likely to crash together than go up together. This intrigues us to explore if the foreign exchange markets exhibit similar pattern, and in other words, whether the assumption that the chosen exchange rates have a symmetric dependence structure is consistent with the data. One of the key findings of this paper is that, without being conditional on the common factors, the dependence among the chosen currencies is strongly asymmetric and the unconditional Gumbel copula is preferable. In contrast, another key finding is that, conditional on the common factors, the dependence among the chosen currencies is weakly asymmetric and the conditional Gaussian copula modelling hypothesis is more appropriate.

This paper may serve as a brief introduction to the concept of factor copulas and their modeling and estimation methods. These techniques determine more accurate dependent relationships between currencies and better assist the risk assessment, portfolio management, option pricing, and hedging at government, corporate, and individual investor levels. Since 2001, the literature on copulas has grown quickly in the fields of finance and economics. There are several directions for future research and applications on conditional copula modeling that we would like to point out here. First, it may be of interest to apply the factor copula modelling method to study dependence relationships among the foreign exchange markets and the financial stock markets. Factor copula modeling could also be applied to risk management and asset pricing. In addition, the copulas we used and other copulas could be extended to model the joint distribution of returns, volume, and duration between transactions for foreign exchange rates. Finally, this factor copula tools can also assist researchers to better measure the risks involved if the currencies such as Chinese Yuan go floating.

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