"Is Deferring to the Wiser the Optimal Strategy?"
Asymmetric Information and Bank Runs

by

Michael Brolley

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Abstract

This paper studies the issue of bank runs in a dynamic model with aggregate uncertainty. The information structure is asymmetric, with the bank having additional private information about the measure of impatient agents, which is modeled by a Markov chain. The additional information permits the bank to more accurately predict the measure of impatient agents in the current period than does the public information alone. Moreover, there exists a unique banking equilibrium where the demand deposit contract achieves more efficient risk-sharing that the contract implied by the public information. The frequency at which a prediction is inaccurate enough to yield no payment for patient agents who do not withdraw early is also studied. A numerical exercise shows that the event occurs less frequently under the bank's prediction when compared to a prediction based solely on public information.

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As usual, all errors and omissions are my own.

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”Wisdom and knowledge shall be the stability of thy times.”

- Isaiah 33:6

1 Introduction

This paper studies the issue of bank runs and their prevention, in a dynamic model with aggregate uncertainty where the information structure of the model is asymmetric. I consider an environment where public information about the state of the financial system exists, and is available to both agents and the bank. In addition to the public information, the bank also has access to private information about the state of the financial system. Within this environment, I examine whether the bank's access to private information yields a unique banking equilibrium where the demand deposit contract achieves more efficient risk-sharing that the contract implied by the public information alone. I also discuss the constraints that are placed upon the optimal contract by the asymmetric information structure. The model builds on the framework of Diamond and Dybvig (1983).

Diamond and Dybvig begin the seminal work by warning that while bank runs had successfully been avoided in the United States following the end of the Great Depression in 1930s, the state of Savings and Loan institutions in the early 1980s made the study of bank runs an important contemporary policy issue (Diamond and Dybvig, 1983). Their warnings were well-founded, as the 1980s and 1990s brought forth the failure of 745 Savings & Loan institutions in the United States, costing the U.S. taxpayer an estimated $124.6 billion (USD) (USGAO, 1996). While bank runs have largely been avoided since the end of the S&L Crisis, (which, admittedly, was not too long ago,) history has begun to repeat itself with yet another systemic financial crisis. Beginning in 2007, the Subprime Mortgage Crisis has seen bank runs on Northern Rock in the U.K., Washington Mutual in the U.S., as well as a run on the securities of investment bank Bear Sterns. In light of these current events, I believe that the study of bank runs has become a current policy issue once again.

In this paper, I introduce an asymmetric information structure that favours the bank into the Diamond and Dybvig (1983) (herein DD (1983),) framework. Furthermore, I take a different approach to modeling aggregate uncertainty, one which amalgamates a number of extensions on their work and introduces elements of stochastic control theory to the banking panic literature. The main feature of the model is that the measure of impatient agents is considered to be stochastic, and modeled by a Markov chain. Naturally, this implies a second feature of the model, that the Diamond and Dybvig framework is considered to be dynamic, where the Markov chain evolves inter-game but not intra-game. This approach makes the supposition that while the measure of impatient agents may be stochastic, it depends on past history.

The model considers a single, socially-benevolent bank that is primarily concerned with the stability of the bank, and secondly, the welfare of its depositors. I make the assumption that the bank has additional private information about the state of the financial system, but their information set is not complete. Agents and banks only have access to an observation about the state of the financial system in the previous period. The observation is noisy, public information which can be said to come from a public survey on 'impatient agent status'. Sampling and truth-telling errors contribute to the noise in the survey's observation. Privately, however, the bank has knowledge of the process that determines the observation and the state of the financial system, (the Markov chain,) but not the current state of the financial system itself.

The model produces three important results. Firstly, I find that there exists a demand deposit contract that yields a banking equilibrium that is unique. Second, the aforementioned contract, which utilizes the bank's prediction of x_t, yields fewer 'pseudo-bank-runs'[1] than the contract implied by the agents' limited public information alone. Lastly, while there exists a unique banking equilibrium, if the prediction of x_t by the bank is considerably higher than the noisy observation implies, the bank is motivated to issue a less efficient contract. Based on the responses of the agents to the most efficient contract, the bank will opt to issue a contract with a lower, less optimal rate of return for impatient agents.

This paper brings together two areas of research from two different sub-disciplines: the study of banking panics as a subset of Monetary Economics, and Kalman Filtering, a technique for estimation of partially-observed Markov Decision processes, which is a product of Control Theory. Therefore, although the paper is written with a monetary focus, a brief discussion will be included on the development of Kalman Filtering and its fields of usage.

Within the literature of banking panics, DD (1983) is often referred to as the seminal work, and this paper will not differ from that general sentiment. I will, however, begin by noting the work of Bryant (1980) as a predecessor to DD (1983) (as Green and Lin (2000) do), and its contribution to the discussion of panics. Bryant finds, in his expansion of the Samuelson (1958) pure consumption-loans model, that the liquidity disparity between the assets and liabilities of the bank can yield a bank run as a possible equilibrium. While Bryant makes mention of government-backed

[1] I define a 'pseudo-bank-run' as the occasion where the bank's estimate of the measure of impatient agents is inaccurate enough that patient agents who withdraw in subperiod 2 receive nothing. Thus, they are affected in the same way as they would have been, had there been a bank run in the first subperiod.

deposit insurance as a mechanism to eliminate the undesirable outcome, the topic is not treated in detail. Instead, Bryant suggests the need for a reserve requirement as a method for preventing bank runs.

DD (1983) consider a similar asymmetric information problem of unobservable risk among agents, with the added dimension of aggregate uncertainty. Without aggregate uncertainty, DD show that while a bank run is a possible equilibrium of their post-deposit game, the full-information optimal risk-sharing banking equilibrium is the unique outcome of the pre-deposit game. Under these conditions, they also cite the policy of suspension of convertibility as a method for preventing bank-runs in a post-deposit game situation. Furthermore, under aggregate uncertainty, DD show that government-backed deposit insurance can yield the optimal risk-sharing equilibrium. It is this powerful result that has arguably established the reputation of DD (1983) as a defining work in the bank run literature.

Wallace (1988) lays a counterclaim to the result of DD (1983) that government-backed deposit insurance can prevent a bank run *ex-ante*. Wallace argues that the treatment of the bank and the government with respect to the "sequential service constraint" is uneven –that banks are subject to the constraint, while the government is not. He provides a re-evaluation of deposit insurance as a policy for eliminating the bank run equilibrium under aggregate uncertainty and finds that the result in DD (1983) does not hold when governments are also subject to the sequential service constraint.

As an alternative to the DD (1983) model, Green and Lin (2000, 2003) consider a finite-trader version of the seminal work with aggregate uncertainty that satisfies Wallace (1988), and, in addition, permits the bank to issue a broader range of contracts[2], which include those contingent on the bank's withdrawal history at the time of each withdrawal. Green and Lin show that the model has no bank-run equilibrium, which leads them to conclude that the bank-run equilibrium present in DD (1983) may be merely a product of the model's environment[3]. Furthering the result discrepancy between DD (1983) and Green and Lin (2000, 2003), Peck and Shell (2007) expand on the Green-Lin modification of the DD model to show that a bank-run equilibrium in the post-deposit game exists even when the banks are permitted to issue a broader range of contracts. The result also holds for suspension regimes that may be contingent on the withdrawal history at the time of each withdrawal request.

Extending the work of DD (1983) in a different direction, Temzelides (1997) studies a multiple-bank repeated-deposit-game environment to analyze the contagion effects of a single bank failure on the financial system. Temzelides introduces a number of features into the DD (1983) model, notably repeating the deposit game. In this paper, I will use the repeated deposit game framework to define an evolution process for the measure of impatient agents in each period. Unlike Temzelides, however, I will limit my discussion to pure strategy equilibria and disregard the feature that the agents' strategies evolve based on the outcome of the game each period. I leave those extensions to future research.

The information structure of the model studied in this paper is such that the bank has access to more information than agents, and that the agent's information set is nested in the bank's information set. By defining the information structure in this way, the state and observation processes of the Markov chain form a (Partially-

[2] Green and Lin (2000, 2003), as well as other papers in this area, refer to these contracts as 'mechanisms'. The two terms are used synonymously throughout this paper.
[3] In Andolfatto, Nosal, Wallace (2007) the work of Green and Lin (2000, 2003) is further generalized.

Observed) Markov Decision Process (POMDP), a coupling of state equations of the form, $x_t = f(x_{t-1}, w_{t-1})$, and observation equations of the form, $y_{t-1} = g(x_{t-1}, v_{t-1})$. The system of equations model the evolution of the state of the system, and a noisy observation process. POMDPs are widely studied in the Engineering Sciences as such models have numerous applications in the area of computer networking. I will note here, however, that POMDPs studied in the Engineering Sciences are commonly considered to be controlled POMDPs, that is, the decision-maker has the ability to control the system, usually at some cost. With a control included, the study of these POMDPs often falls into the area of Control Theory. To predict the true value of the state of the system from a POMDP, Rudolph E. Kalman developed a recursive solution technique popularly known as the 'Kalman Filter', which estimates the state of the POMDP by way of minimizing the mean-squared error (Kalman, 1960). The use of Kalman Filtering has been widely popular in the area of navigation (Bishop and Welch, 2006). Defining the information asymmetry in this way, the problem resembles a POMDP, (albeit an uncontrollable one,) permitting the use of Kalman Filtering to provide a more accurate description of the true state of the system than the noisy observation indicates.

The paper is organized into the following sections: section (2) describes the model environment, and solves the model for the unique subgame perfect Nash equilibrium. Section (3) describes 'pseudo-bank-runs' in detail and how the bank can prevent them. Constraints imposed on the bank's demand deposit contract by the information structure of the model are also discussed. The closing remarks in section (4) provide suggestions for future research.

2 The Model

In this section, I outline and examine a version of the DD (1983) model, which, as was noted briefly in the previous section, closely resembles the repeated version of the DD (1983) model that was studied in Temzelides (1997). The section begins with a description of the model environment, followed by the Nash equilibrium solutions of the post-deposit game, and the unique subgame perfect Nash equilibrium solution to the pre-deposit game.

2.1 Environment

Below, a description of the environment is provided in detail.

Time: The DD game is repeated over time periods indexed as $t \in \mathbb{Z}_+$, where the total of such periods are infinite. Each time period is divided into three subperiods, denoted $\{t_s \mid s \in \{0,1,2\}\}$. These three subperiods reflect the three components of the standard DD game, in which agents decide their strategies for; investment at $s=0$; early withdrawal at $s=1$; and late withdrawal at $s=2$.

Population: The number of agents that make up the economy is assumed to be constant, large and discrete. The total number of agents is denoted by $N=\{1,2,\ldots,n\}$ and the number of patient agents by $M=\{1,2,\ldots,m\}$. By measure, the fraction of impatient agents is denoted by $x=\dfrac{m}{n}$. Furthermore, the measure of the population that is impatient is stochastic, but Markov. The evolution process of x_t is defined by the following Markov chain,

$$x_t = (1 + 1_{\{w_{t-1} \le 0\}} w_{t-1}) x_{t-1} + (1 - x_{t-1}) \cdot 1_{\{w_{t-1} \ge 0\}} w_{t-1}$$

where w_{t-1} is a zero-mean Gaussian error.

Endowment: Each agent is endowed with one unit of goods that is costlessly storable across all subperiods, but not across time periods. At the end of subperiod 2, the good is either consumed or perishes.

Technology: In addition to the opportunity of costlessly storing their endowment across subperiods, agents have the opportunity to invest in a productive technology in subperiod 0 that produces the following returns across subperiods 1 and 2,

	$s=0$	$s=1$	$s=2$
withdrawal in $s=1$	-1	0	$R > \dfrac{1}{\rho}$
withdrawal in $s=2$		1	0

Where $\rho \in (1/R, 1)$ is the discount factor on goods held in the second subperiod. As the return schedule above stipulates, an investment of an agent's entire endowment can produce *either* one subperiod 1 good, or $R > 1$ subperiod 2 goods. Thus, as the agents can do no worse with the productive technology in comparison to the storage technology, storage is strategically dominated by the productive technology in subperiod 0. However, investment in the productive technology can only begin at subperiod 0, and thus patient agents may still elect to use the storage technology if they choose to withdraw in subperiod 1, as they would wish to consume that good in subperiod 2.

Preferences: All agents, i, have identical ex-ante preferences represented by the following utility function,

$$u(c^i) = \log(c^i);\ u(\bullet) \text{ is } C^2,\ -\frac{c \cdot u''(c)}{u'(c)} > 1,\ \forall c$$

The superscripts and subscripts on the consumption variable, c_{st}^i, refer to the following: i, the agent's type; s, the subperiod of consumption, and; t, the period of consumption. Each agent is subject to a preference shock in subperiod 1 that dictates the agents type from the type set, $\Theta = \{\theta_1, \theta_2\}$. Agents that are of type θ_1 receive utility only from consuming in subperiod 1, and thus are deemed *impatient*, whereas, agents that are of type θ_2 receive utility only from consuming in subperiod 2 and thus are considered *patient.* Furthermore, there exists no mechanism through which an agent can learn the outcome of the shock in subperiod 0. Formally, an agent's preferences are represented by,

$$U(c_1, c_2; \theta) = \begin{Bmatrix} u(c_1^1) & \text{if } \theta = \theta_1 \\ \rho u(c_2^2) & \text{if } \theta = \theta_2 \end{Bmatrix}$$

where $u(\bullet)$ is bounded by the conditions given above.

Isolation: Agents are isolated from each other in subperiod 1, for all time periods, *t.* I make this assumption to preclude the creation of asset markets amongst the agents in subperiod 1 (Wallace, 1988). Temzelides argues that the validity of this assumption stems from the needs of agents to hold liquid assets so as to consume when access to asset markets may be difficult.

Banks: The economy consists of a single bank that is "reborn" every period. That is, there is no record-keeping of transactions from previous periods, nor do the outcomes of previous periods carry over (such as debt resulting from a bank-run outcome.) The bank issues demand deposit contracts at each time stage in subperiod zero to all agents who deposit in the bank. The terms of the contract are as follows:

$$\begin{array}{ccc} s=0 & s=1 & s=2 \\ -1 & E[r(x_t)] & 0 \\ & 0 & f(E[r(x_t)]) \end{array}$$

$$\text{where } 1 < E[r(x_t)] < R,\ \ f(E[r(x_t)]) \in [0, R)$$

That is, the rates of return of the contract depend on the expectation of the measure of impatient agents at time *t*. To maintain the social benevolence of the bank, as in the DD model, I set two goals towards which the bank will strive to achieve: solvency, and welfare maximization of its depositors. Here, solvency is a binary condition, toward which the bank always seeks to be able to pay all agents the amount pre-specified by the demand deposit contract issued in subperiod zero. The bank also seeks to maximize the welfare of the agents in the economy, which implies posting a demand deposit contract such that the bank remains solvent, thereby satisfying the bank's primary goal. This holds from the fact that the welfare of agents in a bank-run equilibrium (even under a contract that would yield the optimal risk-sharing allocation,) is strictly dominated by a demand deposit contract which maximizes welfare given a stable banking equilibrium that may not yield the optimal risk-sharing allocation.

Lastly, the bank serves individuals who withdraw sequentially, on a first-come-first-served basis.

Asymmetric Information: Agents and the bank have access to a noisy observation, y_{t-1}, that describes the proportion of impatient agents in period $t-1$. The observation can be thought of as the product of a survey on the number of impatient agents in the economy, which has a natural noise term comprised of truth-telling and sampling errors. Despite both agents and the bank having access to the observation, only the bank knows the process that determines the measure of impatient agents in the current period, x_t, and the noisy observation, y_{t-1}, herein lying the information asymmetry. Together, both the x_t and y_{t-1} form a Markov decision process that is described as,

$$x_t = (1 + 1_{\{w_{t-1} \le 0\}} w_{t-1}) x_{t-1} + (1 - x_{t-1}) \cdot 1_{\{w_{t-1} \ge 0\}} w_{t-1}$$
$$y_{t-1} = (1 - 1_{\{v_{t-1} \le 0\}} v_{t-1}) x_{t-1} + (1 - x_{t-1}) \cdot 1_{\{v_{t-1} \ge 0\}} v_{t-1}$$

where $x_t \in (0,1)$ represents the fraction of impatient agents in the current period; y_{t-1}, the observation of the previous period's impatient investors; v_{t-1}, w_{t-1}, are zero-mean Gaussian noise terms.

The disparity in information access allows the bank to make a more precise prediction of x_t, while agents are limited to the following prediction,

$$\mathrm{E}[x_t \mid y_{t-1}] = \mathrm{E}[y_t \mid y_{t-1}] = y_{t-1}$$

Furthermore, agents are aware that the bank has access to additional private information; however, they do not know the details of the information.

Timeline:

The timeline of period, t, is set out below, outlining the events that occur within each subperiod. The timeline for each period is identical.

- Subperiod zero
  - Agents receive their endowments
  - Agents and the bank observe the noisy observation of the number of impatient agents from the previous time period.

- The bank posts a demand deposit contract based on the observation, its rational expectation of the agents' belief function, and its prediction of the measure of impatient agents in the current period.

- Subperiod one
  - Agents realize their types from the type set, $\Theta = \{\theta_1, \theta_2\}$.
  - Type θ_1 agents withdraw
  - Type θ_2 agents choose whether to withdraw or wait until subperiod two.
- Subperiod two
  - Type θ_2 agents who still have claims on deposits withdraw from the bank.

2.2 The Bank's Contract Design Problem

In subperiod zero, after observing y_{t-1}, the bank must construct a demand deposit contract that stipulates the rates paid to those who withdraw in each of subperiods 1 and 2. As it is the bank's objective to maximize the welfare of all agents who deposit in the bank, the bank faces the following contract design problem,

$$W(x_t) = \max_{c_{1t}^1, c_{2t}^1, c_{1t}^2, c_{2t}^2, z_t} \mathrm{E}\left[x_t \cdot u(c_{1t}^1) + (1 - x_t) \cdot \rho u(c_{2t}^2) \right]$$

$$s.t \qquad z_t \geq x_t \cdot c_{1t}^1 + (1 - x_t) \cdot c_{1t}^2 \tag{1}$$

$$\left(1 - z_t\right) \cdot R \geq x_t \cdot c_{2t}^1 + (1 - x_t) \cdot c_{2t}^2 \tag{2}$$

where z_t represents the number of productive technology liquidations by the bank in subperiod one. The above contract design problem must also satisfy the following participation constraint,

$$\mathrm{E}\left[x_t \cdot u(c_{1t}^1) + (1 - x_t) \cdot \rho u(c_{1t}^2 + c_{2t}^2) \right] \geq \mathrm{E}\left[x_t \right] \cdot u(1) + (1 - \mathrm{E}\left[x_t \right]) \cdot \rho u(R) \tag{3}$$

In this subsection, I determine the conditions required to design the optimal demand deposit contract by solving the welfare-maximization problem defined above. To solve the problem, we begin by taking first-order conditions with respect to the choice-variable set $\left\{c_{1t}^1, c_{2t}^1, c_{1t}^2, c_{2t}^2\right\}$,

$$(c_{1t}^1):\ \mathrm{E}\left[x_t \cdot u'(c_{1t}^1) - \lambda_{1t} \cdot x_t\right] = 0 \Rightarrow \mathrm{E}\left[u'(c_{1t}^1)\right] = \lambda_{1t} \Rightarrow \lambda_{1t} > 0 \tag{4}$$

$$(c_{2t}^2):\ \mathrm{E}\left[(1-x_t) \cdot \rho u'(c_{2t}^2) - \lambda_{2t} \cdot (1-x_t)\right] = 0 \Rightarrow \rho \mathrm{E}\left[u'(c_{2t}^2)\right] = \lambda_{2t} \Rightarrow \lambda_{2t} > 0 \tag{5}$$

$$(c_{1t}^2):\ -\lambda_{1t} \cdot \mathrm{E}\left[x_t\right] < 0 \Rightarrow c_{1t}^2 = 0 \tag{6}$$

$$(c_{2t}^1):\ -\lambda_{2t} \cdot \mathrm{E}\left[1-x_t\right] < 0 \Rightarrow c_{2t}^1 = 0 \tag{7}$$

Equations (4) and (5) dictate that both constraints (1) and (2) must hold with equality. Moreover, by equations (6) and (7) we see that the contract design problem can be simplified to,

$$W(x_t) = \max_{c_{1t}^1, c_{2t}^2, z_t} \mathrm{E}[x_t \cdot u(c_{1t}^1) + (1-x_t) \cdot \rho u(c_{2t}^2)]$$

$$c_{1t}^1 = \frac{z_t}{x_t} \tag{8}$$

$$s.t \qquad c_{2t}^2 = \frac{(1-z_t) \cdot R}{1-x_t} \tag{9}$$

$$c_{st}^i \geq 0,\ \forall i, s, t \tag{10}$$

Next, we consider the necessary condition for the optimal choice of z_t, the number of liquidations in subperiod one. By inserting equations (8) and (9) into the contract design problem above and taking first-order conditions with respect to z_t, we obtain,

$$(z_t):\ \mathrm{E}\left[\frac{x_t \cdot u'(c_{1t}^1)}{x_t} + \left(1-x_t\right) \cdot \rho u'(c_{2t}^2) \cdot \frac{-R}{\left(1-x_t\right)}\right] = 0 \tag{11}$$

$$\Leftrightarrow \mathrm{E}\left[u'(c_{1t}^1)\right] = \rho R \cdot \mathrm{E}\left[u'(c_{2t}^2)\right]$$

The condition above therefore implies that $c_{1t}^{1} < c_{2t}^{2}$ must hold in the optimal demand deposit contract. Taking note of the aforementioned necessary condition, (which we will use momentarily to prove an important claim,) we add both the equality constraints together, simplifying the problem to a one-variable optimization problem in c_{1t}^{1}, which I redefine as r. The demand deposit contract stipulates that agents who withdraw in subperiod one can expect to receive the rate, r. Herein, I will refer to r as the 'demand deposit rate'. The contract design problem can now be represented as,

$$W(r) = \max_{r} \ \mathrm{E}\left[x_t \cdot u(r) + (1 - x_t) \cdot \rho u\left(\frac{(1 - x_t \cdot r) R}{1 - x_t} \right) \right] \tag{12}$$

I now state and prove the following claim about the solution to the contract design problem.

Claim 1: *The solution to the contract design problem satisfies the following conditions: i) The optimal contract is unique; ii)* $1 < c_{1t}^{1} < c_{2t}^{2} < R$ *; iii) The banking contract is preferred to autarky, and thus all agents deposit in the bank initially.*[4]

Proof: The proof of the above claim will be sectioned into two steps. First, by showing conditions i) and ii), and secondly, showing that conditions i) and ii) imply condition iii). I begin by combining the two equality constraints and generating from them the relation,

$$c_{1t}^{1} = \frac{1}{x_t} \cdot \left(1 - (1 - x_t) \cdot \frac{c_{2t}^{2}}{R} \right) \tag{13}$$

[4] We may also write condition iii) of claim 1 in terms of the participation constraint, namely, $\mathrm{E}[x_t \cdot u(c_{1t}^{1}) + (1 - x_t) \cdot \rho u(c_{1t}^{2} + c_{2t}^{2})] > \mathrm{E}[x_t] \cdot u(1) + (1 - \mathrm{E}[x_t]) \cdot \rho u(R)$.

Next, the term c_{1t}^{1} in equation (11) is replaced by the aforementioned relation, thereby generating the function,

$$f\left(c_{2t}^{2}\right)=\mathrm{E}\left[u'\left(\frac{1}{x_t}\cdot\left(1-\left(1-x_t\right)\cdot\frac{c_{2t}^{2}}{R}\right)\right)-\rho R\cdot u'\left(c_{2t}^{2}\right)\right]=0 \tag{14}$$

which, for a general $c\in\left[1,R\right]$, is defined as,

$$f\left(c\right)=\mathrm{E}\left[u'\left(\frac{1}{x_t}\cdot\left(1-\left(1-x_t\right)\cdot\frac{c}{R}\right)\right)-\rho R\cdot u'\left(c\right)\right]=0 \tag{15}$$

Now, to determine the function's behaviour about $c\in\left[1,R\right]$, we take the first-order condition of $f\left(c\right)$,

$$f'\left(c\right)=\mathrm{E}\left[u''\left(\frac{1}{x_t}\cdot\left(1-\left(1-x_t\right)\cdot\frac{c}{R}\right)\right)\cdot\frac{-\left(1-x_t\right)}{x_t R}-\rho R\cdot u''\left(c\right)\right]>0 \tag{16}$$

By condition (16), and the conditions imposed on $u\left(\bullet\right)$, it is clear that $f\left(c\right)$ is a continuous, monotonically increasing function in c, $\forall x_t\in\left(0,1\right)$. Equation (15) indicates that the optimal c^* satisfies the relation that $f\left(c^*\right)=0$. To show condition i) and ii), we first evaluate equation (15) at the point where $c_{1t}^{1}=c_{2t}^{2}=c$. This yields,

$$f\left(c\right)=\mathrm{E}\left[u'\left(c\right)-\rho R\cdot u'\left(c\right)\right] \tag{17}$$

$$\Leftrightarrow \mathrm{E}\left[u'\left(c\right)-\rho R\cdot u'\left(c\right)\right]=\mathrm{E}\left[u'\left(c\right)\left(1-\rho R\right)\right]<0$$

by the assumption that $\rho R>1$. Next, we evaluate equation (15) at the point $c=R$,

$$f\left(c\right)=\mathrm{E}\left[u'\left(1\right)-\rho R\cdot u'\left(R\right)\right]>0^{5} \tag{18}$$

By the intermediate value theorem and the functional properties of $f\left(c\right)$, the function in equation (17) must pass through $f\left(c\right)=0$ at a unique c^* such that it satisfies the condition $1<\frac{1}{x_t}\cdot\left(1-\left(1-x_t\right)\cdot\frac{c}{R}\right)<c<R$, which proves conditions i) and ii) of

[5] The proof for $u'\left(1\right)>\rho R\cdot u'\left(R\right)$ can be found in Diamond and Dybvig (1983), *JPE*, pp. 408.

claim 1. Furthermore, by the result that the contract produced by $\left(c_{1t}^{1}, c_{2t}^{2}\right)$ is uniquely optimal, it is preferred to autarky, and thus the participation constraint holds with strict inequality. •

The formulation of the contract design problem in equation (12) shows that the bank's decision problem in the pre-deposit game is merely a single-variable optimization problem in r. This simplification arises from the result that the participation constraint holds with inequality in the optimal demand deposit contract, and therefore all agents deposit in the bank in subperiod zero. Therefore, there is no need to consider the subgame where agents choose whether to deposit in the bank in subperiod zero. Before solving for the optimal demand deposit rate, however, the patient agent's withdrawal decision problem in subperiod one must be considered. The solution to this problem defines a representative best-response function to the bank's proposed demand deposit contract for all patient agents. Under rational expectations, banks expect this best-response function, and utilize it in solving for the welfare-maximizing r. The next subsection solves for the Nash equilibria of the patient agent's withdrawal problem in the post-deposit game.

2.3 The Post-Deposit Game

The agent's decision problem subgame has two components. First, given the agent has deposited in the bank, and taking the demand deposit rate as given, the agent chooses whether to withdraw in subperiod one, once their type is realized. Second, taking the demand deposit contract as given, the agent chooses whether to deposit in the bank. As the design of the optimal demand deposit contract guarantees in the previous subsection, all agents deposit in the bank in subperiod zero, so there is no need for a treatment of the subgame that considers the agents' deposit decision problem. Therefore, this subsection considers only whether agents, taking the bank's demand deposit rate as given, should withdraw in subperiod one, once their types are realized. In terms of notation, I define the withdrawal probabilities of agents of individual type θ_2 agents as ω, and the aggregate probability of withdrawal of type θ_2 agent in subperiod one as Ω. For completeness, I define the withdrawal probabilities of agents of individual type θ_1 agents as δ, and the aggregate probability of withdrawal of type θ_1 agent in subperiod one as Δ. I note, however, that because type θ_1 agents prefer to consume only in subperiod one, agents who realize that they are of type θ_1 withdraw in subperiod one with certainty, and thus $\delta = \Delta = 1$ always.

After observing their type, type θ_2 agents choose whether or not to withdraw in subperiod one, taking Ω as given. In choosing whether to withdraw early, type θ_2 agents formulate their withdrawal decisions based on what they observed from the noisy observation, y_{t-1}, in subperiod zero. With this information, type θ_2 agents believe that they will receive a positive payment in subperiod 2 if the following inequality is satisfied:

$$y_{t-1} + (1 - y_{t-1}) \cdot \Omega \leq \frac{1}{r} \tag{19}$$

$$\Rightarrow \Omega \leq \bar{\Omega} = \frac{\frac{1}{r} - y_{t-1}}{1 - y_{t-1}}$$

Condition (19) presents the agent with the following choice problem, where α represents the probability of receiving payment in the first subperiod for those who withdraw. For $\Omega > \bar{\Omega}$, $\alpha \in (0,1)$, whereas for $\Omega \leq \bar{\Omega}$, $\alpha = 1$, trivially.

$$W(r, y_{t-1}, \Omega) = \begin{cases} \max\limits_{\omega} \omega \cdot u(r) + (1-\omega) \cdot u\left(\dfrac{\left[1-(y_{t-1}+(1-y_{t-1})\cdot\Omega)\cdot r\right]\cdot R}{1-y_{t-1}-(1-y_{t-1})\cdot\Omega} \right) & \text{if } \Omega \leq \bar{\Omega} \\ \max\limits_{\omega} \omega \cdot \left[\alpha \cdot u(r) + (1-\alpha)\cdot u(0)\right] + (1-\omega)\cdot u(0) & \text{if } \Omega > \bar{\Omega} \end{cases}$$

which implies that if $\Omega > \bar{\Omega}$, then $\omega = 1, \forall r \in [1, R]$, as,

$$\alpha \cdot u(r) + (1-\alpha)\cdot u(0) > u(0),\ \forall r > 0,\ \alpha \in (0,1)$$

Taking the first-order condition of $W(\omega, r, y_{t-1}, \Omega \mid \Omega \leq \bar{\Omega})$ with respect to ω, we note the following condition for the optimal choice of ω,

$$\frac{\partial W(r, y_{t-1}, \Omega)}{\partial \omega} = u(r) - u\left(\frac{\left[1-(y_{t-1}+(1-y_{t-1})\cdot\Omega)\cdot r\right]\cdot R}{1-y_{t-1}-(1-y_{t-1})\cdot\Omega} \right) = 0 \tag{20}$$

which implies that the choice function for ω is distributed about the following equality,

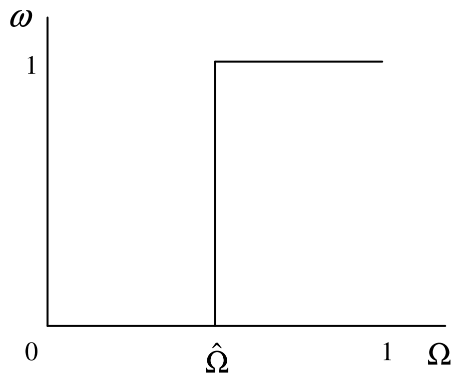
$$r = \frac{\left[1-(y_{t-1}+(1-y_{t-1})\cdot\Omega)\cdot r\right]\cdot R}{1-y_{t-1}-(1-y_{t-1})\cdot\Omega} \tag{21}$$

Solving for Ω, the distribution function, $\omega = f(\Omega)$ is described as,

$$\omega = \begin{cases} 0 & \Omega < \hat{\Omega} \\ \in [0,1] & \Omega = \hat{\Omega} \\ 1 & \Omega > \hat{\Omega} \end{cases} \tag{22}$$

$$\text{where } \hat{\Omega} = \frac{R - r\cdot(1-y_{t-1}+y_{t-1}\cdot R)}{(1-y_{t-1})\cdot r\cdot(R-1)}$$

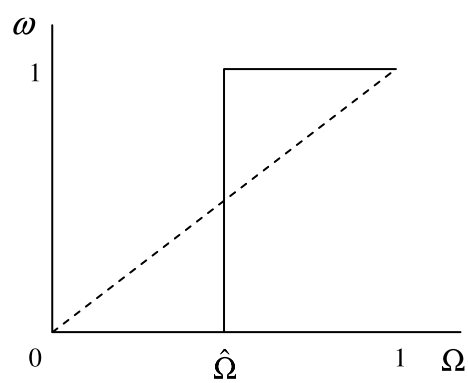
The $\hat{\Omega}$ term in the distribution function above indicates the point at which a patient agent is indifferent between withdrawing in subperiod one, and waiting until subperiod two. Graphically, equation (22) represented in $(\Omega, \omega) - space$ below:



Focusing only on symmetric equilibria, the symmetric Nash equilibrium of the post-deposit game is defined below.

Definition 1: *A Nash equilibrium of the post-deposit game consists of the set* $(\omega, \Omega, y_{t-1}, r)$ *such that: i) Given* (Ω, y_{t-1}, r), *the agent's choice of* ω *solves the agent's maximization problem; ii) All agents' choices of* ω *are symmetric, and thus,* $\omega = \Omega$.

Under these conditions, the game admits three symmetric Nash equilibria. Below, I outline these equilibria in detail by mapping the solutions to the agent's maximization problem in $(\Omega, \omega) - space$ [condition i) of the equilibrium definition] and note the points where condition ii) is satisfied.



As seen in the figure above, there exist three symmetric equilibria, namely,

- (0,0) – Stable banking pure strategy equilibrium
- $(\hat{\Omega}, \hat{\Omega})$ – Mixed strategy equilibrium
- (1,1) – Bank-run pure strategy equilibrium

In the next subsection, I determine the subgame perfect Nash equilibria of the pre-deposit game by evaluating the bank's decision problem at the two pure-strategy Nash equilibria of the post-deposit game. I leave the consideration of the mixed-strategy equilibrium to future research.

2.4 The Pre-Deposit Game

In subperiod zero, prior to agents realizing their types and choosing whether to withdraw in the first subperiod, the bank must post a banking contract so that agents may decide whether to deposit in the bank. From the contract design problem, the optimal contract design guarantees that all agents will deposit in the bank, and therefore the bank need only to choose the optimal demand deposit rate, r, such that the expected welfare of agents is maximized. Thus, the bank solves the following problem,

$$W(x_t, \rho, \Omega, y_{t-1}, R) = \max_r \mathrm{E}[(x_t + (1-x_t)\cdot\Omega)\cdot u(r) + \rho\cdot(1 - x_t - (1-x_t)\cdot\Omega)\cdot u(c_{2t}^2)]$$

$$\text{where, } c_{2t}^2 = \frac{(1-(x_t+(1-x_t)\cdot\Omega)\cdot r)\cdot R}{1-x_t-(1-x_t)\cdot\Omega}$$

The following two claims show the existence of two Nash equilibrium solutions to the pre-deposit game. Each solution corresponds to a pure strategy Nash equilibrium of the agent's post-deposit game.

Claim 2: *There exists a stable banking equilibrium such that: i) All agents will deposit in the bank at subperiod 0; ii)* $\omega = 0, \forall$ *agents st.* $\theta = \theta_2$, *at subperiod 1; iii)* $\delta = 1, \forall$ *agents s.t* $\theta = \theta_1$, *at subperiod 1.*

Proof: In the post-deposit game stable banking equilibrium, $\Omega = 0$. The problem then reduces to,

$$W(x_t, \rho, y_{t-1}, R) = \max_r \mathrm{E}[x_t \cdot u(r) + \rho \cdot (1 - x_t) \cdot u(c_{2t}^2)]$$

$$c_{2t}^2 = \frac{(1 - x_t \cdot r) \cdot R}{1 - x_t}$$

Note here that, because the participation constraint in the contract design problem holds with strict inequality, the optimal contract is always preferred by the agent to autarky, and thus at subperiod zero, all agents deposit in the bank. Furthermore, it is always optimal for agents of type θ_1 to withdraw in subperiod 1, so conditions i) and iii) of the equilibrium definition are already satisfied. I now proceed to show that ii) is also satisfied.

Taking first-order conditions with respect to r, we obtain,

$$\frac{\partial W(r; x_t, \rho, y_{t-1}, R)}{\partial r} = \mathrm{E}[x_t \cdot u'(r) + \rho \cdot (1 - x_t) \cdot u'(c_{2t}^2) \cdot c_{2t}^{2\prime}(r)] = 0 \tag{23}$$

$$\Leftrightarrow \mathrm{E}[x_t] \cdot u'(r) - \rho R \cdot \mathrm{E}[x_t \cdot u'(c_{2t}^2)] = 0$$

$$\Leftrightarrow \mathrm{E}[x_t] \cdot u'(r) = \rho R \cdot \mathrm{E}[x_t \cdot u'(c_{2t}^2)]$$

Equation (23) gives the condition for the optimum r^*.

Taking second-order conditions,

$$\frac{\partial^2 W(r; x_t, \rho, y_{t-1}, R)}{\partial r^2} = \mathrm{E}[x_t \cdot u''(r) + \rho \cdot (1 - x_t) \cdot u''(c_{2t}^2) \cdot (c_{2t}^{2\prime}(r))^2] < 0 \tag{24}$$

$$\Leftrightarrow \mathrm{E}[x_t \cdot u''(r) + \rho \cdot u''(c_{2t}^2) \cdot (x_t \cdot R)^2] < 0$$

$$\Leftrightarrow \mathrm{E}[x_t] \cdot u''(r) + \rho R^2 \cdot \mathrm{E}[x_t^2 \cdot u''(c_{2t}^2)] < 0$$

By the above conditions, it is clear that the welfare function is well-defined, continuous and concave. Given these properties, we proceed to show existence of an $r^* \in (1, 1/x_t)$ that solves equation (23). To do so, we evaluate $W'(r)$ over the set of points $\{0, 1, 1/x_t\}$, to show that $\partial W(1)/\partial r > 0$, and $\partial W(1/x_t)/\partial r < 0$. Then, by the intermediate value theorem, there exists an $r^* \in (1, 1/x_t)$ such that $\partial W(r^*)/\partial r = 0$.

Evaluating $W'(r)$ at the points $r = \{0, 1/x_t\}$,

$$W'(0) = \mathrm{E}\left[x_t \cdot u'(0) - \rho R \cdot x_t \cdot u'\left(\frac{R}{1-x_t}\right)\right] = \mathrm{E}\left[x_t \cdot \infty - \rho R \cdot x_t \cdot u'\left(\frac{R}{1-x_t}\right)\right] = \infty \tag{25}$$

$$W'\left(\frac{1}{x_t}\right) = \mathrm{E}\left[x_t \cdot u'\left(\frac{1}{x_t}\right) - \rho R \cdot x_t \cdot u'(0)\right] = \mathrm{E}\left[x_t \cdot u'\left(\frac{1}{x_t}\right) - \rho R \cdot x_t \cdot (\infty)\right] = -\infty \tag{26}$$

which implies that $W'(r^*) = 0$ lies within the interval $(0, 1/x_t)$. Now, by showing only that $W'(1) > 0$, the conditions for contract optimality will be satisfied.

Evaluating $W'(r)$ at $r = 1$,

$$W'(1) = \mathrm{E}[x_t] \cdot u'(1) - \rho R \cdot \mathrm{E}[x_t] \cdot u'(R)$$

$$\Leftrightarrow W'(1) = \mathrm{E}[x_t]\left(u'(1) - \rho R \cdot u'(R)\right) \equiv F(R) \tag{27}$$

To evaluate the behaviour of $F(R)$, we take the first-order condition of equation (27),

$$F'(R) = -\rho \cdot u'(R) - \rho R \cdot u''(R) \tag{28}$$

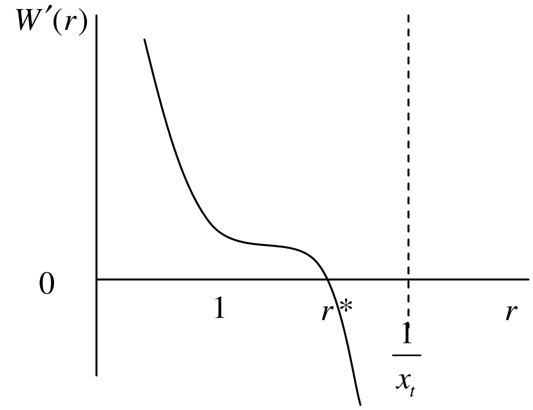
By my assumption that, $-c \cdot u''(c)/u'(c) > 1$, the following must be true about $u(R)$,

$$-R \cdot u''(R) > u'(R),$$

This implies that,

$$F'(R) > 0 \Rightarrow F(R) > F\left(\frac{1}{\rho}\right) > 0$$

By plotting the function $W'(r)$ in $(r, W'(r)) - space$ with the information obtained from the points $\{0, 1, 1/x_t\}$, it can be seen graphically that $\{W'(r^*) = 0\} \in (1, 1/x_t)$.



Therefore, by the intermediate value theorem, $W'(r)$ must pass through $W'(r) = 0$ for some $r \in (1, 1/x_t)$. This is the r^* that solves the bank's decision problem such that $\omega = 0, \forall\ \theta_2$ at subperiod 1, and therefore condition ii) of the equilibrium is satisfied. Thus, the stable banking equilibrium of the pre-deposit game exists. •

Claim 3: *There exists a bank run equilibrium such that: i) All agents will deposit in the bank at subperiod 0; ii)* $\omega = 1, \forall$ *agents,* $\theta \in \Theta$, *at subperiod 1.*

Proof: To prove the existence of the bank-run Nash equilibrium, we begin by again noting that condition i) is already satisfied, as the participation constraint still holds with ">". Therefore, all agents deposit in the bank initially. Given that $\Omega = 1$ in this equilibrium, a type θ_2 agent choosing whether or not to withdraw in subperiod one faces the following problem: to withdraw in subperiod 1, the value function for an agent, (which is based on their position in line, j,) is given by,

$$V_{1t}(r) = \begin{cases} r & \forall j \;\; s.t \;\; x_{tj} \cdot r \leq 1 \\ 0 & \forall j \;\; s.t \;\; x_{tj} \cdot r > 1 \end{cases}$$

whereas, to wait and withdraw in subperiod 2, $V_{2t}(r) = 0$. Clearly, $V_{1t}(r) \geq V_{2t}(r)$, $\forall j$, and so it is optimal for any type θ_2 who believes that $\Omega = 1$ to withdraw in subperiod 1. By this argument, conditions i-ii) are satisfied, and thus the bank-run Nash equilibrium of the pre-deposit game exists. •

Although both Nash equilibria of the bank's decision problem exist, as shown in the two previous claims, I argue that the stable banking equilibrium is the unique subgame perfect Nash equilibrium of the pre-deposit game. The bank run equilibrium ensures every agent only a risky return with a mean of one, a return which is dominated by autarky. Therefore, if one considers that the outcome of the pre-deposit game must coincide with the expectations of the agents depositing in the bank, the stable banking equilibrium outcome becomes an equilibrium outcome that is self-fulfilling, as no agent would deposit in the bank expecting a bank run, when they could improve on that contract by investing in the productive technology themselves.

3 The Stable Banking Equilibrium

In the previous section I determined that under the stable banking equilibrium of the post-deposit game there exists an optimal demand deposit rate, $r^* = \{1, 1/x_t\}$, such that there exists a stable banking equilibrium in the pre-deposit game, and moreover, is unique. In the subsections to follow, I analyze how the bank can use its knowledge of the Markov decision process to set the rate, r^*, such that it yields a demand deposit contract that is as close to achieving the optimal risk-sharing allocation as possible, and what effect this may have on patient agents in terms of the payment they receive upon their withdrawal.

3.1 ...and Pseudo-Bank-Runs

When the bank posts the deposit contract in subperiod 0, it consists of two rates: one for impatient agents, and one for patient agents. While the agents do not know at the time of depositing what type of agent they will be in subperiod one, they choose whether or not to deposit in the bank based on their expected utility of the posted contract. As discussed earlier, the bank expects this and sets the contract in such a way that the participation constraint holds with strict inequality. Furthermore, in the stable banking equilibrium, all patient agents are playing the pure strategy "withdraw in subperiod 2", as they deposit not anticipating a bank run. It is not always guaranteed, however, that they will receive the amount in subperiod 2 that is specified by the demand deposit contract posted by the bank in subperiod 0. The reason being that the value for x_t that yields the demand deposit contract rate is only a prediction for the true value of x_t. Therefore, in the case where the prediction is wildly incorrect, it is possible that the patient agent receives considerably less (or more,) than specified in the contract. In fact, there may be occasions where the agent receives nothing at all. The latter case I have earlier defined as a 'pseudo-bank-run'. To be more precise about this concept, I make the following definition:

Definition 2: *A stable banking equilibrium is called a 'pseudo-bank-run' if it satisfies the condition,* $1 - x_t \cdot r^* \geq 0$.

The event is defined in this way because while the bank is not actually run with any intent by the patient agents, any patient agents waiting until the second subperiod to withdraw would receive the same payment as they would under the bank-run equilibrium.[6]

In this subsection, I consider the frequency of these 'pseudo-bank-runs' by appealing to a numerical simulation of the model economy. For comparison, the frequency of occurrence of these pseudo-bank-runs is analyzed under two estimators of x_t: the prediction as produced by the bank's superior estimation technology; and the prediction generated solely by the noisy observation, y_{t-1}. To determine whether the superior estimating technology is an improvement on the noisy observation as predictor, I compare their performances towards minimizing the occurrence of pseudo-bank-runs. While the bank may use any number of estimating technologies to improve on the accuracy of the noisy observation as an estimator, this paper considers only the technique of Kalman Filtering. Before analyzing the numerical simulation, the mechanics of this technique are described in brief.

Recall the Markov chain that governs the evolution of x_t is defined by the system of equations,

$$
\begin{aligned}
x_t &= (1+1_{\{w_{t-1}\le 0\}} w_{t-1})x_{t-1} + (1-x_{t-1})\cdot 1_{\{w_{t-1}\ge 0\}} w_{t-1} \\
y_{t-1} &= (1-1_{\{v_{t-1}\le 0\}} v_{t-1})x_{t-1} + (1-x_{t-1})\cdot 1_{\{v_{t-1}\ge 0\}} v_{t-1}
\end{aligned}
$$

While the technique of Kalman filtering was created for use with controlled Markov chains[7], the technique is still able to improve upon the noisy observation as a predictor, as the procedure takes both the noisy observation and the Markov process into account when forming its prediction. The process begins with a guess as to the

[6] Our definition of a 'pseudo-bank-run' does not stray too far from the scenario considered in Carmona (2007). Carmona considers case where the number of actual early withdrawers is so large that the bank may not have enough resources to pay the entire true measure of impatient types who withdraw in subperiod one. As a result, all patient agents withdrawing in subperiod two would receive nothing.

[7] A controlled Markov chain is a Markov chain that can be acted upon by a controller in the current period to affect the evolution of the chain in the next period (usually at some cost.)

initial state, x_0. In this paper, the guess and the noisy observation, y_0, are set to equal the true initial state, x_0, for simplification purposes only. Given this guess, the bank sets its expectation of x_0 as a linear function of the observation, y_0, defined by,

$$\mathrm{E}[x_0 \mid y_0] = \alpha_0 y_0 + \beta_0 \tag{29}$$

and defines the error of it's guess as,

$$x_0 - \mathrm{E}[x_0 \mid y_0] = x_0 - \alpha_0 y_0 + \beta_0 \tag{30}$$

The goal of the bank is then to minimize the squared error of this guess by solving,

$$\min_{\alpha_0, \beta_0} \left(x_0 - \mathrm{E}[x_0 \mid y_0] \right)^2$$

which admits the solution,

$$(\alpha_0, \beta_0) = \left(\frac{\mathrm{E}\left[x_0^2\right]}{\mathrm{E}\left[v_0^2\right] + \mathrm{E}\left[x_0^2\right]}, 0 \right)$$

and so the expectation of x_0 simplifies to,

$$\mathrm{E}[x_0 \mid y_0] = \left(\frac{\mathrm{E}\left[x_0^2\right]}{\mathrm{E}\left[v_0^2\right] + \mathrm{E}\left[x_0^2\right]} \right) \cdot y_0 \tag{31}$$

With this, we can solve for the expected squared error, which is defined as,

$$\Sigma_0 := \mathrm{E}\left[\left(x_0 - \mathrm{E}[x_0 \mid y_0] \right)^2 \right] = \frac{\mathrm{E}\left[x_0^2\right] \cdot \mathrm{E}\left[v_0^2\right]}{\mathrm{E}\left[v_0^2\right] + \mathrm{E}\left[x_0^2\right]} \tag{32}$$

If the process is iterated for t periods, the following general formula can be obtained for $\mathrm{E}[x_t \mid I_{t-1}]$, (where I_{t-1} is all information available to the bank at period t-1,)

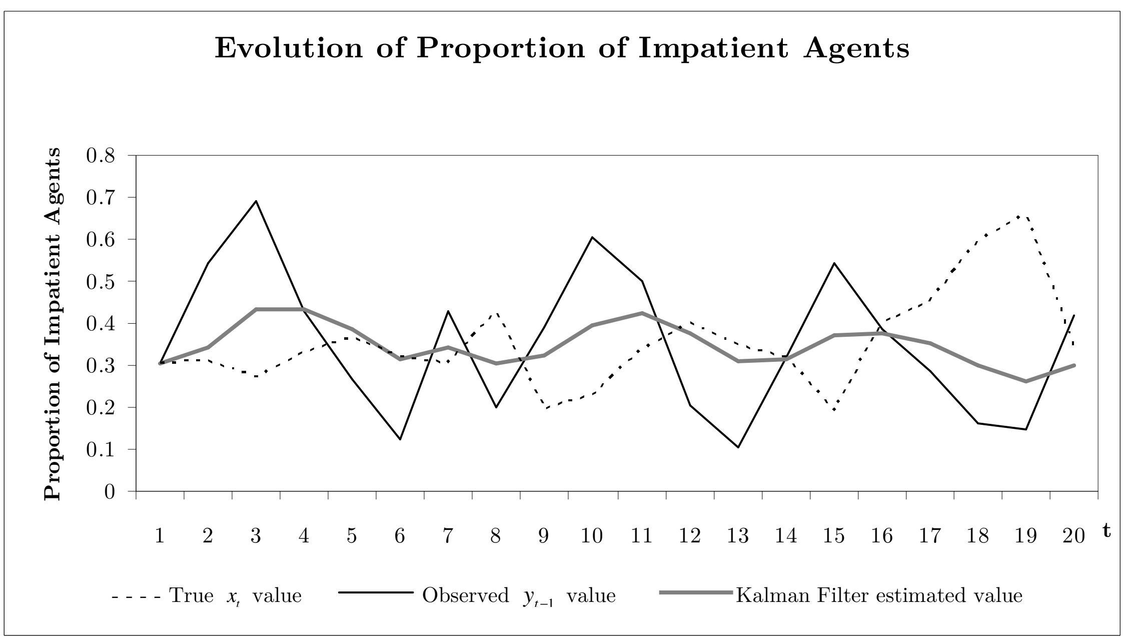
$$\mathrm{E}[x_t \mid y_{t-1}] = \mathrm{E}[x_{t-1} \mid y_{t-1}] = \mathrm{E}[x_{t-2} \mid y_{t-2}] + \alpha_{t-1} \cdot \left(y_{t-1} - \mathrm{E}[x_{t-2} \mid y_{t-2}] \right) \tag{33}$$

$$\alpha_{t-1} = \frac{\Sigma_{t-2} + E\left[w_{t-2}^2\right]}{\Sigma_{t-2} + E\left[w_{t-2}^2\right] + E\left[v_{t-1}^2\right]}$$

where,

$$\Sigma_{t-2} = \frac{\left(\Sigma_{t-3} + \mathrm{E}\left[w_{t-3}^2\right]\right)\left(\mathrm{E}\left[v_{t-2}^2\right]\right)}{\Sigma_{t-3} + \mathrm{E}\left[w_{t-3}^2\right] + \mathrm{E}\left[v_{t-2}^2\right]}$$

To show that using the Kalman Filter to predict the value of $\mathrm{E}\left[x_t \mid y_{t-1}\right]$ yields a more accurate prediction than simply using the information provided through the noisy observation, I simulate the partially-observed Markov decision process over 20 time periods, apply the Kalman filter technique, and compare it's results with those generated by the noisy observation. I set $\sigma_{v_{t-1}}^2 > \sigma_{w_{t-1}}^2$, thereby making the observation process more volatile than the process that governs the proportion of impatient agents each period. With these values, I compute the Kalman Filter as defined above, and the results for its prediction of the true x_t are compared to the true x_t and the agents' prediction (which is just the observation value, y_{t-1},) in the graph below;



It can be seen clearly from the previous figure that the Kalman Filter predictions are considerably closer to the true x_t value, and provides a consistently better predictor than the noisy observation alone. I now determine whether using the Kalman Filter prediction to obtain the optimal r^* yields fewer pseudo-bank-runs. Recall that for a pseudo-bank-run to occur, the discrepancy between the actual measure of impatient agents and the one predicted by the bank must be large enough to satisfy the condition, $1 - x_t \cdot r^* \geq 0$. The simulation finds that by using the noisy observation as a predictor, two pseudo-bank-runs occur in time periods 18 and 19. The Kalman Filter prediction, however, is closer to the true value of x_t in every time period, and the numerical analysis[8] shows that the Kalman Filter prediction does not yield any pseudo-bank-runs over the 20-period simulation. It should be noted, however, that the Kalman Filter prediction departs enough from the true value of x_t in period 18 that a patient agent withdrawing early in that period would have received more than waiting until subperiod 2, albeit still receiving more than their initial endowment. In time period 19, though, the Kalman Filter prediction not only avoids a pseudo-bank-run, but provides patient agents who wait until subperiod 2 with a rate greater than that which they would have received had they withdrawn in subperiod 1.[9] Thus, it is clear from this numerical exercise that the bank's prediction using the Kalman Filter generates a contract that avoids pseudo-bank-runs better than those generated by the noisy observation alone.

Before proceeding to the following subsection, it is important to note that the choice to use Kalman Filtering as the technique for predicting the measure of impa-

[8] We display the impatient withdrawal rates and the ex-post patient withdrawal rates for the periods 1-20 for both the Kalman Filter and the observation-implied deposit contracts in the Appendix.

[9] While our simulation does not admit a case where the contract produced by the Kalman Filter estimate yields a pseudo-bank-run, such an event is possible, if the noisy observation differs from the true state by a very large amount.

tient agents in the current period does not preclude the possibility other techniques may produce similar or better results.

3.2 ... and the Optimal Demand Deposit Rate

In the previous subsection, it was shown through a numerical exercise that the Kalman Filter estimator used by the bank to determine the expected measure of impatient agents in the current period is a relatively more efficient estimator of the true measure of impatient agents than the noisy observation, y_{t-1}. However, because of the information asymmetry in the model, there are occasions where the bank has an incentive to deviate from issuing the demand deposit rate implied by their prediction of x_t. As banks can expect $\hat{\Omega}(r)$, they are aware that $\hat{\Omega}(r)$ admits a bank-run equilibrium ex-ante for all demand deposit rates within the interval,

$$r \in \left(\frac{R}{1 - y_{t-1} + y_{t-1}R}, \frac{1}{x_t} \right]$$

Therefore, it is in the best interest of the bank to issue a demand deposit contract with a demand deposit rate equal to,

$$r^* = \frac{R}{1 - y_{t-1} + y_{t-1}R}$$

as this is the rate that will maximize the expected welfare of the agents while still admitting the stable banking equilibrium as an ex-ante Nash equilibrium of the game. As no agent will deposit in the bank expecting a bank run, the existence of a stable banking equilibrium makes the outcome self-fulfilling. Therefore, the private information possessed by the bank permits it to post a contract that will prevent the bank-run ex-ante (but not always pseudo-bank-runs, as noted in the previous section). It is clear, however, that the information asymmetry, and the inability of the bank to signal the credibility of their superior information, yields a sub-optimal outcome that is a less efficient risk-sharing allocation for all rates such that,

$$r^* \in \left\{ \left(\frac{R}{1 - y_{t-1} + y_{t-1}R}, \frac{1}{x_t} \right], \ s.t \ \frac{R}{1 - y_{t-1} + y_{t-1}R} < \frac{1}{x_t} \right\}$$

Thus, if one considers more patient agents as being a "good state" of the economy, it can be said that, based on their prediction of x_t, banks can optimally post contracts when the state of the economy is "poor". Yet, in extremely "good" states of the economy, agents are unlikely to believe these optimistic contracts, forcing the bank to post a contract that is suboptimal, based on their prediction of the state of the economy.

4 Conclusion

In this paper I depart from the classic DD (1983) framework by considering a measure of impatient agents that is stochastic over time, and modeled by a Markov chain. Further, the bank and agents have noisy public information about the past history of the measure, but only the bank has private information about the Markov process. The results of this paper generate three important conclusions. First, there exists a demand deposit contract which admits a unique banking equilibrium. Second, this contract yields fewer pseudo-bank-runs than the contract implied by the public information alone. Lastly, if public information about the measure of impatient agents is considerably lower than the bank's prediction, the bank is motivated to issue a less efficient demand deposit contract.

By introducing the Markov process into DD (1983), the model is open to a handful of extensions for future research, two of which I will note here. Firstly, and perhaps most simply, the model can be extended to a multiple bank economy, where the information sets differ amongst the banks. Secondly, in addition to the multiple bank extension, one might consider the feature of 'accumulated debt', whereby the balance of each bank carries over to the next period. With this feature, a bank run would result in the affected bank owing outstanding deposits to its patient agents in the next period. One could explore the desirability of inter-bank lending towards ensuring individual bank stability, and how the risk of financial contagion, as in Temzelides (1997), may be a factor in the desire for inter-bank information sharing.

This paper begins what I hope will be a wealth of adaptations of stochastic control theory to the banking panic literature, and to the seminal Diamond and Dybvig framework in particular. Further research in this area may be able to generate a clearer understanding of how those with larger information sets (commercial banks as compared to agents; central banks as compared to commercial banks) may choose to share private information so as to direct the stability of the banking system.

5 References

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6 Appendix

Below I display the impatient and patient rate tables generated by the Kalman Filter prediction as well as the noisy observation. r_K* refers to the demand deposit rate as determined using the Kalman Filter prediction. r_y* is determined using the noisy observation.

Kalman-Filter-Implied Rates

t	r_K*	$c_2^2 = \frac{(1-x_t \cdot r_K*)R}{(1-x_t)}$	t	r_K*	$c_2^2 = \frac{(1-x_t \cdot r_K*)R}{(1-x_t)}$
1	1.48	3.96	11	1.66	3.31
2	1.57	3.71	12	1.74	2.53
3	1.33	4.38	13	1.47	3.75
4	1.58	3.57	14	1.37	4.13
5	1.54	3.45	15	1.57	4.33
6	1.50	3.83	16	1.57	3.10
7	1.46	4.00	17	1.55	2.75
8	1.54	2.96	18	1.53	1.07
9	1.56	4.32	19	1.38	1.39
10	1.46	4.32	20	1.44	3.88

Observation-Implied Rates

t	r_y*	$c_2^2 = \frac{(1-x_t \cdot r_y*)R}{(1-x_t)}$	t	r_y*	$c_2^2 = \frac{(1-x_t \cdot r_y*)R}{(1-x_t)}$
1	1.54	3.83	11	1.33	4.16
2	1.28	4.38	12	1.66	2.79
3	1.17	4.68	13	1.84	2.77
4	1.4	4.01	14	1.53	3.75
5	1.57	3.36	15	1.28	4.67
6	1.79	3.15	16	1.44	3.54
7	1.4	4.13	17	1.55	2.75
8	1.66	2.50	18	1.74[10]	0.0
9	1.43	4.48	19	1.73[11]	0.0
10	1.24	4.65	20	1.42	3.93

[10] Note: At this rate, the bank is under a pseudo-bank-run and some impatient agents will not receive payment.