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Signally by Jump Bidding in Private Value Auctions

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Abstract

This paper examines how a bidder can benefit from jump bidding by using the jump bid as a signal of a high valuation which causes other bidders to drop out of the auction earlier than they would otherwise. The information contained in a jump bid must be sufficient to induce a discrete change in the bidding behaviour of the other bidders. In an auction for a single item, a jump bid signals both the identity and the high valuation of a bidder. The existence of a beneficial jump bid equilibrium requires a gap in the distribution of the jump bidder and her identity must be concealed. Concealing the identity of the bidders permits the jump bidder to signal more information through the jump bid and thus she can benefit more from it. In an auction for multiple items, the jump bid signals a high valuation by the jump bidder. This causes a discrete change in the bidding behaviour of the other bidder since it causes this bidder to reduce her demand. In both a one-object and multiple-object auctions, a seller may expect less revenue in a jump bid equilibrium than a non-jump bid equilibrium.

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1 Introduction

The effort to design an efficient auction mechanism for the distribution of spectrum rights in the United States and other countries has caused economic theorists to weigh the benefits and costs associated with several alternative auction formats.[1] Of particular importance was the choice of how much information to release to the auction participants during the course of the bidding. If bidders' valuations are affiliated, it is desirable for the seller to reveal as much information as possible so as to assist sensible bidding and reduce the winner's curse (Milgrom and Weber [9]). However, if there is a relatively small number of bidders, full information release may increase the possibility of bidder collusion or predatory bidding. To discourage such behaviour, the seller may choose to conceal the identities of the bidders.

The auctions for spectrum rights in the United States have undergone format changes reflecting the above concerns. For example, in the nationwide narrowband PCS auction the Federal Communications Commission (FCC) decided to conceal the identities of the bidders. After each round of bidding, the high bid on each licence was posted but not the name of the firm that made the bid—only the bidders' confidential bidder numbers were used. In subsequent auctions however, all information regarding bidder identities was revealed to the auction participants.

The choice of the amount of information to release seems to have had a dramatic impact on bidding behaviour. For instance, in the nationwide narrowband PCS auction there were a surprising number of 'jump bids'; that is, bids greater than the minimum amount necessary to raise the standing high bid. Cramton [4] recounts that of the 196 new high bids in the auction, 96 (49%) were jump bids. Jump bidding is in stark contrast to most theoretical formulations of auctions. Traditionally, models presume bidders bid the minimum increment above the previous high bid to avoid the risk of bidding more than is necessary to win the auction. Cramton [4] rationalizes jump bidding as follows:

> The basic idea is that the jump bid conveys information about a bidder's valuations. It is a message of strength, conveying that the bidder has a high value for the particular license. Moreover, it conveys this message in a credible way. Jump bidding has a cost—it exposes the bidder to the possibility of leaving money on the table. It is precisely this cost that makes the communication credible. A bidder with a low value would not find it in its

[1] See Chakravorti et al. [3], Cramton [4, 5], McAfee and McMillan [8].

best interest to make a large jump bid. The gain, increasing the chance of winning the license, would not exceed the cost, the risk of overbidding.

Bidding behaviour in the nationwide narrowband auction was also likely influenced by the presence of asymmetric bidders. For example, of the 25 bidders, Cramton [4] describes a few bidders as likely having high values because of their large market share, prior product development, or other advantages.

In this paper we use two different formulations, a one-object and a multiple-object model, to explore a bidder's rationale for jump bidding. In both cases a bidder jump bids to signal that she has a high valuation which causes a discrete change in the bidding behaviour of the other bidders. In order for a jump bid to be effective, it must be credible and convey sufficient information to be profitable to the jump bidder. In the one-object model, the jump bid signals both the identity of the bidder as well as the fact that she has a high valuation. The existence of a jump bid equilibrium requires certain restrictions on the distribution of the jump bidder. In particular, a gap must be present in her distribution and it must be stochastically dominated by her rivals' distributions over low valuations. Furthermore, the seller must conceal the identities of the bidders. With some information concealed, the jump bid reveals more information and thus the jump bidder can benefit more from it.

In the multiple-object model the conditions required for a jump bid equilibrium are quite different although the motivation is the same as in the one-object model. In this case, the existence of a jump bid equilibrium requires the jump bidder to demand fewer items than her rival and her distribution must be strictly concave. By jump bidding, a bidder induces her rival to strategically reduce her demand for the objects she would like to win so that she pays lower prices for the objects she does in fact win.

In the one-object case, there are two types of bidders that draw their valuations from a different distribution. This reflects the fact that some bidders are likely to value the object more than other bidders perhaps because of differences in the number and type of other objects that they presently own. For example, firms that own cellular licences in regions contiguous to areas where PCS licences are being auctioned, may place high values on the PCS licences because they could provide them with room to expand their mobile telephone network.

In the multiple-object case, there are also two types of bidders that draw their valuations from a different distribution. However, they also have different demands for the objects up for bid. Certain bidders may wish to win all the objects up for bid while other bidders may desire only a few.

A couple of recent papers have also explored the issue of jump bidding. Daniel and Hirschleifer [6] examine a model where the bidders bid in a sequence and it is costly to submit a bid or bid revision. They focus on an equilibrium in which the first bidder either passes or reveals her valuation by jumping immediately to a bid whose level is an increasing function of her valuation. Each successive bidder quits if her valuation is less than that revealed by an earlier bidder, or she jump bids to a substantially higher bid that reveals sufficient information to force all preceding bidders to quit. In contrast to traditional bidding models, large jumps occur at every stage of the bidding even in the limit as bid costs go to zero. Furthermore, bidders with low valuations will sometime postpone making a bid in order to learn more about their rivals.

Avery [2] examines the role of jump bidding in a model of common value. He solves for equilibria of sequential bid auctions when jump bidding strategies may be employed to intimidate one's opponents. In these equilibria, jump bids serve as correlating devices that select asymmetric equilibria to be played sequentially. Bidders benefit from jump bidding compared to the symmetric equilibrium of a sealed bid, second-price auction.

One of our results confirms that of Daniel and Hirschleifer in the sense that when bidders are modelled as drawing their valuations from a continuous probability distribution and bidding costs are very small, bidders do not benefit from making jump bids and the seller does not suffer a loss in revenue. Daniel and Hirschleifer show that jump bidding can benefit a bidder only when there are positive bidding costs. In a model with no bidding costs, we show that jump bidding can benefit a bidder and reduce the seller's revenue when there is a discontinuity in buyers' distributions or if bidders differ in the number of objects they would like to win.

The paper is structured as follows. In section 2 we show that if all bidders are modelled as drawing their valuations for a single object from a continuous probability distribution, then beneficial jump bid equilibria cannot exist. This result motivates our development of a new theoretical formulation in section 3 where bidders are assumed to be asymmetric. In section 5 we extend our framework to include an auction for multiple objects. Section 6 provides a few concluding remarks. The proofs of all results are relegated to a technical appendix.

2 Preliminaries

In this section we define the different strategies that bidders may use when bidding. Using these strategies we show that the traditional theoretical formulation of a one-object auction with bidders having continuous distributions fails to explain the phenomenon of jump bidding.

Standard auction theory predicts that in an English open-outcry auction where bidders possess independent private values, each bidder continues to submit bids in the smallest allowable increment until her valution is reached at which point she drops out of the auction. Given this bidding behaviour, the winner of the auction is the bidder with the highest valuation and she pays an amount equal to the second highest valuation.[2] This outcome depends critically on the absence of bidding costs. With no bidding costs, a bidder always has an incentive to cast another bid although she may know with certainty that a rival has a higher valuation and therefore, will win the object. In this paper, although we do not introduce bidding costs explicitly, we assume that if a bidder expects a non-positive payoff from continuing to bid, she immediately drops out of the auction. This assumption is motivated by the notion that it takes time and energy for a bidder to continue in an auction and it is likely that she will choose to drop out if she realizes that she cannot win with positive probability.

For simplicity, we model an auction as a two stage process where in the first stage, bidders may make jump bids. In the second stage, we model the auction as a continuous English auction starting with the highest bid in the first stage.

If a bidder makes a jump bid in the first stage, then she is said to be using a *Jump Bidding Strategy* (JBS). These come in two flavors; a *Separating Jump Bidding Strategy* and a *Pooling Jump Bidding Strategy*.

Definition 1. A **Separating Jump Bidding Strategy** (SJBS) consists of a function $B(v)$ (where $B'(v) > 0$) such that if the bidder's valuation is equal to v then she jump bids to $B(v)$ in the first stage. A **Pooling Jump Bidding Strategy** (PJBS) consists of a pair (a, b) such that if the bidder's valuation is greater than or equal to b then she jump bids to a in the first stage. Conversely, if her valuation is less than b then she does not jump bid in the first stage.

In this paper, we only consider strategies in the form of Definition 1 and assume that only one bidder intends to use it. This may appear to be

[2] Daniel and Hirschleifer [6] refer to this as the *ratchet* solution.

a serious restriction on a bidder's choice of actions, however, we will show subsequently that the jump bid equilibrium in the strategy space outlined above is a separating equilibrium so that even if firms are permitted to jump bid later in the auction they will choose not to do so as no information will be contained in these jumps. Therefore, our restriction on jump bids occurring only in the first stage will be shown to be inconsequential for the equilibrium we derive. It is important to note, however, that other equilibria may exist in strategies with jump bids later in the auction, but due to the complexity of the analysis we do not attempt to characterize them.

Using the strategies defined above, we next show that no *beneficial* jump bid equilibria exist when the bidders are modelled as drawing their valuations from a continuous probability distribution. By the term beneficial, we are referring to equilibria where a bidder expects to receive a greater expected payoff from jump bidding than by not jump bidding. We include this analysis to show that conventional auction models without bidding costs cannot explain the phenomena of jump bidding.

Consider n bidders, $n \geq 2$, who wish to buy a single object that is up for bid. The bidders all have independent private values which are drawn from the continuous distribution $F(v)$.[3] For simplicity, we set the seller's valuation for the object to zero so that she is willing to accept any positive bid.

Proposition 1. *Given $n \geq 2$ bidders with independent private values drawn from a continuous distribution, a bidder cannot achieve a strictly greater expected payoff by jump bidding than by not jump bidding.*

Since with independent private values a jump bid cannot change a bidder's probability of winning—the object will always go to the bidder with the highest valuation—a bidder can benefit from jump bidding only if she expects to pay less by jump bidding than by not jump bidding. If a bidder does not jump bid and wins, she expects to pay the next largest valuation conditional on her own being the largest. However, a bidder cannot credibly jump bid to an amount less than this because such a bid would be mimicked by another bidder with a slightly smaller valuation than her own. Therefore, a separating jump bid equilibrium does not exist.

A similar argument can be used to show that a pooling equilibrium does not exist. The expected gain from signalling information through the jump bid is exactly offset by the expected cost of bidding more than necessary to win the object.

[3]The results easily extend to the case of bidders having different distributions each of which is continuous.

These results are reaffirmed by Daniel and Hirschleifer [6]. They show that if there are bidding costs then an equilibrium exists where a bidder receives a strictly greater payoff from making a jump bid. However, as the bidding costs approach zero, the payoff from jump bidding approaches zero.

Since most of the results in auction theory are derived from models with continuous distributions, these models cannot explain the phenomena of jump bidding without introducing bidding costs.

Given the anlaysis above, it is apparent that in order to explain the phenomena of jump bidding and determine its impact on seller revenue we need to develop an alternative theoretical framework. With this in mind we now turn to section 3.

3 The One-Object Case

3.1 The Model

There is one seller who wishes to sell a single non-divisable object. We assume that the seller's valuation for the object is zero which implies she is willing to accept any strictly positive bid. The seller faces $n+1$ bidders; one of them we refer to as the *special* bidder and the others we call *ordinary* bidders. The special bidder draws her valuation from the distribution $G(w)$ with a support over two non-overlapping intervals, $[\underline{v}, \tilde{v}]$ and $[\tilde{v}+\tau, \overline{v}]$ where $\tau > 0$. Each ordinary bidder draws her valuation independently from the continuously differentiable distribution $F(v)$ with the support $[\underline{v}, \overline{v}]$.

The two interval support for the special bidder's distribution can be motivated in a variety of ways. For example, one can suppose that ordinary bidders may be aware that the special bidder owns an object that is similar to the object up for bid but do not know whether the objects are complements or substitutes. If they are complements, then it is likely that the special bidder has a large valuation, $w \in [\tilde{v}+\tau, \overline{v}]$. Conversely, if they are substitutes then the special bidder is more likely to have a low valuation, $w \in [\underline{v}, \tilde{v}]$.

An example of this possibility is a firm's valuation for a particular spectrum license. If the special bidder already owns a spectrum licence in a similar frequency band and geographic area as the license up for bid, she may value an additional license quite highly if this license complements her existing one by bringing much needed capacity to her business. On the other hand, the special bidder's existing license may provide sufficient spectrum capacity for her needs. In this case, the license up for bid is a substitute for the one the special bidder already owns, hence she may place a small value on acquiring an additional license. Continuing with this example, the special bidder may already own a cellular license in the same geographic area as a PCS license. At the time of the MTA broadband auction in the US, it was unclear whether cellular and PCS technology were complements or substitutes.[4]

Another reason behind a non-overlapping two interval support arises from the uncertainty that ordinary bidders may have concerning the special bidder's previous R&D efforts. If the special bidder has successfully developed some valuable new technology it is likely that she will place a high value on the object.[5] Conversely, if the special bidder's R&D efforts have been unsuccessful then she will likely have a low valuation.

3.2 Equilibrium

In this section we show, using this simple model, that jump bidding can benefit a special bidder with a high valuation if bidder identities are concealed.

Consider a special bidder with a valuation $w \in [\tilde{v}+\tau, \overline{v}]$. Suppose she bids α in the opening round where $\alpha < w$, and this bid signals that her valuation is greater than or equal to $\tilde{v}+\tau$. Observing this α, all ordinary bidders with valuations less than or equal to $\tilde{v}+\tau$ drop out of the auction—their probability of winning the auction is equal to zero. If the largest ordinary bidder valuation is greater than $\tilde{v}+\tau$, then the special bidder upon winning the auction, expects to pay the largest ordinary bidder valuation. Therefore, if a special bidder with valuation w jump bids, her expected payoff is given by

$$F^n(\tilde{v}+\tau)(w-\alpha) + \int_{\tilde{v}+\tau}^{w} (w-\xi)\, dF^n(\xi).$$

If the special bidder chooses not to jump bid, her expected payoff depends on the amount of information bidders are given about their rivals. First, consider the scenario of complete information release where bidders are informed as to the identities of the other bidders as well as the number that drop out as the bidding proceedes. If the largest ordinary bidder valuation is less than or equal to $\tilde{v}$ or is greater than or equal to $\tilde{v}+\tau$, then the special bidder expects to pay the largest ordinary bidder valuation conditional on her own being the largest. If the largest ordinary bidder valuation lies in the interval $[\tilde{v}, \tilde{v}+\tau]$, then all ordinary bidders will drop out when the

[4] See Chakravorti et al. [3].

[5] In the case of radio spectrum, the development of a new digital compression technique could dramatically increase the value of a spectrum license.

bidding reaches $\tilde{v}$ since at this point they must infer that the special bidder's valuation is greater than or equal to $\tilde{v}+\tau$. In this instance, the special bidder pays an amount $\tilde{v}$ in return for the object. Given this reasoning, the special bidder's expected payoff under full information release is given by

$$\int_{\underline{v}}^{\tilde{v}} (w-\xi)\, dF^n(\xi) + \int_{\tilde{v}+\tau}^{w} (w-\xi)\, dF^n(\xi) + [F^n(\tilde{v}+\tau) - F^n(\tilde{v})](w-\tilde{v}).$$

Next, consider the expected payoff to the special bidder when bidder identities are concealed but bidders are still able to observe when other bidders drop out of the auction. We refer to this case as that of partial information release. In this instance, the special bidder upon winning the auction is most likely to pay an amount equal to the largest ordinary bidder valuation conditional on her own being the largest. However, she can pay less than this amount if *all* the ordinary bidders have valuations within the interval $[\tilde{v}, \tilde{v}+\tau]$. Given this event, when the bidding reaches the point $\tilde{v}$ every ordinary bidder must infer that the special bidder's valuation is greater than or equal to $\tilde{v}+\tau$ and therefore, they will all drop out. Given this reasoning, the special bidder's expected profit under partial information release is given by

$$\int_{\underline{v}}^{w} (w-\xi)\, dF^n(\xi) + \int_{\tilde{v}}^{\tilde{v}+\tau} (\xi-\tilde{v})\, d[F(\xi) - F(\tilde{v})]^n.$$

Finally, consider the case where bidders do not know the identities of their rivals or whether other bidders have dropped out of the auction. In this case of no information release, a bidder does not know at any particular moment how many rivals she is currently competing with. This implies that the special bidder, if she wins the auction, expects to pay an amount equal to the largest ordinary bidder valuation conditional on her own valuation being the largest among all the bidders. Therefore, the special bidder's expected profit under no information release is given by

$$\int_{\underline{v}}^{w} (w-\xi)\, dF^n(\xi).$$

Since the special bidder's expected payoff when jump bidding is strictly decreasing in the amount of the jump bid, there exists a unique jump bid for each information structure, where she is indifferent between jump bidding and not jump bidding. Setting the special bidder's expected profit from jump bidding equal to that from not jump bidding and solving for the jump bid where she is indifferent between these two strategies gives

$$\alpha^{\mathrm{f}} = \tilde{v} - \frac{\int_{\underline{v}}^{\tilde{v}} (\tilde{v} - \xi)\, dF^n(\xi)}{F^n(\tilde{v} + \tau)},$$

$$\alpha^{\mathrm{p}} = \tilde{v} + \tau - \frac{\int_{\underline{v}}^{\tilde{v}+\tau} F^n(\xi)\, d\xi + \int_{\tilde{v}}^{\tilde{v}+\tau} (\xi - \tilde{v})\, d[F(\xi) - F(\tilde{v})]^n}{F^n(\tilde{v} + \tau)},$$

$$\alpha^{\mathrm{n}} = \tilde{v} + \tau - \frac{\int_{\underline{v}}^{\tilde{v}+\tau} F^n(\xi)\, d\xi}{F^n(\tilde{v} + \tau)},$$

where 'f', refers to full information release, 'p', for partial information release and 'n', for no information release. Note that all three expressions α^{f}, α^{p} and α^{n} do not depend on the precise valuation of the special bidder—it only matters that it is in the interval $[\tilde{v} + \tau, \overline{v}]$. The reason for this is that the payoffs from jump bidding and not jump bidding only differ in the event that the largest ordinary bidder's valuation lies below $\tilde{v} + \tau$. Hence, only the probability below $\tilde{v} + \tau$ matters.

The relationship among the three expressions, α^{f}, α^{p} and α^{n} is characterized in the following proposition.

Proposition 2. *The maximum jump bid a special bidder with a valuation in the interval $[\tilde{v}+\tau, \overline{v}]$ is willing to make in the case of no information release is greater than or equal to that in the case of partial information release which in turn is greater than or equal to that in the case of full information release. That is, $\alpha^n \geq \alpha^p \geq \alpha^f$.*

This result arises solely from the differences in the special bidder's expected payoffs under the various information structures when she does not jump bid. These differences in turn, are caused by what the largest valuation ordinary bidder with a valuation in the interval $[\tilde{v}, \tilde{v} + \tau]$ is able to learn during the course of the bidding. In the case of no information release, each ordinary bidder participates in the auction until the bidding reaches the level of her valuation and then she drops out. However, in the cases of partial information release and full information release, ordinary bidders might drop out before their valuations are reached. With full information release, the ordinary bidder with the largest valuation will drop out of the auction when the bidding reaches $\tilde{v}$. With partial information release, the ordinary bidder with the largest valuation will drop out of the auction only if *all* bidders are active when the bidding reaches $\tilde{v}$. Clearly, if there is a possibility that bidders drop out of the auction before their valuations are reached, this benefits the special bidder since in the event of winning the auction she will not have to pay an amount equal to the valuation of the highest ordinary bidder. Therefore, the special bidder's expected payoff is greatest when all information is released, and it is smallest when no information is released. The case of partial information release lies between these two extremes. This ranking of expected payoffs dictates the ranking of maximum jump bids that the special bidder is willing to make. For example, since the special bidder expects the highest expected payoff when all information is released, she is unwilling to make a large jump bid in this case.

In order to demonstrate that a jump bid equilibrium exists, we must show that an ordinary bidder and a low valuation special bidder will not mimic the jump bid of a high valuation special bidder. To show this we require the following technical conditions on $G(w)$ and $F(v)$ to hold

Condition 1.

$$\int_{\underline{v}}^{\tilde{v}+\tau} \left[\frac{F(\xi)}{F(\tilde{v}+\tau)}\right]^{n-1} \left[\frac{G(\xi)}{G(\tilde{v}+\tau)} - \frac{F(\xi)}{F(\tilde{v}+\tau)}\right] d\xi > \int_{\tilde{v}}^{\tilde{v}+\tau} (\xi - \tilde{v})\, d\left[\frac{F(\xi)-F(\tilde{v})}{F(\tilde{v}+\tau)}\right]^{n},$$

and

Condition 2.

$$\int_{\underline{v}}^{\tilde{v}+\tau} \left[\frac{F(\xi)}{F(\tilde{v}+\tau)}\right]^{n-1} \left[\frac{G(\xi)}{G(\tilde{v}+\tau)} - \frac{F(\xi)}{F(\tilde{v}+\tau)}\right] d\xi > 0.$$

Note that Condition 2 is a necessary condition for Condition 1. Basically, both conditions state that for valuations below $\tilde{v}+\tau$, the special bidder's distribution, $G(w)$, is first-order stochastically dominated by the ordinary bidders' distribution $F(v)$. This accords well with the complement/substitute justification given earlier for the split distribution of the special bidder. For example, if the special bidder owns an object which is a substitute for the one up for bid, she may value owning another one for strategic purposes or as a backup for the object she already owns.[6] It is likely that if a bidder values the object for these reasons, her valuation is less than a bidder that requires the object to undertake her core business.

[6] If the objects are spectrum licenses a firm may want to acquire an additional license in order to exclude competitors.

The importance of Conditions 1 and 2 for the subsequent analysis will be made clear below.

The following proposition characterizes equilibria where a high valuation special bidder jump bids in the opening round in order to signal information to the other bidders.

Proposition 3. *Given a special bidder with a valuation, $w \in [\tilde{v}+\tau, \overline{v}]$, a jump bid equilibrium does not exist when all information is released. In the case of partial information release, any $\alpha \in [\overline{\beta}, \alpha^p]$ can be used to construct a jump bid equilibrium where*

$$\overline{\beta} = \tilde{v} + \tau - \frac{\int_{\underline{v}}^{\tilde{v}+\tau} G(\xi) F^{n-1}(\xi)\, d\xi}{G(\tilde{v}) F^{n-1}(\tilde{v}+\tau)},$$

and Condition 1 holds. In the case of no information release, any $\alpha \in [\overline{\beta}, \alpha^n)$ can be used to construct a jump bid equilibrium if Condition 2 holds.

If all information is released, the special bidder does not benefit from jump bidding. In order to prevent a special bidder with a low valuation from mimicking the jump bid, the jump bid must be set so large that a high valuation special bidder earns the same expected profit as she would by not jump bidding. This result is similar in spirit to that of section 2.

If there is partial information release, then there is a range of possible jump bids that constitute equilibria. Since all jump bids within the interval $[\overline{\beta}, \alpha^{\mathrm{p}})$ signal exactly the same information, it seems appropriate for the special bidder to select the jump bid that is least costly (i.e., $\overline{\beta}$). Since a jump bid of α^{p} gives the special bidder the same expected payoff as that of not jump bidding, it's clear that she benefits from jump bidding for jump bids strictly less than α^{p}.

If no information is released, the range of possible jump bids that constitute equilibria is $[\overline{\beta}, \alpha^{\mathrm{n}})$. This range is potentially larger than that associated with partial information release since $\alpha^{\mathrm{n}} \leq \alpha^{\mathrm{p}}$. Again, there is one jump bid that is least costly for the special bidder and it is identical to that of the partial information case (i.e., $\overline{\beta}$).

The set of jump bid equilibria in the cases of no and partial information release, exist only if conditions 1 and 2 hold. To understand the need for these conditions, consider the case of no information release and the special bidder and the ordinary bidder with the largest valuation both have valuations in the interval $[\tilde{v}+\tau, \overline{v}]$. If the special bidder wins the auction without jump bidding, she will pay an amount equal to the largest ordinary bidder's valuation. If the largest ordinary bidder wins, she will pay the larger of the second largest ordinary bidder's valuation or the special bidder's valuation. Condition 2 is a sufficient condition for the largest ordinary bidder to expect a higher payoff than the special bidder since it is likely that the ordinary bidder's valuation is smaller than that of the second highest ordinary bidder (i.e., $F(v)$ stochastically dominates $G(w)$ for valuations below $\tilde{v} + \tau$). This deters the highest valuation ordinary bidder from making large jump bids (i.e., $\overline{\beta} < \alpha^{\mathrm{n}}$).

In the case of partial information release, the condition needs to be strengthened to Condition 1. This is because the expected payoff to the special bidder from not jump bidding with partial information release in increased relative to the scenario of no information release. The special bidder's expected payoff from not jump bidding increases because there is a small probability that she will pay less than the largest ordinary bidder valuation if she wins the auction. Recall, this will only happen if all ordinary bidder valuations lie in the interval $[\tilde{v}, \tilde{v} + \tau]$. Because the expected payoff to the special bidder from not jump bidding is increased relative to the no information release scenario, the amount she is willing to jump bid decreases. To ensure that an ordinary bidder does not have an incentive to mimic the special bidder's jump bid, $F(v)$ must stochastically dominate $G(w)$ by a sufficient amount (i.e., Condition 1 must hold).

The following corollory addresses the issue of the seller's expected revenue.

Corollory 1. The seller expects less revenue if jump bid equilibria exist than if jump bid equilibria do not exist.

Regardless of the bidding strategy adopted by the special bidder, the object is always acquired by the bidder with the largest valuation. Therefore, jump bidding does not effect auction efficiency but only the amount that the special bidder pays in the event she wins the auction. Given the state where the special bidder's valuation is within the interval $[\tilde{v} + \tau, \overline{v}]$ and the largest ordinary bidder's valuation is within the interval $[\tilde{v}, \tilde{v} + \tau]$, the special bidder will receive a strictly greater payoff from jump bidding than by not jump bidding. In all other states the special bidder receives the same expected payoff from jump bidding as from not jump bidding. Since all states occur with positive probability, it follows that the expected revenue to the seller must clearly decrease when jump bid equilibria exist.

4 A Numerical Example

To make our ideas more concrete we offer the following numerical example.

Suppose that the ordinary bidders' valuations are uniformly distributed over the unit interval while the special bidder's distribution is given by

$$G(w) = \begin{cases} \sqrt{w} & \text{for } 0 \leq w \leq \tilde{v} < 1, \\ 1 & \text{for } w = 1. \end{cases}$$

If the special bidder has a high valuation (i.e., $w = 1$), she may wish to signal this by means of a jump bid if either no information or only partial information concerning the bidders is released. If in a jump bid equilibrium the ordinary bidders observe a jump bid, then they immediately drop out of the auction since they each have a zero probability of winning the object.

Given this setup, (11) (see the Appendix) holds if and only if $\mathcal{H}(\tilde{v}) \geq 0$ where,

$$\begin{aligned} \mathcal{H}(\tilde{v}) &= \frac{\int_0^{\tilde{v}} \xi^{n-1}\sqrt{\xi} + \int_{\tilde{v}}^1 \xi^{n-1}\sqrt{\tilde{v}}\, d\xi}{\sqrt{\tilde{v}}} - \left[\int_0^1 \xi^n \, d\xi + \int_{\tilde{v}}^1 (\xi - \tilde{v})\, d(\xi - \tilde{v})^n\right], \\ &= \frac{1}{n} - \left[\frac{1}{n} - \frac{1}{n+1/2}\right]\tilde{v}^n - \frac{1}{n+1} - \frac{n}{n+1}(1-\tilde{v})^{n+1}, \\ &= \frac{1}{n(n+1)} - \frac{\tilde{v}^n}{n(2n+1)} - \frac{n(1-\tilde{v})^{n+1}}{n+1}. \end{aligned}$$

It can be shown that $\mathcal{H}(\tilde{v})$ is strictly concave with $\mathcal{H}(0) < 0$ and $\mathcal{H}(1) > 1$. This implies that there exists a value, v^*, where for all $\tilde{v} > v^*$, $\mathcal{H} > 0$ and (11) holds.

5 The Multiple-Object Case

5.1 The Model

In this section, jump bidding is examined in the context of an auction for multiple objects. To keep the analysis as simple as possible we focus on the case where there are two identical objects for sale. As before, we set the seller's valuation for each object to zero which implies the seller is willing to accept any strictly positive bid.

There is one bidder, who we call the *special* bidder, who wishes to purchase only one of the objects. This bidder draws her valuation from the distribution function $G(w)$ with a support over the unit interval. We assume that $G(w)$ is strictly concave which implies that the density $g(w)$ is strictly decreasing. In loose terms, this means that the special bidder is more likely to have a low valuation for an object rather than a high one.

There is also one bidder, who we call the *ordinary* bidder, who wishes to purchase both objects up for bid. This bidder draws her valuation from the distribution $F(v)$ with a support over the unit interval.[7]

In a simultaneous auction bidders are confronted with a difficult strategic decision. If a bidder bids for all the objects she would like to win, she runs the risk of driving up prices on the objects she does in fact end up winning. Ausebel and Cramton [1] document this inefficiency in multi-unit auctions in several different settings. The problem confronting bidders who want more than one object, is knowing when to stop bidding for several objects and settle for fewer objects instead. Weber [10] provides anecdotal evidence of firms strategically reducing their demands in the FCC's MTA broadband auction.

Consider the ordinary bidder who wants both objects. Define the function $s(v)$ as the point where an ordinary bidder with valuation v will stop bidding for both objects and settle for one instead. At an interior solution, this function is implicitly defined by

$$v - s = 2\left[\frac{G(v) - G(s)}{1 - G(s)}\right]\left[v - \frac{\int_s^v \xi \, dG(\xi)}{G(v) - G(s)}\right], \tag{1}$$

where the LHS is her payoff from stopping at the bid level s to claim one item and the RHS is her expected payoff from continuing to bid for both objects.

Lemma 1. The function $s(v)$ is increasing over the interval $(\tilde{v}, 1]$ where $\tilde{v}$ is defined as $\tilde{v}/2 = \int_0^{\tilde{v}} G(\xi)\, d\xi$. Over the interval $[0, \tilde{v}]$ the function $s(v)$ is equal to zero and $s(1) = 1$.

Note that because $G(w)$ is strictly concave, $\tilde{v}$ decreases as $G''(w)$ decreases for all $w \in [0, 1]$.

5.2 Equilibrium

We are looking for an equilibrium in the form of a couple (b^*, w^*) where a special bidder with a valuation equal to or greater than w^* jump bids to an amount b^*. If the special bidder's valuation is less than w^* then instead of jump bidding, she bids in small increments.

Consider the case where the special bidder has a valuation less than w^* (and therefore, does not jump bid) and the ordinary bidder has a valuation

[7] The fact that both bidder's valuations are drawn from distributions with the same support is a useful simplification. Different supports could be incorporated into the analysis without changing the fundamental results.

greater than or equal to w^*. In this case, if the bidding reaches a level s then the ordinary bidder is indifferent between continuing to bid for two objects or settling for one instead if the following condition holds

$$v - s = 2\left[v - \frac{\int_s^{w^*} \xi \, dG(\xi)}{G(w^*) - G(s)}\right].$$

The LHS is her payoff if she quits bidding immediately and settles for one object while the RHS is her expected payoff from continuing to bid for both objects. Upon re-arrangement this condition becomes

$$\frac{v+s}{2} = \frac{\int_s^{w^*} \xi \, dG(\xi)}{G(w^*) - G(s)}.$$

Given that $G(w)$ is strictly concave and $v > w^*$, it is clear by inspection that the LHS is always greater than the RHS for all $s, w^* \in [0, 1]$. This shows that an ordinary bidder with a valuation equal to or greater than w^* will always continue to bid for both objects if she infers that the special bidder's valuation is less than w^*.

Now consider the case where both the ordinary and special bidder's valuations are less than w^*. Define the function $\bar{s}(v; w^*)$ as the point where the ordinary bidder will stop bidding for both objects and settle for one instead. At an interior solution, the function $\bar{s}(v; w^*)$ is implicitly defined by

$$v - s = 2\left[\frac{G(v) - G(s)}{G(w^*) - G(s)}\right]\left[v - \frac{\int_s^{v} \xi \, dG(\xi)}{G(v) - G(s)}\right]. \tag{2}$$

Lemma 2. The function $\bar{s}(v; w^*)$ is increasing over the interval $(\bar{v}(w^*), w^*]$ where $\bar{v}(w^*)$ is defined by $\bar{v}(w^*)G(w^*)/2 = \int_0^{\bar{v}(w^*)} G(\xi) \, d\xi$. The function $\bar{s}(v; w^*)$ is equal to zero over the interval $[0, \bar{v}(w^*)]$ and $\bar{s}(v; w^*) = w^*$.

Note that by inspection of (1) and (2), it is evident that $\bar{s}(v; w^*) \geq s(v)$ for all $v \in [0, w^*]$.

Now consider the case where both bidders have valuations greater than w^*. Upon observing the special bidder's jump bid, the ordinary bidder can immediately drop out of the auction and receive one object for a payoff of v, or she can continue bidding for both objects from the point w^*. Clearly, if $s(v) \leq w^*$ then the ordinary bidder will settle for one object. However, if $s(v) > w^*$ she may or may not settle for one object. An ordinary bidder with valuation v (and $s(v) > w^*$) is indifferent between continuing to bid and quitting if the following holds

$$v = 2\left[\frac{\int_{w^*}^{s(v)} (v-\xi)\, dG(\xi)}{1-G(w^*)}\right] + \left[\frac{1-G(s(v))}{1-G(w^*)}\right][v - s(v)].$$

This condition can be re-written using the implicit definition of $s(v)$ from (1) as

$$v = 2\left[\frac{\int_{w^*}^{v} (v-\xi)\, dG(\xi)}{1-G(w^*)}\right]. \tag{3}$$

For a given w^* let $\hat{v}(w^*)$ denote the ordinary bidder with the largest valuation that would settle for one object rather than continue to bid for both.

Lemma 3. The function $\hat{v}(w^*)$ is increasing over the interval $(0, \tilde{w}^*]$ where $\tilde{w}^*$ is defined by $\int_{\tilde{w}^*}^{1} \xi\, dG(\xi)/[1 - G(\tilde{w}^*)] = 1/2$. Moreover, $\hat{v}(0) = \tilde{v}$ and $\hat{v}(\tilde{w}^*) = 1$.

Note that because $G(w)$ is strictly concave and continuous, $\tilde{w}^*$ must be strictly greater than zero but less than 1/2. Also, as $G''(w)$ decreases for all $w \in [0,1]$, $\tilde{w}^*$ increases.

Given the equilibria we have described, a special bidder with valuation w^* must be indifferent between jump bidding and not jump bidding. This condition is given by

$$(w^* - b^*)F(\hat{v}(w^*)) = \int_0^{w^*} (w^* - \bar{s}(v;\xi))\, dF(\xi),$$

where $\hat{v}(w^*)$ is determined by (3). This condition can be re-written as

$$b^* = \left[1 - \frac{F(w^*)}{F(\hat{v}(w^*))}\right] w^* + \left[\frac{F(w^*)}{F(\hat{v}(w^*))}\right]\left[\frac{\int_0^{w^*} \bar{s}(v;\xi)\, dF(\xi)}{F(w^*)}\right]. \tag{4}$$

Note that the jump bid, b^*, is a weighted average of w^* and the expected value of $\bar{s}(v; w^*)$ given that the ordinary bidder's valuation is less than w^*. This serves to demonstrate that the more information that is signaled by way of the jump bid (i.e., the larger the difference between w^* and $\hat{v}(w^*)$), the closer that b^* is to w^*. All (w^*, b^*) pairs satisfying the above relationship constitute valid jump bid equilibria.

5.3 Expected Revenue

In this section we examine whether the expected revenue of the seller decreases when jump bid equilibria exist.

Consider a special bidder with a valuation below w^*. Her expected payoff in a non-jump bid equilibrium is given by

$$\int_0^w (w - s(\xi))\, dF(\xi), \tag{5}$$

while in a jump bid equilibrium it is given by

$$\int_0^w (w - \bar{s}(v;\xi))\, dF(\xi). \tag{6}$$

Taking the difference between (5) and (6) gives

$$\int_0^w (\bar{s}(v;\xi) - s(\xi))\, dF(\xi) \equiv \Psi. \tag{7}$$

If in a jump bid equilibrium the special bidder has a low valuation, then she does not jump bid which signals this unfavourable information to the ordinary bidder which makes the ordinary bidder bid more aggressively. More aggressive bidding by the ordinary bidder aids economic efficiency since it makes it more likely that the ordinary bidder will win both objects when in fact she values each one more than does the special bidder. A gain in efficiency benefits the seller since she expects to sell the objects at a greater price.

Next, consider a special bidder with a valuation greater than or equal to w^*. In a non-jump bid equilibrium her expected payoff is given by (5). In a jump bid equilibrium it is given by

$$(w - b^*)F(\hat{v}(w^*)) + \int_{\hat{v}(w^*)}^{w} (w - s(\xi))\, dF(\xi). \tag{8}$$

Taking the difference between (8) and (5), the special bidder receives a greater expected payoff from jump bidding if

$$b^* < \frac{\int_0^{\hat{v}(w^*)} s(\xi)\, dF(\xi)}{F(\hat{v}(w^*))}.$$

This states that a special bidder benefits from jump bidding for jump bids less than the expected value of $s(v)$ conditional on the ordinary bidder's valuation less than $\hat{v}(w^*)$. Note that for the range of v above $\hat{v}(w^*)$ there is no difference in the expected payoffs from jump bidding or not.

Substituting for b^* from (4) and rearranging terms gives

$$\frac{\int_0^{\hat{v}(w^*)} s(\xi)\, dF(\xi)}{F(\hat{v}(w^*))} - \left[1 - \frac{F(w^*)}{F(\hat{v}(w^*))}\right] w^* \\ - \left[\frac{F(w^*)}{F(\hat{v}(w^*))}\right]\left[\frac{\int_0^{w^*} \bar{s}(v;\xi)\, dF(\xi)}{F(w^*)}\right] \equiv \Theta. \quad (9)$$

If $\Theta > 0$ then a special bidder with a valuation equal to or greater than w^* expects to profit from jump bidding.

To determine if the special bidder prefers a jump bid equilibrium over a non-jump bid equilibrium we need to consider her *ex ante* expected payoff—her expected payoff before she knows her valuation. A jump bid equilibrium is preferred to a non-jump bid equilibrium if

$$\Pr\{w \geq w^*\}\mathcal{E}\{\Theta | w \geq w^*\} > \Pr\{w < w^*\}\mathcal{E}\{\Psi | w < w^*\},$$

or upon substitution

$$[1 - G(w^*)]\Theta > \int_0^{w^*} \int_0^{w} (\bar{s}(v;\xi) - s(\xi))\, dF(\xi)\, dG(w). \quad (10)$$

Given the generality of the model, (10) may or may not hold in practice. It will depend on the particular functional forms of $G(w)$ and $F(v)$.

Given the zero-sum nature of the auction, if the special bidder prefers a jump bid equilibrium then the seller expects less revenue. The expected gain to the seller from the aggressive bidding of the ordinary bidder in the case the special bidder's valuation being less than w^*, is exceeded by the expected loss if the special bidder's valuation is greater than w^*. Note that if the special bidder expects to benefit in a jump bid equilibrium, the efficiency of the auction is also reduced. This is because it is more likely that each bidder will win one object despite the ordinary bidder valuing each object more than the special bidder.

5.4 A Numerical Example

To aid understanding, we offer the following numerical example. The special bidder draws her valuation from the distribution $G(w) = 2w - w^2$ which is clearly strictly concave. The ordinary bidder draws her valuation from the distribution $F(v) = v^\alpha$ where $\alpha \geq 1$.

Given the properties of $G(w)$, we have

$$s(v) = \begin{cases} 0 & \text{for } 0 \le v \le \tilde{v}, \\ v(1+\sqrt{3}) - \sqrt{3} & \text{for } \tilde{v} < v \le 1, \end{cases}$$

and

$$\bar{s}(v; w^*) = \begin{cases} 0 & \text{for } 0 \le v \le \bar{v}(w^*), \\ v - \sqrt{3(2(w^* - v) - (w^{*2} - v^2))} & \text{for } \bar{v}(w^*) < v \le w^*, \end{cases}$$

where $\tilde{v} = \sqrt{3}/(1+\sqrt{3}) \approx 0.634$ and $\bar{v}(w^*) = 3/2 - (\sqrt{3}/2)\sqrt{2w^{*2} - 4w^* + 3}$.

For ease of analysis, we choose to focus on the jump bid equilibrium where $w^* = \tilde{w}^*$ so that $\hat{v}(w^*) = 1$. For this example, $\tilde{w}^* = 1/4$.

Consider first the case where $\alpha = 1$ so that $F(v)$ is uniform. For this parameter value the jump bid associated with $w^* = 1/4$ is $b^* \approx 0.189$. However, the special bidder is worse off *ex ante* in this jump bid equilibrium compared to the non-jump bid equilibrium. The calculation of Θ shows that it is slightly negative ($\Theta = -0.003$).

Now consider the case where $\alpha = 2$ so that $F(v)$ is convex. The jump bid associated with $w^* = 1/4$ in this case is $b^* \approx 0.235$. Moreover, the special bidder is better off *ex ante* in this jump bid equilibrium than in the non-jump bid equilibrium. From (10) $\Theta \approx 0.05 > 0.000002 \approx \Psi$.

6 Conclusion

In this paper we examine a bidder's rationale for jump bidding. We show that in a one-object auction a bidder benefits from jump bidding only if her distribution is discontinuous and if bidder identities are concealed. The more information that a seller conceals, the higher the jump bid a bidder is willing to make in order to signal her valuation and the larger is the set of jump bid equilibria. In the case of an auction for two objects, we show that a bidder who wants only one object can expect a strictly greater expected payoff from jump bidding if the jump bid causes the other bidder to reduce her demand. Finally, we show that the expected revenue of the seller decreases when a jump bid equilibrium exists.

We have explored the role of jump bidding when bidders consider making a jump bid only in the first round. It is possible that other jump bid equilibria exist that involve bidders making jump bids in later rounds but we have not tried to characterize them. The existence of equilibria constructed from strategies of this type is left for future work. Moreover, our model considers the case where bidder valuations are statistically independent and private information. An interesting avenue for future research is to explore the interaction between jump bidding and information release in the context of the common value auction paradigm.

Throughout our analysis we have focused on the role of jump bidding in a single isolated auction. If a sequence of auctions were scheduled, a bidder may jump bid in an early auction to signal that she is a very aggressive bidder to her rivals. This may allow her to gain a reputation for being a tough bidder which may cause her rivals not to enter subsequent auctions.[8] This possible role for jump bidding is very plausible and promises to be an interesting area for future research.

References

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A Technical Appendix

Proof of Proposition 1. A bidder benefits from jump bidding if she expects to receive a greater payoff from using that strategy than from not jump bidding. If a bidder does not jump bid and wins, she expects to pay an amount equal to the second highest valuation conditional on her own being the largest. For a bidder with valuation v, this is given by

$$A(v) = \frac{\int_{\underline{v}}^{v} \xi \, dF^{n-1}(\xi)}{F^{n-1}(v)}.$$

We prove the proposition in three stages. First, we show that no separating jump bid equilibria exist. Second, we show that no pooling equilibria exist. Finally, we show that no partially pooling equilibria exists.

(i) Suppose that in equilibrium, a bidder with valuation v makes a jump bid of $B(v)$ where $B(v)$ is strictly increasing. Therefore, given a bidder's choice of jump bid, all other bidders infer the bidder's true valuation. If $B(v)$ is the equilibrium jump bid function, then the expected profit of a bidder with valuation v^* jump bidding to $B(v)$ is given by

$$F^{n-1}(v)[v^* - B(v)].$$

This must be maximized at $v = v^*$ if $B(v)$ is the equilibrium jump bid function. Taking the derivative with respect to v and setting $v = v^*$ in the first-order conditions gives

$$[F^{n-1}(v^*)]'[v^* - B(v^*)] - F^{n-1}(v^*)B'(v^*) = 0.$$

Adding $F^{n-1}(v^*)$ on both sides gives

$$[F^{n-1}(v^*)[v^* - B(v^*)]]' = F^{n-1}(v^*).$$

Integrating both sides from $\underline{v}$ to v gives

$$F^{n-1}(v)[v - B(v)] - F^{n-1}(\underline{v})[\underline{v} - B(\underline{v})] = \int_{\underline{v}}^{v} F^{n-1}(v^*) \, dv^*.$$

It is straightforward to argue that $F^{n-1}(\underline{v}) = 0$ so that the second term on the LHS vanishes. Solving for $B(v)$ and integrating by parts gives $B(v) = A(v)$. Therefore, this jump bid equilibrium is not beneficial to the buyer or the seller. Note that $A(v)$ is also equal to the bidding function in a first-price sealed-bid auction.

(ii) Consider the possibility of a pooling equilibrium. In this equilibrium there exists a b where all bidders with valuations greater than or equal to b jump bid to a. Those bidders with valuations below b do not jump bid. It is clear from part (i) above that $a = B(b)$, otherwise a bidder with a valuation slightly smaller than b will choose to mimic the jump bid. To show that no jump bid equilibrium exists we need to demonstrate that a bidder with valuation v, where $v > b$, does not expect to profit from jump bidding. This implies that her expected profit from jump bidding must equal that from not jump bidding. This condition is given by

$$\begin{aligned} F^{n-1}(b) &\left[v - \frac{\int_{\underline{v}}^{b} \xi \, dF^{n-1}(\xi)}{F^{n-1}(b)} \right] \\ &+ [F^{n-1}(v) - F^{n-1}(b)] \left[v - \frac{\int_{b}^{v} \xi \, dF^{n-1}(\xi)}{F^{n-1}(v) - F^{n-1}(b)} \right] \\ &\qquad = F^{n-1}(v) \left[v - \frac{\int_{\underline{v}}^{v} \xi \, dF^{n-1}(\xi)}{F^{n-1}(v)} \right], \end{aligned}$$

where the LHS is the expected profit from jump bidding. Simplifying both sides of this expression gives

$$\int_{\underline{v}}^{v} \xi \, dF^{n-1}(\xi) = \int_{\underline{v}}^{b} \xi \, dF^{n-1}(\xi) + \int_{b}^{v} \xi \, dF^{n-1}(\xi),$$

which clearly holds with equality.

(iii) Finally, consider the possibility of a general partial pooling equilibrium. Let v_1 be the lowest valuation associated with any jump bid of any bidder. Let bidder 1 be the bidder with the lowest jump bid valuation and let the corresponding jump bid be b_1. Bidder 1 with valuations higher than v_1 may also jump bid to b_1. Seeing a bid of b_1, only bidders with valuations lower than v_1 will drop out since bidders with valuations higher than v_1 still have a chance to win.

If bidder 1 with valuation v_1 wants to jump bid to b_1, then the payoff from jump bidding must not be lower than from not jump bidding. This means the following condition must hold

$$F^{n-1}(v_1)(v_1 - b_1) \geq F^{n-1}(v_1)\left[v_1 - \frac{\int_{\underline{v}}^{v_1} \xi \, dF^{n-1}(\xi)}{F^{n-1}(v_1)}\right],$$

which implies that

$$b_1 \leq \frac{\int_{\underline{v}}^{v_1} \xi \, dF^{n-1}(\xi)}{F^{n-1}(v_1)}.$$

A bidder 1 with a valuation lower than v_1, say $\tilde{v}_1$, chooses not to jump bid. This must mean that this bidder's expected payoff from jump bidding does not exceed that from not jump bidding. This implies the following condition must hold

$$F^{n-1}(v_1)(\tilde{v}_1 - b_1) \leq F^{n-1}(\tilde{v}_1)\left[\tilde{v}_1 - \frac{\int_{\underline{v}}^{\tilde{v}_1} \xi \, dF^{n-1}(\xi)}{F^{n-1}(\tilde{v}_1)}\right].$$

Solving for b_1 gives

$$b_1 > \frac{\tilde{v}_1[F^{n-1}(v_1) - F^{n-1}(\tilde{v}_1)]}{F^{n-1}(v_1)} + \frac{\int_{\underline{v}}^{\tilde{v}_1} \xi \, dF^{n-1}(\xi)}{F^{n-1}(v_1)}.$$

The RHS of this expression is increasing in $\tilde{v}_1$. As $\tilde{v}_1$ approaches v_1 the two bounds of b_1 coincide which means that bidder 1 with valuation v_1 cannot profit from jump bidding. □

Proof of Proposition 2. By direct inspection it is clear that $\alpha^{\mathrm{p}} \leq \alpha^{\mathrm{n}}$. To complete the proof we need to show that $\alpha^{\mathrm{f}} \leq \alpha^{\mathrm{p}}$. Using the expressions in the text, α^{p} is greater than $jumpF$ if the following condition holds

$$\int_{\underline{v}}^{\tilde{v}+\tau} \xi \, dF^n(\xi) - \int_{\tilde{v}}^{\tilde{v}+\tau} (\xi - \tilde{v}) \, d[F(\xi) - F(\tilde{v})]^n > \tilde{v}F^n(\tilde{v}+\tau) - \int_{\underline{v}}^{\tilde{v}} (\tilde{v} - \xi) \, dF^n(\xi).$$

This can be re-written as

$$\int_{\underline{v}}^{\tilde{v}+\tau} \xi \, dF^n(\xi) - \int_{\tilde{v}}^{\tilde{v}+\tau} (\xi - \tilde{v}) \, d[F(\xi) - F(\tilde{v})]^n$$

$$= \int_{\underline{v}}^{\tilde{v}} \xi \, dF^n(\xi)$$

$$+ \int_{\tilde{v}}^{\tilde{v}+\tau} \left\{ \xi n \left[F^{n-1}(\xi) - [F(\xi) - F(\tilde{v})]^{n-1} \right] + \tilde{v} n [F(\xi) - F(\tilde{v})]^{n-1} \right\} dF(\xi)$$

$$> \int_{\underline{v}}^{\tilde{v}} \xi \, dF^n(\xi)$$

$$+ \int_{\underline{v}}^{\tilde{v}+\tau} \left\{ \tilde{v} n \left[F^{n-1}(\xi) - [F(\xi) - F(\tilde{v})]^{n-1} \right] + \tilde{v} n [F(\xi) - F(\tilde{v})]^{n-1} \right\} dF(\xi)$$

$$= \tilde{v} F^n(\tilde{v} + \tau) - \int_{\underline{v}}^{\tilde{v}} (\tilde{v} - \xi) \, dF^n(\xi).$$

This condition always holds provided $\tau > 0$. □

Proof of Proposition 3. To prove that a jump bid equilibrium exists, we need to show that a low valuation special bidder and an ordinary bidder do not have an incentive to mimic the jump bid.

First, consider a special bidder with a valuation $w' \in [\underline{v}, \tilde{v}]$. If she mimics the jump bid her expected payoff is given by

$$F^n(\tilde{v} + \tau)(w' - \alpha),$$

while if she does not jump bid her expected payoff is given by

$$\int_{\underline{v}}^{w'} (w' - \xi) \, dF^n(\xi).$$

Note that the special bidder's expected profit from using either strategy does not depend on the amount of information released. The special bidder with valuation w' will not jump bid provided the jump bid is sufficiently high so as to make her expected payoff from jump bidding less than from not jump bidding. This implies that there exists a point $\underline{\alpha}$ where a low valuation special bidder would not mimic jump bids greater than this amount. Setting the special bidder's expected payoff from jump bidding equal to her expected payoff from not jump bidding and solving for the jump bid gives

$$\underline{\alpha} = w' - \frac{\int_{\underline{v}}^{w'} (w' - \xi) \, dF^n(\xi)}{F^n(\tilde{v} + \tau)}.$$

Note that this expression is increasing in w' and therefore, is largest when $w' = \tilde{v}$. Also note that if $w' = \tilde{v}$, then $\underline{\alpha} = \alpha^{\mathrm{f}}$. This implies that a special bidder with a high valuation (i.e., $w \in [\tilde{v} + \tau, \overline{v}]$) cannot profit from jump bidding when all information is disclosed because any jump bid strictly less than α^{f} will be mimicked by a special bidder with valuation $\tilde{v}$. When some information is concealed, a low valuation special bidder is unwilling to jump bid as high as a high valuation special bidder (i.e., $\underline{\alpha} \leq \alpha^{\mathrm{f}} \leq \alpha^{\mathrm{p}} \leq \alpha^{\mathrm{n}}$).

When some information is concealed, we must also consider the possibility of an ordinary bidder mimicking the jump bid. First, consider an ordinary bidder with valuation $v \in [\tilde{v} + \tau, \overline{v}]$. If she jump bids her expected profit is

$$G(\tilde{v} + \tau) F^{n-1}(\tilde{v} + \tau)(v - \alpha) + \int_{\tilde{v}+\tau}^{v} (v - \xi)\, d[G(\xi) F^{n-1}(\xi)],$$

while if she does not jump bid it's given by

$$\int_{\underline{v}}^{v} (v - \xi)\, d[G(\xi) F^{n-1}(\xi)].$$

She is indifferent between jump bidding and not jump bidding for a jump bid equal to

$$\overline{\beta} = \tilde{v} + \tau - \frac{\int_{\underline{v}}^{\tilde{v}+\tau} G(\xi) F^{n-1}(\xi)\, d\xi}{G(\tilde{v} + \tau) F^{n-1}(\tilde{v} + \tau)}.$$

Note that this expression is independent of the ordinary bidder's valuation.

Next, consider an ordinary bidder with valuation $v' \in [\underline{v}, \tilde{v} + \tau]$. If this bidder mimics the jump bid her expected profit is

$$G(\tilde{v} + \tau) F^{n-1}(\tilde{v} + \tau)(v' - \alpha),$$

while if she does not jump bid it's given by

$$\int_{\underline{v}}^{v'} (v' - \xi)\, d[G(\xi) F^{n-1}(\xi)].$$

She is indifferent between jump bidding and not jump bidding for a jump bid equal to

$$\underline{\beta} = \frac{\int_{v'}^{\tilde{v}+\tau} v'\, d[G(\xi) F^{n-1}(\xi)] + \int_{\underline{v}}^{v'} \xi\, d[G(\xi) F^{n-1}(\xi)]}{G(\tilde{v} + \tau) F^{n-1}(\tilde{v} + \tau)}.$$

By inspection, $\underline{\beta} < \overline{\beta}$, which means that if an ordinary bidder with a valuation greater than $\tilde{v} + \tau$ chooses not to jump bid, one with a smaller valuation will not as well.

For a jump bidding equilibrium to exist with partial information release, we must show that $\overline{\beta}$ is less than α^{p}. This is true if the following expression is satisfied

$$\frac{\int_{\underline{v}}^{\tilde{v}+\tau} G(\xi)F^{n-1}(\xi)\, d\xi}{G(\tilde{v}+\tau)F^{n-1}(\tilde{v}+\tau)} > \frac{\int_{\underline{v}}^{\tilde{v}+\tau} F^n(\xi)\, d\xi + \int_{\tilde{v}}^{\tilde{v}+\tau} (\xi - \tilde{v})\, d[F(\xi) - F(\tilde{v})]^n}{F^n(\tilde{v}+\tau)}. \tag{11}$$

With some re-arrangement it is easy to show that this expression holds if Condition 1 is satisfied. Similarly, it can be shown that $\overline{\beta}$ is less than α^{n} if Condition 2 is satisfied. □

Proof of Lemma 1. Since (1) satisfies the conditions of the implicit function theorem, $s(v)$ exists. Obviously, $s(0) = 0$. Setting $v = 1$ in (1) and re-arranging gives

$$\frac{1-s}{2} = 1 - \frac{\int_s^1 \xi\, dG(\xi)}{1 - G(s)}.$$

Clearly, if $G(w)$ is strictly concave then this expression can never hold with equality for any s in the interval $[0, 1]$. The RHS is always greater than the LHS for all s. This implies that $s(1) = 1$ or that an ordinary bidder with the highest possible valuation will always wish to keep bidding for two objects.

Next, consider the slope of $s(v)$. Equation (1) can be re-written as follows

$$(v - s)[1 + G(s)] = 2 \int_s^v G(\xi)\, d\xi. \tag{12}$$

Totally differentiating this gives

$$\frac{ds}{dv} = \frac{1 + G(s) - 2G(v)}{1 - (v - s)g(s) - G(s)}.$$

We show that ds/dv is positive by showing that both the numerator and denominator of this expression are both negative.

The denominator is negative since with $G(w)$ concave we have

$$G(\xi) - G(s) < g(s)(\xi - s),$$

for all $\xi > s$. Integrating both sides from s to v gives

$$\int_s^v [G(\xi) - G(s)]\, d\xi > g(s) \frac{(v-s)^2}{2},$$

which gives

$$2 \int_s^v G(\xi)\, d\xi < 2G(s)(v-s) + g(s)(v-s)^2.$$

From (12) we have

$$(v-s)[1 + G(s)] < 2G(s)(v-s) - G(s)(v-s)^2.$$

Cancelling the term $v - s$ from both sides we obtain the denominator above and hence it must be negative.

To see that the numerator is negative, substitute the expression for $1 + G(s)$ from (12) into the numerator and re-arrange. The numerator is then negative if

$$\int_s^v G(\xi)\, d\xi < G(v)(v-s),$$

which holds given that $G(w)$ is an increasing function and $v > s$.

Finally, if we set $s = 0$ in (12) and re-arrange we have

$$\frac{1}{2} = \frac{\int_0^v G(\xi)\, d\xi}{v}.$$

The RHS is increasing in v and

$$\lim_{v \to 0} \frac{\int_0^v G(\xi\, d\xi}{v} = 0,$$

and

$$\lim_{v \to 1} \frac{\int_0^v G(\xi)\, d\xi}{v} = 1.$$

Therefore, there exists a unique v, call it $\tilde{v}$, such that the expression holds.

Clearly then, $s(v)$ is increasing for $v > \tilde{v}$ and equal to zero for $0 \leq v \leq \tilde{v}$. □

Proof of Lemma 2. Since (2) satisfies the conditions of the implicit function theorem $\bar{s}(v; w^*)$ exists.

Obviously, $\bar{s}(v; 0) = 0$. Setting $v = w^*$ in (2) and re-arranging gives

$$\frac{w^* - s}{2} = w^* - \frac{\int_s^{w^*} \xi \, dG(\xi)}{G(w^*) - G(s)}.$$

Clearly, given that $G(w)$ is strictly concave, the expression above can never hold with equality for s in the interval $[0, w^*]$. The RHS is greater than the LHS for all s. This implies that $\bar{s}(v; w^*) = w^*$.

Next, consider the slope of $\bar{s}(v; w^*)$. Equation (2) can be re-written as follows

$$(v - s)[G(w^*) + G(s)] = 2 \int_s^v G(\xi) \, d\xi. \tag{13}$$

Totally differentiating gives

$$\frac{d\bar{s}}{dv} = \frac{G(w^*) + G(s) - 2G(v)}{G(w^*) - (v - s)g(s) - G(s)}.$$

We show that $d\bar{s}/dv$ is positive by showing that both the numerator and denominator are negative. The denominator is negative since with $G(w)$ concave we have

$$G(\xi) - G(s) < g(s)(\xi - s),$$

for all $\xi > s$. Integrating both sides from s to v gives

$$\int_s^v [G(\xi) - G(s)] \, d\xi > g(s) \frac{(v - s)^2}{2},$$

which gives

$$2 \int_s^v G(\xi) \, d\xi < 2G(s)(v - s) + g(s)(v - s)^2.$$

From (13) we have

$$(v - s)[G(w^*) + G(s)] < 2G(s)(v - s) - G(s)(v - s)^2.$$

Cancelling the term $v - s$ from both sides we obtain the denominator above and hence it is negative.

To see that the numerator is negative, substitute for $G(w^*) + G(s)$ from (13) into the numerator and re-arrange. The numerator is negative if

$$\int_s^v G(\xi)\, d\xi < G(v)(v - s),$$

which holds given that $G(w)$ is an increasing function and that $v > s$.

Finally, if we set $s = 0$ in (13) and re-arrange, we have

$$\frac{G(w^*)}{2} = \frac{\int_0^v G(\xi)}{v}.$$

Since the RHS is increasing in v and

$$\lim_{v \to 0} \frac{\int_0^v G(\xi)\, d\xi}{v} = 0,$$

and

$$\lim_{v \to 1} \frac{\int_0^v G(\xi)\, d\xi}{v} = 1,$$

there exists a unique v, call it $\bar{v}(w^*)$, such that the expression above holds.

Clearly then, $\bar{s}(v; w^*)$ is increasing for $v > \bar{v}(w^*)$ and equal to zero for $0 \le v \le \bar{v}(w^*)$. □

Proof of Lemma 3. By definition $\hat{v}(0) = \tilde{v}$. If we set $v = 1$ in (3) and solve for w^*, then the condition holds for a w^*, call it $\tilde{w}^*$, defined by

$$\frac{\int_{\tilde{w}^*}^1 \xi\, dG(\xi)}{1 - G(\tilde{w}^*)} = \frac{1}{2}.$$

Totally differentiating (3) gives

$$\frac{dv}{dw^*} = \frac{(v - 2w^*)g(w^*)}{(v - s(v))g(s(v))s'(v) + G(s(v)) - G(w^*) - s'(v)[1 - G(s(v))]}.$$

Since $s(v) > w^*$, $G(s(v)) - G(w^*) > 0$. The denominator will be positive if

$$g(s(v)) > \frac{1 - G(s(v))}{v - s(v)},$$

which holds because $G(w)$ is strictly concave and $v > s(v)$. The numerator is positive since for $\hat{v}(w^*) > 0$ it must be the case that $v > 2w^*$, otherwise the ordinary bidder would not have a positive expected payoff from continuing to bid for both objects. Therefore, $\hat{v}(w^*)$ is an increasing function with $\hat{v}(0) = \tilde{v}$ and $\hat{v}(\tilde{w}^*) = 1$. □