Queen's Economics Department Working Paper No. 961

Equilibrium Valuation of Options on the Market Portfolio with Stochastic Volatility and Return Predictability

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8-1997

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This version: August, 1997

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Abstract

This paper uses an extension of the equilibrium model of Lucas (1978) to study the valuation of options on the market portfolio with return predictability, endogenous stochastic volatility and interest rates. Equilibrium conditions imply that the mean-reverting of the rate of dividend growth induces the predictable feature of the market portfolio. Although the actual drift of the price for the market portfolio does not explicitly enter into the option price formula when the equivalent martingale pricing principle is used, parameters underlying the predictable feature affect option prices through their influence on endogenized volatility and interest rates. Equilibrium conditions also review that there is strong interdependence between the equilibrium price process for the market portfolio and its volatility process, both of which are induced by the process for aggregate dividend. Closed-form pricing formulas for options on the market portfolio incorporate both stochastic volatility and stochastic interest rates. With realistic parameter values, numerical examples show that stochastic volatility and stochastic interest rates are both necessary for correcting the pricing biases generated by the Black-Scholes model. In addition, Closed-form solutions for European bond option prices are obtained, which encompass the Vasicek (1977) model and the Cox-Ingersoll-Ross (1985) model. In this sense, the current model provides a consistent way to price options written on the market portfolio and the bonds.

* This is a significant expansion on an earlier paper entitled "General Equilibrium Valuation of Options with Stochastic Volatility: A One Factor Model". I thank Phelim Boyle, Ritchie Y. He, John Hull, Robert Jarrow, Raymond Kan, Graig Lewis, Tom McCurdy, Alan White and participants of the finance workshop at the University of Toronto for helpful comments and valuable suggestions. I also gratefully acknowledge the Social Sciences and Humanities Research Council of Canada for financial support. All remaining errors are mine alone.

1. Introduction

The Black-Scholes (1973) formula is still rightly regarded by both practitioners and academics as the premier model of option valuation. However, it has some well-known deficiencies when matched with market prices. In particular, the model overprices out-of-the-money call options and underprices in-the-money calls.[1] These deficiencies are usually ascribed to the strong assumption in the Black-Scholes model (BS henceforth) that the price of the asset underlying the derivative security follows a geometric Brownian motion with constant interest rate and volatility. The majority of empirical works indicates that volatility is not constant (e.g. Rosenberg 1972, Oldfield, Rogalski and Jarrow 1977). Consequently, many theoretical option models have allowed for non-constant volatility while maintaining the assumption of constant interest rates. Notable examples are Merton (1976), who adds a jump risk into the diffusion process used by Black and Scholes for the stock price, Hull and White (1987) and Stein and Stein (1991), who assume that the volatility is driven by a state variable which is different from the one in the stock price process.

These modifications of the BS model are important attempts to incorporate non-constant volatility in option prices but they are not satisfactory for three reasons. First, they fail to incorporate the influence of stock return predictability on option prices. The predictability of financial asset returns has been documented by Bekaert and Hodrick (1992), Breen, Glosten and Jagannathan (1989), Campbell and Hamao (1992), Fama and French (1988a, 1988b) and Ferson and Harvey (1991). The predictability is typically reflected in the drift of the stock price and can be reviewed as a mean-reverting process. In partial equilibrium models, Grundy (1991) states that the BS model can be consistent with the stock return predictability, while Lo and Wang (1995) argue that the predictability can provide information on the forecast of the volatility. It will be interesting to investigate how the predictability affects option prices when the interest rate and volatility are

[1] This evidence is documented by MacBeth and Merville (1979). Similar findings are provided by Lauterbach and Schultz (1990) in an empirical study on warrants.

both endogenous.

Second, the typical assumption on the stochastic volatility is that it is uncorrelated with aggregate consumption. This assumption is problematic when the security under consideration is the market portfolio.[2] A considerable amount of evidence has shown that the non-constant volatility of stock prices is highly correlated with the volatility of the market as a whole. For example, Wiggins (1987) has found a significantly negative correlation between volatility movements and stock prices for highly aggregated stock indices. The estimated correlation for S&P 500 is -0.79 for an 8-day interval in which the volatility is assumed constant. Intuitively, the volatility and price of the market portfolio are both driven by the same fundamental forces such as aggregate consumption or aggregate dividend. Thus, it is desirable to specify only the processes for these fundamental forces, drive endogenously the processes for the market portfolio and its volatility, and then examine the relation between these two processes.

The third unsatisfactory feature of non-constant volatility models is that they have assumed a constant interest rate. However, the volatility of the market portfolio is in general negatively related to the interest rate (see Bailey and Stulz 1989). To reflect this negative correlation, one must abandon the assumption of constant interest rates and simultaneously analyze the effects of stochastic volatility and stochastic interest rates. Finally, previous models of non-constant volatility have introduced a non-traded source of risk such as jumps or stochastic volatility and hence lost their completeness, i.e., the ability to hedge options with the underlying asset. Such ability is desirable for hedging and risk management in the business world, as stated in Dupire (1994).

To eliminate these unsatisfactory features, I use a continuous-time extension of the Lucas (1978) equilibrium model. As in Lucas (1978), the economy is a pure exchange economy in which there is a single representative agent with an infinite lifetime horizon. In the financial market, this agent can instantaneously trade a single risky stock, pure discount bonds and other contingent claims

[2] In a parallel fashion, Naik and Lee (1990) argue that the jump risk in the market portfolio should be correlated with aggregate consumption and dividend.

written on this risky stock and the discount bonds. The risky stock can be viewed as the market portfolio whose dividend is the only exogenous source of uncertainty. By appealing to the study of Marsh and Merton (1987) on the dynamic behavior of aggregate dividend, the rate of dividend growth is modelled as an affine-class of mean reverting process. The general pricing equation (the Euler equation) is derived by solving the representative agent's maximization problem. Under the specification of dividend process and the agent's preference, the processes for the market portfolio, its volatility, the spot interest rate and the bond price are derived in equilibrium.

Equilibrium results indicate that the mean-reverting feature of the rate of dividend growth generates the predictability of the stock return. Endogenizing the return of the market portfolio, its volatility and spot interest rates enables me to determine whether the BS model is consistent with a mean-reverting actual stock return. It is shown that the predictability of the return to the market portfolio requires either the volatility and/or the spot interest rate to be stochastic. Thus, the BS model with constant volatility and spot interest rate is not consistent with the predictability of the asset return on the market portfolio. In addition, I analyze how the predictable features affect the option prices in an environment where the spot interest rate and the volatility of the market portfolio are mean-reverting. Although the actual drift rate does not explicitly appear in the option pricing formula when the equivalent martingale pricing principle is used, fundamental forces that affect the drift also affect option prices through their effects on the endogenous interest rate and volatility.

It is also shown that the process of the market volatility and its price process are not independent. The two processes are negatively correlated, as indicated by empirical evidence, when reasonable conditions are imposed on the parameters underlying the dividend process. Finally, the equilibrium spot interest rate exhibits the mean-reverting feature. As a result, stochastic interest rates and stochastic volatility are both incorporated into the European stock option formulas which include the BS model as a special case. The predictability affects stock option prices through both the stochastic interest rate and stochastic volatility. Since changes in parameters underlying the predictable features generate opposite impacts on the interest rate and volatility simultaneously, an induced increase in the volatility or interest rate by these fundamental parameters does not necessarily increase the call price.[3]

In addition to its simplicity and analytical tractability, the current stock option model adequately corrects the BS pricing biases very well under reasonable parameter values. Numerical exercises show that the current model provides higher prices for in-the-money calls and lower prices for out-the-money calls than the BS model. Also, call prices given by the current model generate a realistic pattern of implied volatility which is consistent with the empirical study on S&P 500 index European option by Wiggins (1987) and Dumas, Fleming and Whaley (1995). These results indicate that eliminating the unsatisfactory features can significantly improve some aspects of the BS stock option model.

The equilibrium approach for option valuation is shared with Bailey and Stulz (1989) who price options written on stock indices, Naik and Lee (1990) who address the systematic jump risks in the market portfolio, and Amin and Ng (1993) who focus on individual stock option prices with systematic volatility. The current model differs from these models by explicitly modelling the predictability of stock returns and stochastic interest rates. In addition, this paper provides closed-form formulas for European bond option prices which encompass the Vasicek (1979) model and the Cox-Ingersoll-Ross model (1985, CIR henceforth). It provides a consistent way to price options on the market portfolio and bonds in a one-factor context.

On the effects of predictability of the asset return on option prices, this paper is related to Grundy (1991) and Lo and Wang (1995). Using partial equilibrium frameworks, these studies have attempted to understand some basic issues such as whether the BS risk-neutral log-normal assumption is consistent with a trending Ornstein-Uhlenbeck process for the actual stock return

[3]The common belief is that an increase in stock volatility will be accomplished by an increase in call price according to the BS model. Bailey and Stulz (1989) show that this common belief is not necessarily supported in an equilibrium framework. The results in the current paper confirms the observation made by Bailey and Stulz (1989).

(Grundy 1991) and how parameters underlying such a process affect the BS formulas (Lo and Wang 1995). These studies employ the BS environment where interest rates and volatility are constant. In contrast, the current paper uses an equilibrium framework where both interest rates and volatility are endogenous and stochastic, which shows that the BS model with constant volatility and interest rate is inconsistent with stock return predictability.

The remainder of this paper is organized as follows. Section 2 describes the economy, presents the general equilibrium results and then analyzes the dynamics of the price of the market portfolio, its volatility, the bond price and the spot interest rate. Section 3 examines the relations among the spot interest rate, stock return predictability and its volatility. Section 4 derives the equilibrium pricing formulas for options written on the market portfolio and bonds, and examines the effects of predictability on option prices. In addition, comparative statics are performed for both types of options. Section 5 numerically compares the option pricing for the market portfolio with the BS model and examines the pattern of the implied volatility for the market portfolio. Section 6 concludes the paper and the appendices provide necessary proofs.

2. The Economy

2.1. Structure of the Economy

Consider a continuous-time extension of the Lucas (1978) pure exchange economy in which there is a representative investor with an infinite lifetime horizon. In the financial market, the representative agent can trade a single risky stock, pure discount bonds and a finite number of other contingent claims at any time. The risky stock can be viewed as the market portfolio, whose total supply is normalized to one share and its dividend stream $\{\delta_t\}$ can be understood as the aggregate dividends in the economy. The contingent claims and the riskless bond are all in zero net supply. I assume that the aggregate dividend process is exogenously given by a Markov process on a given probability space $(\Omega, \mathcal{F}, \mathcal{P})$. The fundamental uncertainty in the model is completely described by the process for the aggregate dividend. Denote the security prices at time t by a vector X_t and the corresponding vector of dividends by q_t. The cumulative dividends up to t are defined as $D_t \equiv \int_0^t q_\tau d\tau$.

The representative agent's information structure is given by the filtration $\mathcal{F}_t \equiv \sigma(\,\delta_\tau, 0 \leq \tau \leq t)$. His preference is described by a smooth time-additive expected utility function:

$$V(c) = E\int_0^\infty U(c_t, t)\mathrm{d}t,$$

where $U : \mathcal{R}_+ \times (0,\infty) \to \mathcal{R}$ is smooth on $(0,\infty)\times(0,\infty)$ and, for each $t \in (0,\infty)$, $U(\cdot,t) : \mathcal{R}_+ \to \mathcal{R}$ is increasing, strictly concave, and has a continuous derivative $U_c(\cdot,t)$ on $(0,\infty)$. Initially, the agent is endowed with one share of the risky stock. Denote his portfolio holdings at time t as $\theta_t = (\theta_t^s, \theta_t^B, \theta_t^{x'})$, where θ_t^s, θ_t^B and $\theta_t^{x'}$ represent the number of shares invested in the risky stock, the discount bond and other contingent claims, respectively. The agent's consumption over time is financed by a continuous trading strategy $\{\theta_t, t \geq 0\}$. The agent's decision problem is to choose such a trading strategy so as to maximize his expected lifetime utility. Precisely, he solves[4]

$$\max_{\{c_t,\theta_t\}} E\int_0^\infty U(c_t, t)\mathrm{d}t \tag{2.1}$$

s.t.

$$\int_0^t c_\tau \mathrm{d}\tau = \theta_0 \cdot X_0 + \int_0^t \theta_\tau \cdot \mathrm{d}D_\tau + \int_0^t \theta_\tau \cdot \mathrm{d}X\tau - \theta_t \cdot X_t. \tag{2.2}$$

The first order conditions give the usual stochastic Euler equation:

$$X_t = \frac{1}{U_c(c_t,t)} E_t\left(\int_t^\infty U_c(c_\tau,\tau)\mathrm{d}D_\tau\right).$$

Thus, the price of any security equals the expected discounted sum of its dividends, with the marginal rate of substitution being the stochastic state price deflator.

In equilibrium, the financial market clears and so the demand for the stock equals the supply of shares, which is one share. Also, equilibrium prices are such that the representative agent holds nothing of the other claims because the corresponding net supply is zero. In addition, the goods

[4]All the expectations in this paper are taken with respect to the filtration specified earlier. The budget constraint is similar to that defined in Duffie (1992) for the security market equilibrium. This Euler equation approach is also adopted in Naik and Lee (1990). The partial differential equation satisfied by the price of any asset can also be derived through the optimal control rule. Such an analysis is omitted here but available upon request.

market clears so that consumption equals dividends generated from the risky stock. Therefore, the equilibrium price of the risky stock, denoted $S_t(\delta_t)$, is

$$S_t(\delta_t) = \frac{1}{U_c(\delta_t, t)} E_t \left(\int_t^{\infty} U_c(\delta_\tau, \tau) \delta_\tau \mathrm{d}\tau \right), \quad \forall \quad t \in (0, \infty). \tag{2.3}$$

For a riskless bond paying 1 unit of consumption goods at T and 0 at all other time, its equilibrium price at time t , denoted $B_t(T, \delta_t)$, is

$$B_t(T, \delta_t) = \frac{1}{U_c(\delta_t, t)} E_t \left(U_c(\delta_T, T) \right), \quad \forall \quad t \in (0, T). \tag{2.4}$$

For any contingent claim i with a payoff q_T^i at maturity T, its price at time t , denoted $F_t(T)$, is

$$F_t(T, \delta_t) = \frac{1}{U_c(\delta_t, t)} E_t \left(U_c(\delta_T, T) q_T^i \right), \quad \forall \quad t \in (0, T). \tag{2.5}$$

In particular, $q_T^i = \max\{S_T(\delta_T) - K, 0\}$ for a European call option written on the risky stock, and $q_T^i = \max\{B_T(\overline{T}, \delta_T) - K, 0\}$ for a European call option written on the riskless bond whose maturity is $\overline{T} \geq T$. K denotes the striking price for both options.

2.2. Equilibrium Prices under a Specific Dividend Process

To facilitate discussion and to obtain closed form solutions, let us restrict attention to a specific dividend process for the market portfolio. The specific process is chosen by appealing to the study of Marsh and Merton (1987) on the dynamic behavior of aggregate dividends (see also Lintner 1956 and Fama and Babiak 1968). Their estimation results suggest that changes in the rate of dividend tend to conform with the following description:[5]

$$\ln(\mathrm{div}_t) - \ln(\mathrm{div}_{t-1}) = \text{speed of adjustment} \times (\text{target ratio} \times \text{change in stock price}_t - \ln \mathrm{div}_{t-1}).$$

Their regression results also show that the random components in the change of dividend growth exhibit heteroskedasticity. In order to be consistent with these findings, the process for aggregate dividends is assumed as follows:

[5]Lintner (1956) and Fama and Babiak (1968) study the dividend behavior for individual stocks. They use the accounting earnings variable instead of the changes in stock prices.

Assumption 1. *The rate of aggregate dividend growth evolves according to the following stochastic process:*

$$d\ln\delta = (\beta_1 - \alpha_1 \ln\delta)dt + \sqrt{\beta_2 + \alpha_2 \ln\delta}\; dz, \tag{2.6}$$

where dz is the standard Wiener process. In addition, restrict $0 \geq \alpha_2 \geq -2\alpha_1$.

The drift in (2.6) captures the mean-reverting feature of the rate of dividend growth while the volatility structure $\beta_2 + \alpha_2 \ln\delta$ corresponds to a GARCH model. Note that (2.6) is an extension of the process assumed for the single state variable in CIR (1985). It is the continuous-time counterpart of a first-order autoregressive process in discrete-time where the randomly changing *rate of dividend* is pulled toward a long-run mean, β_1/α_1. Parameter α_1 determines the speed of mean reversion. The restriction $0 \geq \alpha_2 \geq -2\alpha_1$ is imposed in order to guarantee the realistic negative relationship between the stock price and its volatility, as well as the negative relationship between the bond price and the spot interest rate (see later discussion).

In order to describe the distribution of δ_T conditional on (δ_t, t), I take a linear transformation $Y(\delta) = \beta_2 + \alpha_2 \ln\delta$. By Ito's Lemma, we have

$$\mathrm{d}Y = (\alpha_1\beta_2 + \alpha_2\beta_1 - \alpha_1 Y)\mathrm{d}t + \alpha_2\sqrt{Y}\mathrm{d}z. \tag{2.7}$$

The process implied by (2.7) has the following properties: (i) Y is strictly positive if $2(\alpha_1\beta_2 + \alpha_2\beta_1) \geq \alpha_2^2$ and $\alpha_1 > 0$; (ii) the variance of Y increases when Y increases; and (iii) Y_T conditional on (Y_t, t) has a non-central χ^2 distribution with the following density function:[6]

$$f(Y_T, T; Y_t, t) = a(t,T)e^{-(x+\lambda)}\left(\frac{x}{\lambda}\right)^{\frac{1}{2}(v-1)} I_{v-1}(\sqrt{4x\lambda}) \tag{2.8}$$

where

$$\begin{aligned} a(t,T) &\equiv \frac{2\alpha_1}{\alpha_2^2(1-e^{-\alpha_1(T-t)})}, \qquad & v &\equiv \frac{2(\alpha_1\beta_2 + \alpha_2\beta_1)}{\alpha_2^2}, \\ \lambda &\equiv a(t,T)Y_t e^{-\alpha_1(T-t)}, \qquad & x &\equiv a(t,T)Y_T. \end{aligned} \tag{2.9}$$

[6]See Johnson and Kotz (1970) for the non-central χ^2 distribution and Feller (1951) for the corresponding the probability transition function.

In this description, 2λ is the non-central parameter, $2v$ is the degree of freedom, and $I_{v-1}(\cdot)$ stands for the modified Bessel function of the first kind of order $v-1$.[7] In the steady state as $T \to \infty$, the density function (2.8) converges to $f(Y_\infty, \infty; Y_t, t) = \bar{a}\Gamma(v)^{-1}e^{-\bar{a}Y_\infty}(\bar{a}Y_\infty)^{v-1}$, a central χ^2 distribution. $\bar{a} = \lim_{T\to\infty} a(t,T) = \frac{2\alpha_1}{\alpha_2^2}$ and the degree of freedom is v.

The conditional expected mean and variance of $\ln \delta_T$ can be computed as:

$E(\ln \delta_T \mid \ln \delta_t) = \ln \delta_t e^{-\alpha_1(T-t)} + \frac{\beta_1}{\alpha_1}(1 - e^{-\alpha_1(T-t)})$,

$Var(\ln \delta_T \mid \ln \delta_t) = \frac{\alpha_2}{\alpha_1}\ln \delta_t(e^{-\alpha_1(T-t)} - e^{-2\alpha_1(T-t)}) + \frac{\beta_2}{2\alpha_1}(1 - e^{-2\alpha_1(T-t)}) + \frac{\alpha_2\beta_1}{2\alpha_1^2}(1 - e^{-\alpha_1(T-t)})^2$.

When the reversion rate α_1 goes to 0, the mean converges to $\ln \delta_t + \beta_1(T-t)$ and the variance converges to $(\alpha_2 \ln \delta_t + \beta_2)(T-t) + \alpha_2\beta_1(T-t)^2$. As $T \to \infty$, the steady state mean and variance are β_1/α_1 and $(\alpha_1\beta_2 + \alpha_2\beta_1)/2\alpha_1^2$, respectively.

For analytical tractability, I adopt the typical logarithmic assumption on the agent's preference.[8]

Assumption 2. *The representative agent's period utility is described by*

$$U(c_t, t) = e^{-\rho t}\ln c_t, \tag{2.10}$$

where $\rho > 0$ is the rate of time preference.

Based on Assumptions 1 and 2, the equilibrium price for any financial asset can be solved through the Euler equation. The equilibrium stock price S_t can be easily computed from (2.3):

Proposition 2.1. *Under Assumptions 1-2, the equilibrium price of the risky stock at time t, is $S_t = S(\delta_t) = \frac{\delta_t}{\rho}, \quad \forall \quad t \in (0, \infty)$.*

The stock price equals the present value of future dividends discounted at the rate of time preference. That is, the stock generates a constant dividend yield which is equal to the rate of time

[7]The modified Bessel function of the first kind of order q is defined as:

$$I_q(y) = \left(\frac{y}{2}\right)^q \sum_{j=0}^{\infty} \frac{(\frac{y^2}{4})^j}{j!\Gamma(q+j+1)},$$

where $\Gamma(a)$ is expressed as $\Gamma(a) \equiv \int_0^\infty e^{-y}y^{a-1}dy$. (see Johnson and Kotz 1994).

[8]This assumption is also adopted by Merton (1971) and CIR (1985) for a similar reason.

preference. The equilibrium stock price follows a similar process as the dividend:

$$\frac{dS}{S} = (\mu_s - \frac{\delta}{S})dt + \sqrt{V}dz = \left[\beta_1 + \frac{\beta_2}{2} - (\alpha_1 - \frac{\alpha_2}{2})\ln\delta\right]dt + \sqrt{\beta_2 + \alpha_2\ln\delta}\ dz,$$

where μ_s is the expected stock return and V is the variance, explicitly given below. The expected stock return and its process are:

$$\begin{aligned}\mu_s &= \beta_1 + \beta_2/2 - (\alpha_1 - \alpha_2/2)\ln\delta + \rho,\\ d\mu_s &= (\alpha_1\rho + \frac{v\alpha_2^2}{4} - \alpha_1\mu_s)dt - \sqrt{(\alpha_1 - \frac{\alpha_2}{2})(\alpha_2\rho + \frac{v\alpha_2^2}{2} - \alpha_2\mu_s)}dz.\end{aligned}$$

The speed of reversion (α_1) and the long-run mean of the stock return ($\rho + v\alpha_2^2/4\alpha_1$) determine the predictable features of the stock return. Clearly, the predictability is affected by all parameters (ρ, α_1, β_1, α_2, β_2). This result provides a theoretical explanation for the predictability of the stock return. That is, such predictability is induced by the mean-reverting feature of the fundamental rate of dividend growth (See references at the beginning of this subsection).

The dividend process also endogenously generates stochastic volatility that is meaning-reverting and provides a rationale for a similar stochastic volatility process assumed by Stein and Stein (1991). The volatility and its process are:

$$\begin{aligned}V &= \beta_2 + \alpha_2\ln\delta,\\ dV &= (\beta_1\alpha_2 + \alpha_1\beta_2 - \alpha_1 V)dt + \alpha_2\sqrt{V}dz.\end{aligned}$$

The instantaneous variance exhibits the mean-reverting feature, with the speed of adjustment being α_1 and the long-run mean being $(\alpha_1\beta_2 + \alpha_2\beta_1)/\alpha_1$. The mean-reverting feature of the volatility is induced by the heteroskedasticity in the dividend yield. Note that the restriction $\alpha_2 < 0$ ensures a negative correlation between the price of the market portfolio and its volatility.

Since bond prices and the spot interest rate are useful for analyses on option prices, they are determined in the following proposition and corollary (see Appendix A for a proof):

Proposition 2.2. *Under Assumptions 1-2, the equilibrium price of a pure discount bond with maturity T at time $t \leq T$, $B_t(T, \delta_t)$, is*

$$B_t(T,\delta_t) = A(t,T)^v e^{-\left(\rho + \frac{A(t,T)e^{-\alpha_1(T-t)}-1}{T-t}\frac{Y_t}{\alpha_2}\right)(T-t)}, \tag{2.11}$$

where $A(t,T) = \frac{a(t,T)\alpha_2}{a(t,T)\alpha_2+1}$, $a(t,T)$ and v are defined in (2.9).

Corollary 2.3. *Denote the instantaneous interest rate at any time $\tau \in (t,T)$ by $r(\tau)$ and define it implicitly through $B_t(T,\delta_t) = E_t^*(e^{-\int_t^T r(\tau)d\tau})$.*[9] *The spot instantaneous interest rate $r(t)$ and the expected steady state interest rate $\bar{r}$ are*

$$r(t) = \rho + \beta_1 - \alpha_1 \ln \delta_t - \tfrac{1}{2}(\beta_2 + \alpha_2 \ln \delta_t),$$
$$\bar{r} = \lim_{T\to\infty} E_t[r(T)] = \rho - v\alpha_2^2/4\alpha_1.$$

Since the spot interest rate is linear in $\ln \delta$, I can rewrite the bond price in Proposition 2.2 in terms of the spot interest rate. That is,

$$B_t(T,\delta_t) = A(t,T)^v \exp\left(-\rho(T-t) - (r - \rho - \frac{\alpha_2 v}{2})A(t,T)\frac{1-e^{-\alpha_1(T-t)}}{\alpha_1}\right).$$

The restriction $0 \geq \alpha_2 \geq -2\alpha_1$ ensures a negative relation between the bond price and the spot interest rate. Under such restriction, the bond price has appealing properties. For example, the bond price is a decreasing convex function of the interest rate and an increasing (decreasing) function of time (maturity). The bond price is negatively correlated with the aggregate dividend. That is, the common disturbance in aggregate dividends has similar effects on the prices of the stock and the bond. The intuitive explanation is that a high aggregate dividend implies a high stock price, which in turn induces a low demand for the stock as agents look for investment opportunities in the bond market. Consequently, a high demand for bonds will pull down the bond price.

The dynamics of the bond price are described as

$$\frac{dB_t}{B_t} = \left[r - (r - \rho - \alpha_2 v/2)A(t,T)(1 - e^{-\alpha_1(T-t)})\frac{\alpha_2}{\alpha_1}\right]dt + \left[1 - A(t,T)e^{-\alpha_1(T-t)}\right]\sqrt{Y}dz.$$

[9] $E_t^*(\cdot)$ denotes the expectation under equivalent martingale measure.

The volatility of the bond price is $V_B(t,T) = \left[1 - A(t,T)e^{-\alpha_1(T-t)}\right]^2 Y$. As one should expect, the volatility of the bond price equals 0 when the bond is at the maturity. When the maturity goes to infinity, the volatility approaches Y which is the same as that of the stock price. The intuitive reason is that when the bond has an infinite maturity, it is very similar to a stock. In this case, the volatility of bond mimics that of the stock price, which in turn reflects the volatility of the aggregate dividend.

The bond price is usually quoted in terms of the yield-to-maturity, $R(r,t,T)$, which is defined through $e^{-R(r,t,T)(T-t)} = B_t(T,\delta_t)$. We have

$$R(r,t,T) = \rho + \frac{2(r-\rho-\alpha_2 v/2)}{\alpha_2(\alpha_2 a(t,T)+1)(T-t)} - \frac{v\ln(A(t,T))}{T-t}.$$

As the bond approaches the maturity, the yield-to-maturity approaches the spot interest rate. If the maturity goes to infinity, the yield approaches the dividend yield for the risky stock (ρ).[10] When the spot rate is below ρ, the term structure is uniformly increasing. If the spot rate is above $\rho - v\alpha_2^2/4\alpha_1$, the term structure decreases. For any value of the spot rate in between, the yield curve is hump shaped.

The spot interest rate here obeys a mean-reverting process similar to that of $\ln\delta$ since it is linear in $\ln\delta$. Under the restrictions $0 \geq \alpha_2 \geq -2\alpha_1$, the spot interest rate follows:

$$\mathrm{d}r = (\alpha_1\rho - \frac{v\alpha_2^2}{4} - \alpha_1 r)\mathrm{d}t + \sqrt{(\alpha_1 + \frac{\alpha_2}{2})(\alpha_2\rho + \frac{v\alpha_2^2}{2} - \alpha_2 r)}\mathrm{d}z.$$

This mean-reverting process resembles the so-called affine class of the term-structure model.[11] The mean-reverting speed for the spot rate is α_1 and the long-run mean is $\rho - v\alpha_2^2/4\alpha_1$. The Vasicek (1977) model corresponds to $\alpha_2 = 0$ and the CIR (1985) model to $\rho = -\alpha_2 v/2$, respectively.

[10]This also confirms the earlier observation that the bond with infinite maturity is similar to the risky stock.

[11]As stated in Duffie (1992), the process of the affine class of term-structure model is

$$\mathrm{d}r = (a_1 + a_2 r)\mathrm{d}t + \sqrt{b_1 + b_2 r}\,\mathrm{d}z.$$

3. Relations among the Spot Interest Rate, the Predictability of Return of the Market Portfolio and its Volatility

Before getting into the details of pricing options, let us examine the relations among the spot interest rate, the stock return and its volatility. Recent studies in partial equilibrium settings have attempted to find the relationship between the stock return predictability and the BS model. For example, Grundy (1991) states that the BS model can be consistent with a mean-reverting stock return. Lo and Wang (1995) cautioned that the predictability of stock returns could provide additional information on the forecast of the volatility in the BS model. With the current model, one can investigate whether the BS model can be consistent with the stock return predictability in equilibrium where the spot interest rate (r), the drift to the stock price (μ_s) and its volatility (V) are all endogenously determined.

As is typical in such an examination, I first present the equivalent martingale price process for the market portfolio. Using the formulas for μ_s, V and r, we can rewrite $\mu_s = r+\beta_2+\alpha_2 \ln \delta = r+V$. This implies that the market risk of $\mathrm{d}z$, defined commonly as $(\mu_s - r)/\sqrt{V}$, is $\sqrt{V}$. The equivalent martingale process for the stock price is[12]

$$\mathrm{d}S = (r-\rho)S\mathrm{d}t + S\sqrt{V}\mathrm{d}z^*,$$

where $\mathrm{d}z^* = \mathrm{d}z + \sqrt{V}\mathrm{d}t$ is the equivalent martingale Wiener process. The equivalent martingale process for the volatility is

$$\mathrm{d}V = [\beta_1\alpha_2 + \alpha_1\beta_2 - (\alpha_1+\alpha_2)V]\,\mathrm{d}t + \alpha_2\sqrt{V}\mathrm{d}z^*,$$

which differs from the actual volatility process in the reversion speed and the long-run mean.

As the usual argument suggests, the actual drift of the stock price (μ_s) does not explicitly enter the option price formula when the equivalent martingale pricing principle is used. However, it would be erroneous to infer that fundamental forces that affect the drift μ_s do not affect the option

[12]The market risk of $\mathrm{d}z$ can be easily verfied under the partial differential equation approach.

prices. As is apparent from the formulas for r and V, parameters $(\rho, \alpha_1, \beta_1, \alpha_2, \beta_2)$ that underlie the predictability affect both the volatility and the spot interest rate simultaneously. This result, which can be obtained only in an equilibrium model where μ_s, V and r are endogenous, is quite different from and much stronger than that in Lo and Wang (1995), who only argue that the stock return predictability can help the forecast on volatility. The effects of the parameters which affect predictability on the interest rate and volatility can be briefly discussed here. First, the rate of time preference, ρ, affects the spot interest rate positively, but has no effect on the market volatility in this context. Second, α_1 not only determines the speed of adjustments for r and V, but also affects the long-run means for r and V. The parameter α_1 affects the long-run mean of r negatively, but affects that of V positively. In addition, parameters $(\beta_1, \alpha_2, \beta_2)$ influence the spot rate and the market volatility differently. For example, an increase in the long-run mean for r could be resulted from increase in β_1 or decrease in either α_2 or β_2. Also, an increase in the long-run mean for V could be resulted from decrease in β_1 or increase in either α_2 or β_2.

Table 1: Summary of Special Cases for Stock Return, Volatility and Spot Interest Rate

	Stochastic Interest rate: SI	Constant Interest Rate: CI
Stochastic Volatility: SV	Generic Case (SVSI): restrictions: $-2\alpha_1 < \alpha_2 < 0$ $V = \beta_2 + \alpha_2 \ln \delta$ $r = \rho + \beta_1 - \alpha_1 \ln \delta - (\beta_2 + \alpha_2 \ln \delta)/2$ $\mu_s = \rho + \beta_1 - \alpha_1 \ln \delta + (\beta_2 + \alpha_2 \ln \delta)/2$	Special Case 1 (SVCI): restriction: $\alpha_2 = -2\alpha_1$ $V = \beta_2 + \alpha_2 \ln \delta$ $r = \rho + \beta_1 - \beta_2/2$ $\mu_s = \rho + \beta_1 + \beta_2/2 - 2\alpha_1 \ln \delta$
Constant Volatility: CV	Special Case 2 (CVSI): restriction: $\alpha_2 = 0$ $V = \beta_2$ $r = \rho + \beta_1 - \alpha_1 \ln \delta - \beta_2/2$ $\mu_s = \rho + \beta_1 - \alpha_1 \ln \delta + \beta_2/2$	Special Case 3 (CVCI): restrictions: $\alpha_2 = \alpha_1 = 0$ $V = \beta_2$ $r = \rho + \beta_1 - \beta_2/2$ $\mu_s = \rho + \beta_1 + \beta_2/2$

Table 1 presents a summary of the relations among r, μ_s and V under the restriction $0 \geq \alpha_2 \geq -2\alpha_1$. In the generic case (SVSI), μ_s, V and r are all stochastic and mean-reverting. There are three special cases. Case 3 (CVCI) corresponds to the BS model, where volatility V and interest rate r are constant. However, in this case the actual drift μ_s should also be constant, implying that the BS model is not consistent with a mean-reverting drift in the current context. This result differs from Grundy (1991) who states that the BS formulas hold for an Ornstein-Uhlenbeck process for the stock return. To accommodate the predictability of the stock return, either the volatility or the spot interest rate or both must be made stochastic and mean-reverting, as in case 1 (SVCI), case 2 (CVSI) and (SVSI), respectively. Incorporating the predictability of the stock return, stochastic spot interest rate and stochastic volatility can also significantly improve the numerical performance of the option model, as we will show in Section 5 later.

The equilibrium approach also illustrates that the two endogenized equivalent martingale processes ($\mathrm{d}S$ and $\mathrm{d}V$) are inherently interdependent. Such interdependence can not be established in any partial equilibrium option model, where the two processes are exogenous. For example, Hull and White (1987) and Stein and Stein (1991) simply assume the two corresponding processes as

$$\mathrm{d}S = rS\mathrm{d}t + \sqrt{V}S\mathrm{d}z_1^*, \qquad \mathrm{d}V = \theta(V)\mathrm{d}t + \phi(V)\mathrm{d}z_2^*,$$

where the drift in $\mathrm{d}V$ is not related to the stock price process ($\mathrm{d}S$). Moreover, the correlation between $\mathrm{d}z_1$ and $\mathrm{d}z_2$ is usually assumed to be zero. In contrast, the endogenized processes for S and V indicate that the drifts of $\mathrm{d}S$ and $\mathrm{d}V$ are associated in a particular way. Also, the correlation between $\mathrm{d}z_1$ and $\mathrm{d}z_2$ depends on the sign of α_2. They are perfectly negatively correlated if $\alpha_2 < 0$. Although the special relation between $\mathrm{d}z_1$ and $\mathrm{d}z_2$ here depends on the one-factor setting, the general message of our exercise should be valid when the model is extended into a multi-factor setting. These requirements suggest that cross-equation restrictions must be imposed on the coefficients when the processes for S and V are to be estimated.

4. Pricing European Options

4.1. Options Written on the Market Portfolio

Consider the European style stock option. The equilibrium price of such a stock option satisfies the Euler equation (2.5). We can explicitly compute (2.5), since the stock price is linear in the dividend which is a function of Y and since the density function of Y_T conditional on (Y_t, t) is known. For a European call written on the risky stock with a striking price K that matures at time T, its price at time $t \leq T$, denoted $C_t(K,T)$, can be written as

$$C_t(K,T) = E_t\left(\frac{U_c(c_T,T)}{U_c(c_t,t)}\max(S_T - K, 0)\right) = e^{-\rho(T-t)} S_t E_t\left(\frac{1}{\delta_T}\max(\delta_T - \rho K, 0)\right).$$

Similarly, for a European put written on the risky stock with a striking price K that matures at time T, its price at time $t \leq T$, denoted $P_t(K,T)$, can be expressed as

$$P_t(K,T) = e^{-\rho(T-t)} S_t E_t\left(\frac{1}{\delta_T}\max(\rho K - \delta_T, 0)\right).$$

The following proposition summarizes the European stock option price formulas for the generic case (SVSI) with the restriction $-2\alpha_1 < \alpha_2 < 0$ (see Appendix B for a proof).[13]

Proposition 4.1. *Under Assumptions 1-2, the equilibrium stock option prices are:*

$$\begin{aligned} C_t(K,T) = {} & S_t e^{-\rho(T-t)} \sum_{j=0}^{\infty} \frac{e^{-\lambda}\lambda^j}{j!} \frac{\gamma(v+j, a(t,T)Y(\rho K))}{\Gamma(v+j)} \\ & - K B_t(T,\delta_t) \sum_{j=0}^{\infty} \frac{e^{-A(t,T)\lambda}(A(t,T)\lambda)^j}{j!} \frac{\gamma(v+j, \frac{a(t,T)}{A(t,T)} Y(\rho K))}{\Gamma(v+j)}, \end{aligned} \tag{4.1}$$

and

$$\begin{aligned} P_t(K,T) = {} & K B_t(T,\delta_t) \sum_{j=0}^{\infty} \frac{e^{-A(t,T)\lambda}(A(t,T)\lambda)^j}{j!} \frac{\Gamma(v+j, \frac{a(t,T)}{A(t,T)} Y(\rho K))}{\Gamma(v+j)} \\ & - S_t e^{-\rho(T-t)} \sum_{j=0}^{\infty} \frac{e^{-\lambda}\lambda^j}{j!} \frac{\Gamma(v+j, a(t,T)Y(\rho K))}{\Gamma(v+j)}. \end{aligned}$$

where $\Gamma(a,x) \equiv \int_x^{\infty} e^{-y} y^{a-1} dy$, $\gamma(a,x) \equiv \int_0^x e^{-y} y^{a-1} dy$ ($\forall\ x > 0$), and $\gamma(a,x) + \Gamma(a,x) \equiv \Gamma(a)$. The call and put prices satisfy the put-call parity condition for European options on assets with a constant dividend yield.

[13]Note that the domain for δ is $\delta \in (0, e^{-\beta_2/\alpha_2})$ under $\alpha_2 < 0$ since $Y = \beta_2 + \alpha_2 \ln \delta > 0$. However, the domain for δ becomes $\delta \in (e^{-\beta_2/\alpha_2}, \infty)$ under $\alpha_2 > 0$. The option price formulas under condition $\alpha_2 > 0$ are presented in the appendices.

Since the stock option prices are functions of parameters (α_1, β_1, α_2, β_2, ρ), I can explicitly examine how each parameter affects the option prices. To economize on space, I examine only the call option written on the risky stock. Similar analysis can be conducted on puts. Comparative statics show that a high ρ results in a low call price. The intuitive explanation is that, since the stock dividend yield in equilibrium equals the rate of time preference, a high dividend yield (ρ) generates a low stock price which in turn induces a low call price. However, the signs of $\frac{\partial C_t}{\partial \alpha_1}$, $\frac{\partial C_t}{\partial \beta_1}$, $\frac{\partial C_t}{\partial \beta_2}$ and $\frac{\partial C_t}{\partial \alpha_2}$ are all ambiguous, which confirm the discussion on the effects of the predictability in Section 3. Changes in any of the four parameters have opposite effects on the long-run mean of r and V. For example, a high β_1 implies a high long-run mean of r and a low long-run mean of V. Since the spot rate and the volatility are pulled toward their long-run means, a higher β_1 is more likely to result in a higher r and a lower V. A higher r alone generates a higher call while a lower V alone corresponds to a low call price, thus the overall effect on the call price is ambiguous. The effects of α_2 or β_2 can be explained in a similar way. The effect of changing α_1 is even more complicated since such changes not only affect the long-run means of r and V differently but also influence the reversion speeds of both.

Finally, the limit behavior of the call option when S_t becomes very large is intuitive. When $S_t \to \infty$, a call option is almost certain to be exercised. The call option becomes very similar to a forward contract with a delivery price K. That is, $C_t(K,T) \to S_t e^{-\rho(T-t)} - KB_t(T)$, which is confirmed by the limit of (4.1).

4.2. Pricing European Bond Options

Consider the price of European style bond options. Denote by $CB_t(k,T,\overline{T})$ ($PB_t(k,T,\overline{T})$) the value at time t of a call (put) option on a discount bond of a maturity date $\overline{T}$, with a striking price k and an expiration date T. It is understood that $t \leq T \leq \overline{T}$. According to (2.5), we have

$$\begin{aligned} CB_t(k,T,\overline{T}) &= e^{-\rho(T-t)}\delta_t E_t\left(\delta_T^{-1} \times \max(B_T(\overline{T},\delta_T) - k, 0)\right), \\ PB_t(k,T,\overline{T}) &= e^{-\rho(T-t)}\delta_t E_t\left(\delta_T^{-1} \times \max(k - B_T(\overline{T},\delta_T), 0)\right). \end{aligned}$$

Since the bond price $B_T(\overline{T}, \delta_T)$ is a function of δ_T, we can compute the bond option prices in the same way as those of the stock options. The explicit formulas under the generic case (SVSI) are stated in the following proposition (see Appendix C for a proof):

Proposition 4.2. *Under Assumptions 1-2, the equilibrium bond option prices are:*

$$CB_t(k,T,\overline{T}) = B_t(\overline{T},\delta_t) \sum_{j=0}^{\infty} \frac{e^{-D(t,T,\overline{T})\lambda}(D(t,T,\overline{T})\lambda)^j}{j!} \frac{\gamma(v+j, \frac{a(t,T)}{D(t,T,\overline{T})} Y(\overline{k}))}{\Gamma(v+j)}$$

$$-kB_t(T, \delta_t) \sum_{j=0}^{\infty} \frac{e^{-A(t,T)\lambda}(A(t,T)\lambda)^j}{j!} \frac{\gamma(v+j, \frac{a(t,T)}{A(t,T)})Y(\overline{k}))}{\Gamma(v+j)},$$

and

$$P_t(K,T,\overline{T}) = kB_t(T; \delta_t) \sum_{j=0}^{\infty} \frac{e^{-A(t,T)\lambda}(A(t,T)\lambda)^j}{j!} \frac{\Gamma(v+j, \frac{a(t,T)}{A(t,T)} Y(\overline{k}))}{\Gamma(v+j)}$$

$$-B_t(\overline{T}, \delta_t) \sum_{j=0}^{\infty} \frac{e^{-D(t,T,\overline{T})\lambda}(D(t,T,\overline{T})\lambda)^j}{j!} \frac{\Gamma(v+j, \frac{a(t,T)}{D(t,T,\overline{T})} Y(\overline{k}))}{\Gamma(v+j)} \tag{4.2}$$

where $\overline{k}$ is so chosen that $B_T(\overline{T},\overline{k}) = k$ and $D(t,T,\overline{T}) = \frac{A(t,\overline{T})}{A(T,\overline{T})}$. The call and put prices satisfy the put-call parity condition.

As one should expect, the call on bond is an increasing function of the bond price and a decreasing function of the striking price. Also, the call price increases with the maturity of the option. The remaining signs of comparative statics are ambiguous. These general features are similar to those of the call on bonds stated in CIR (1985), since the latter model is a special case of the current model.

5. Performance of the Model for Options on the Market Portfolio

In this section, I use numerical examples to examine the performances of the generic case (SVSI, Table 1) where both the spot interest rate and volatility are stochastic and mean-reverting, then investigate the implied volatility pattern in the current model.

5.1. Comparison with the Black-Scholes' Model

Table 2 compares the generic case SVSI with the BS model for European call options for three different maturities, assuming a stock price of $S_t = 100$. The rate of time preference, ρ, is set to be

4% to match the aggregate dividend yield on stock index. To compute the BS call prices, the risk-free rate is set to be the long run real interest rate $r_{BS} = 4\%$ and the volatility is set to match the observed average volatility $\sigma_{BS} = 20\%$. Choosing α_1 to match the estimate by Marsh and Merton (1987) on the speed of adjustment of aggregate dividends gives $\alpha_1 = 0.25$. To determine β_1, we initialize the spot interest rate and instantaneous volatility to be r_{BS} and σ_{BS}. Thus, $\beta_1 = 0.3666$. Further, $\alpha_2 = -0.1029$ and $\beta_2 = 0.1827$ are identified through the condition on the instantaneous volatility and restrictions to ensure positive Y.

Table 2, Figure 1 and Figure 2 here.

Table 2 shows the price differences between the model SVSI and the BS model for call options on the same stock with different striking prices. The generic case SVSI corrects the BS price biases, providing lower prices for the out-of-the-money calls and higher prices for the in-the-money calls for different maturities. Such biases of the BS model become more pronounced as the time to maturity increases or the degree to which the option is in- or out-of-the-money increases. These numerical results are consistent with the empirical study by MacBeth and Merville (1979) for stock options of large companies during 1976. They also confirm the result in Hull and White (1987) for the case where the Wiener process for the stock price and the process for volatility are negatively correlated.

The current model corrects the BS bias because of the mean-reverting feature of r and V and the negative correlation between the stock price and its volatility. To be specific, consider a situation where the rate of dividend is very high, which indicates a low spot rate. The mean-reverting force will push the spot rate up, which induces a high call price. Also a high rate of dividend results in a high stock price which, in turn, implies a low volatility because of the negative correlation between S and V. In this case, it is unlikely for the stock price to change by a large amount. The joint effect is that a high stock price will imply a higher call value in the current model than in a model with constant interest rates and constant volatility. Numerical exercises in Table 3 show that the effect of mean-reverting interest rates dominates the effect of negative correlation between the stock price and its volatility when the stock price is low.

Figure 1 shows that the price difference is positive and the largest for the deep-in-the-money call, which falls monotonically as the call goes less and less in the money. This pattern resembles the recent empirical results of Dumas, Fleming and Whaley (1995) on the S&P 500 index options. When the call is slightly out-the-money, the price difference is almost 0. When the call moves more to out-of-the-money, the price difference is negative and the absolute difference increases. The influence of the time to maturity on absolute price bias is insignificant. The percentage price correction is illustrated in Figure 2. Note that the largest percentage price correction is for the deep-out-of-the-money option. The time to maturity also affects the percentage price bias. The longer the maturity, the larger the percentage price bias for the same striking price.

Table 3, Figure 3 and Figure 4 here.

Table 3 compares the BS model (case 3 CVCI), case 1 (SVCI), and case 2 (CVSI). A SVCI model generates reasonable price corrections for the in-the-money call, but fails to correct the BS price bias for the out-of-the-money call. On the other hand, a CVSI model generates a uniformly lower call prices than the BS model. This is because the stock price and bond price are negatively correlated. In this case, according to Merton (1973), stochastic interest rates reduce the call price in relation to the BS value with constant interest rates. Thus allowing for a stochastic interest rate alone can only produce reasonable prices for the out-of-the-money call.

Comparing the performance of the generic case in Table 2 and those of the other three cases in Table 3, one concludes that both stochastic volatility and stochastic interest rates are necessary to generate reasonable corrections for the bias in the BS model for in- and out-of-the-money calls.

5.2. The Pattern of Implied Volatility

Table 4 presents the implied volatility of the current model, which is calculated through the BS formula using the option prices of the generic case SVSI given in Table 2. Consistent with empirical results, the pattern of the implied volatility with respect to the striking price is a decreasing function of K/S. For the deep-in-the-money calls, the implied volatility is very large and decreases as the call moves to the deep-out-of-the-money, as illustrated in Figure 5. This phenomenon is consistent with the empirical findings of MacBeth and Merville (1979) and Wiggins (1987). Also, a recent empirical study by Dumas, Fleming and Whaley (1995) on S&P 500 index European option finds the same regularity. In addition, for a given maturity, the average of the implied volatility with different striking prices is higher than the instantaneous volatility.

Table 4 and Figure 5 here.

In summary, call prices given by the generic case SVSI match market prices better than the Black-Scholes formula. Important for correcting the BS price biases are the mean-reverting feature of the spot interest rates, the negative correlation between the stock price and its volatility and the negative correlation between the market volatility and the spot interest rate. These features are also necessary for delivering a realistic pattern of implied volatility. Sensitivity analyses show that these results are robust with respect to changes in parameter values.[14]

6. Conclusion

This paper has used an extension of the equilibrium model of Lucas (1978) to study the valuation of options on the market portfolio with stochastic volatility and predictability of stock return. I have investigated the equilibrium relationship between the price of the market portfolio and its volatility, as well as the relationship between the spot interest rate and the market volatility, in an endowment economy. The only uncertainty in this economy is the aggregate dividend whose growth rate follows an affine class of mean-reverting process. The equilibrium results indicate that the predictability of the stock return can be induced by the mean-reverting feature of the growth rate of aggregate dividends. In contrast to previous analyses that employ partial equilibrium settings, here we show that the BS model is not consistent with the predictability of return on the market portfolio when

[14]For lack of space, sensitivity analyses are not presented here but available upon request.

the interest rate and volatility are endogenously generated from the underlying dividend process. Although the actual drift of the stock price does not explicitly enter the option price formula when the equivalent martingale pricing principle is used, fundamental forces that affect the drift **do** affect option prices through their effects on the endogenous interest rate and volatility. It is also shown that there are strong interdependence between the price process and its volatility process for the market portfolio.

Using the Euler equation, I have derived the pricing formulas for the options on the market portfolio which incorporate both stochastic volatility and stochastic interest rate. Since there is only one source of uncertainty, this model preserves the completeness feature for the hedging and risk management purpose. Numerical examples show that the current model performs better than the BS model with realistic parameter values. They suggest that the option valuation should incorporate both stochastic volatility and stochastic interest rates in order to correct the BS pricing bias. Moreover, stochastic volatility and stochastic interest rate are consistent with predictability of stock return.

In addition to providing pricing equations for options on the market portfolio, I have also derived closed-form formulas for European style bond options in a manner that is consistent with the prices of options written on the market portfolio. The Vasicek (1977) model and the CIR (1985) model can be viewed as special cases. These formulas have potential use for future examinations of the term structure and bond option pricing.

References

[1] Amin, K. I. and V. K. Ng, 1993, "Option Valuation with Systematic Stochastic Volatility," *Journal of Finance*, 48, No. 3, 881-910.

[2] Bailey, W. and R. Stulz, 1989, "The Pricing of Stock Index Options in a General Equilibrium Model," *Journal of Financial and Quantitative Analysis*, 24, 1-12.

[3] Bekaert, G. and R. Hodrick, 1992, "Characterizing Predictable Components in Excess Returns on Equity and Foreign Exchange Markets," *Journal of Finance*, 47, 467-509.

[4] Black, F. and M. Scholes, 1973, "The Pricing of Options and Corporate Liabilities," *Journal of Political Economy*, 81, 637-655.

[5] Breen, W., L. Glosten and R. Jagannathan, 1989, "Economic significance of Predictable Variations in Stock Index Returns", *Journal of Finance*, 44, 1177-1189.

[6] Campbell, J. and Y. Hamao, 1992, "Predictable Stock Returns in the United States and Japan: A Study of Long-term Capital Market Integration," *Journal of Finance*, 47, 43-70.

[7] Cox, C. J. and S. A. Ross, 1976, "The Valuation of Options for Alternative Stochastic Process", *Journal of Financial Economics*, 3, 145-166.

[8] Cox, C. J., Ingersoll, J. and S. A. Ross, 1985, "A Theory of the Term Structure of Interest Rates," *Econometrica*, 53, No. 2, 384-407.

[9] Dumas, B., Fleming, J. and R. E. Whaley, 1995, "Implied Volatility Functions: Empirical Tests," *Manuscript*, Duke University.

[10] Dupire, B., 1994, "Pricing with a Smile," *Risk*, 7, No.1, 18-20.

[11] Duffie, D., 1992, Dynamic Asset Pricing Theory, Princeton University Press, Princeton.

[12] Fama, E. and H. Babiak, 1968, "Dividend Policy: An Empirical Analysis," *Journal of the American Statistical Association*, 63, No.4., 1132-1161.

[13] Fama, E. and K. R. French, 1988a, "Permanent and Temporary Components of Stock Market Prices," *Journal of Political Economy*, 96, No. 1-3, 246-273.

[14] Fama, E. and K. R. French, 1988b, "Dividend Yields and Expected Stock Returns," *Journal of Financial Economics*, 22, 3-26.

[15] Ferson, W. and C. Harvey, 1991, "Sources of Predictability in Portfolio Returns," *Financial Analysts Journal* May-June, 49-56.

[16] Grundy, B., 1991, "Option Prices and the Underlying Asset's Return Distribution," *Journal of Finance*, 46, 1045-1069.

[17] Feller, W., 1951, "Two Singular Diffusion Problems," *Annals of Mathematics*, 54, 173-182.

[18] Hull, J., 1993, Options, Futures and other Derivative Securities, Prentice Hall, Englewood Cliffs, New Jersey.

[19] Hull, J. and A. White, 1987, "The Pricing of Options on Assets with Stochastic Volatilities," *Journal of Finance*, 42, 281-300.

[20] Johnson, N. and S. Kotz, 1994, Discrete Distributions, Houghton Mifflin, Boston

[21] Johnson, N. and S. Kotz, 1970, Continuous Univariate Distributions, Houghton Mifflin, New York.

[22] Lauterbach, B. and P. Schultz, 1990, "Pricing Warrants: An Empirical Study of the Black-Scholes Model and its Alternatives," *Journal of Finance*, 45, No. 4, 1181-1210.

[23] Lintner, J., 1956, "Distribution and Incomes of Corporations among Dividends, Retained Earnings and Taxes," *American Economic Review*, 46, 97-113.

[24] Lo, A. W. and J. Wang, 1995, "Implementing Option Pricing Models When Asset Returns Are Predictable", *Journal of Finance*, 50, No.1, 87 - 129.

[25] Lucas, R. E., 1978, "Asset Prices in an Exchange Economy," *Econometrica*, 46, 1429-1445.

[26] MacBeth, J. D. and L. J. Merville, 1979, "An Empirical Examination of the Black-Scholes Call Option Pricing Model," *Journal of Finance*, 34, 1173-1186.

[27] Marsh, T. A. and R. Merton, 1987, "Dividend Behavior for the Aggregate Stock Market," *Journal of Business*, 60, 1-40.

[28] Merton, R., 1971, "Optimum Consumption Rules in a Continuous-Time Model", *Journal of Economic Theory*, 3, 373 - 413.

[29] Merton, R., 1973, "Theory of Rational Option Pricing, " *Bell Journal of Economics and Management Science*, 4, 49, 141-183.

[30] Merton, R., 1976, "Option Pricing when Underlying Stock Returns are Discontinuous," *Journal of Financial Economics*, 3, 125-144.

[31] Naik, V. and M. Lee, 1990, "General Equilibrium Pricing of Options on the Market Portfolio with Discontinuous Returns," *Review of Financial Studies*, 3, No.4, 493-521.

[32] Oldfield, G., R. Rogalski and R. Jarrow, 1977, "An Auto Regressive Jump Process for Common Stock Returns," *Journal of Financial Economics*, 5, 389-418.

[33] Rosenberg, B., 1972, "The Behavior of Random Variables with Non Stationary Variance and the Distribution of Security Prices," *Research Program in Finance* working paper no.11, University of California.

[34] Stein, E. M. and J. C. Stein, 1991, "Stock Price Distributions with Stochastic Volatility: An Analytic Approach," *Review of Financial Studies*, 4, No. 4, 727-752.

[35] Vasicek, O., 1977, "An Equilibrium Characterization of the Term Structure," *Journal of Financial Economics*, 5, 177-188.

[36] Wiggins, J. B., 1987, "Option values under stochastic volatility: Theory and Empirical Estimates," *Journal of Financial Economics*, 19, 351 - 372.

Appendices

A. Proofs of Proposition 2.2 and Corollary 2.3:

A.1. The Price of Pure Discount Bonds

Proof. By equation (2.4) and the specifications in section 2.2, we can express the pure discount price with maturity T at time $t \leq T$ as

$$B_t(T,\delta_t) = e^{\rho t}\delta_t E_t\left(e^{-\rho T}\delta_T^{-1} \times 1\right) = e^{-\rho(T-t)}\delta_t E_t\left(\delta_T^{-1}\right), \quad \forall \ t \in (0,T).$$

Since $Y = \beta_2 + \alpha_2 \ln \delta$, we use the conditional density function for Y_T in (2.8) and parameters defined in (2.9), thus

$$\begin{aligned} B_t(T,\delta_t) &= e^{-\rho(T-t)} e^{\frac{-\beta_2+Y_t}{\alpha_2}} \int_0^\infty e^{\frac{\beta_2-Y_T}{\alpha_2}} a(t,T) e^{-(x+\lambda)} x^{v-1} \sum_{j=0}^{\infty} \frac{(x\lambda)^j}{j!\Gamma(v+j)} \mathrm{d}Y_T \\ &= e^{-\rho(T-t)+\frac{Y_t}{\alpha_2}} \sum_{j=0}^{\infty} \frac{e^{-\lambda}\lambda^j}{j!\Gamma(v+j)} \Gamma(v+j) \left(\frac{a(t,T)\alpha_2}{a(t,T)\alpha_2+1}\right)^{v+j} \\ &= A(t,T)^v e^{-\left(\rho + \frac{A(t,T)e^{-\alpha_1(T-t)}-1}{T-t}\frac{Y_t}{\alpha_2}\right)(T-t)}, \end{aligned}$$

where $A(t,T) = \frac{a(t,T)\alpha_2}{a(t,T)\alpha_2+1}$. ■

A.2. The Instantaneous Interest Rate

Proof. The instantaneous interest rate is defined through $B_t(T,\delta_t) = E_t^*\left(e^{-\int_t^T r(s)ds}\right)$. Thus $r(t) = -\frac{\mathrm{d}\ln B_t(T,\delta_t)}{\mathrm{d}T}\ |_{T=t}$. Since

$$\frac{\mathrm{d}\ln B_t(T,\delta_t)}{\mathrm{d}T} = -\left(\rho + \frac{v\alpha_2}{2}A(t,T)e^{-\alpha_1(T-t)} - \left(\frac{1}{2}+\frac{\alpha_1}{\alpha_2}\right)Y_t A(t,T)^2 e^{-\alpha_1(T-t)}\right),$$

therefore, the spot instantaneous interest rate is

$$r(t) = \rho + \beta_1 - \alpha_1 \ln \delta_t - \frac{1}{2}(\beta_2 + \alpha_2 \ln \delta_t).$$

It is easy to show that

$$E_t[r(T)] = \rho - \frac{v\alpha_2^2}{4\alpha_1} - \frac{1}{2}(2\alpha_1+\alpha_2)e^{-\alpha_1(T-t)}\left(\ln \delta_t - \frac{\beta_1}{\alpha_1}\right).$$

Then, the expected steady state interest rate is $\bar{r} = \lim_{T\to\infty} E_t[r(T)] = \rho - v\alpha_2^2/4\alpha_1$. ■

B. Proof of Proposition 4.1:

Proof. The domain for δ is $\delta \in (0, e^{-\beta_2/\alpha_2})$ for $\alpha_2 < 0$ since $Y = \beta_2 + \alpha_2 \ln \delta > 0$. As stated in section 4.1, the European call option with a striking price K and maturity T at time $t \leq T$ is computed as

$$\begin{aligned} C_t(K,T) &= e^{-\rho(T-t)} S_t E_t \left(\delta_T^{-1} \times \max(\delta_T - \rho K, 0) \right) \\ &= e^{-\rho(T-t)} S_t \mathrm{Prob}(\delta_T \geq \rho K) - e^{-\rho(T-t)} K \rho S_t \int_{\rho K}^{e^{-\beta_2/\alpha_2}} \delta_T^{-1} g(\delta_T \mid \delta_t) \mathrm{d}\delta_T \,. \end{aligned}$$

The call price of the stock is proven as follows:

$$e^{-\rho(T-t)} S_t \mathrm{Prob}(\delta_T \geq \rho K) = e^{-\rho(T-t)} S_t \mathrm{Prob}[Y_T \leq Y(\rho K)]$$

$$= e^{-\rho(T-t)} S_t \textstyle\int_0^{Y(\rho K)} a(t,T) e^{-(x+\lambda)} x^{v-1} \sum_{j=0}^{\infty} \frac{(x\lambda)^j}{j!\Gamma(v+j)} \mathrm{d}Y_T$$

$$= e^{-\rho(T-t)} S_t \textstyle\sum_{j=0}^{\infty} \frac{e^{-\lambda}\lambda^j}{j!} \frac{\gamma(v+j, a(t,T)Y(\rho K))}{\Gamma(v+j)},$$

and

$$e^{-\rho(T-t)} K \rho S_t \textstyle\int_{\rho K}^{e^{-\beta_2/\alpha_2}} \delta_T^{-1} g(\delta_T \mid \delta_t) \mathrm{d}\delta_T$$

$$= K e^{-\rho(T-t) + \frac{-\beta_2 + Y_t}{\alpha_2}} \textstyle\int_0^{Y(\rho K)} e^{\frac{\beta_2 - Y_T}{\alpha_2}} a(t,T) e^{-(x+\lambda)} x^{v-1} \sum_{j=0}^{\infty} \frac{(x\lambda)^j}{j!\Gamma(v+j)} \mathrm{d}Y_T$$

$$= K e^{-\rho(T-t) + \frac{Y_t}{\alpha_2}} \textstyle\sum_{j=0}^{\infty} \frac{e^{-\lambda}\lambda^j}{j!\Gamma(v+j)} \int_0^{Y(\rho K)} a(t,T) e^{-\frac{Y_T}{\alpha_2}} e^{-a(t,T)Y_T} (a(t,T)Y_T)^{v+j-1} \mathrm{d}Y_T$$

$$= K e^{-\rho(T-t) \frac{Y_t}{\alpha_2}} \textstyle\sum_{j=0}^{\infty} \frac{e^{-\lambda}\lambda^j}{j!\Gamma(v+j)} \gamma[v+j, \frac{a(t,T)}{A(t,T)} Y(\rho K)] A(t,T)^{v+j}$$

$$= K B_t(T, \delta_t) \textstyle\sum_{j=0}^{\infty} \frac{e^{-A(t,T)\lambda}(A(t,T)\lambda)^j}{j!} \frac{\gamma(v+j, \frac{a(t,T)}{A(t,T)} Y(\rho K))}{\Gamma(v+j)}. \blacksquare$$

The European put option price can be computed in a similar way as for the call:

$$\begin{aligned} P_t(K,T) &= e^{-\rho(T-t)} K \rho S_t \int_0^{\rho K} \delta_T^{-1} g(\delta_T \mid \delta_t) \mathrm{d}\delta_T - e^{-\rho(T-t)} S_t \mathrm{Prob}(\delta_T \leq \rho K) \,, \quad \forall \quad t \in (0,T) \\ &= K B_t(T, \delta_t) \sum_{j=0}^{\infty} \frac{e^{-A(t,T)\lambda}(A(t,T)\lambda)^j}{j!} \frac{\Gamma(v+j, \frac{a(t,T)}{A(t,T)} Y(\rho K))}{\Gamma(v+j)} \\ &\quad - e^{-\rho(T-t)} S_t \sum_{j=0}^{\infty} \frac{e^{-\lambda}\lambda^j}{j!} \frac{\Gamma(v+j, a(t,T)Y(\rho K))}{\Gamma(v+j)}. \blacksquare \end{aligned}$$

C. Proof of Proposition 4.2:

Proof. Since

$$B_T(\overline{T}, \delta_T) = A(T,\overline{T})^v e^{-\rho(\overline{T}-T) - \frac{Y_T}{\beta_2}(A(T,\overline{T}) e^{-\alpha(\overline{T}-T)} - 1)},$$

we first compute the present value of $B_T(\overline{T}, \delta_T)$. It is easy to show that

$$E_t[PV(B_T(\overline{T}, \delta_T))] = e^{-\rho(T-t)}\delta_t E_t\left(\delta_T^{-1} B_T(\overline{T}, \delta_T)\right) = B_t(\overline{T}, \delta_t).$$

As stated in section 4.1, the value of a European call option on bond, $CB_t(k, T, \overline{T})$, is

$$\begin{aligned} CB_t(k,T,\overline{T}) &= e^{-\rho(T-t)}\delta_t E_t\left(\delta_T^{-1} \times \max(B_T(\overline{T}, \delta_T) - k, 0)\right) \qquad \forall \quad t \leq T \leq \overline{T} \\ &= e^{-\rho(T-t)}\delta_t \int_{\overline{k}}^{e^{-\beta_2/\alpha_2}} \delta_T^{-1}(B_T(\overline{T}, \delta_T) - k) g(\delta_T \mid \delta_t) \mathrm{d}\delta_T, \quad \forall \quad \alpha_2 < 0, \end{aligned}$$

where $\overline{k}$ is chosen so that $B_T(\overline{T}, \overline{k}) = k$. We have

$$\begin{aligned} & e^{-\rho(T-t)}\delta_t \textstyle\int_{\overline{k}}^{e^{-\beta_2/\alpha_2}} \delta_T^{-1} B_T(\overline{T}, \delta_T) g(\delta_T \mid \delta_t) \mathrm{d}\delta_T \\ &= e^{-\rho(T-t)+\frac{-\beta_2+Y_t}{\alpha_2}} \textstyle\int_0^{Y(\overline{k})} A(T,\overline{T})^v e^{\frac{\beta_2}{\alpha_2}-(\rho(\overline{T}-T))-\frac{Y_T}{\alpha_2}A(T,\overline{T})e^{-\alpha(\overline{T}-T)}} f(Y_T \mid Y_t) \mathrm{d}Y_T \\ &= e^{-\rho(\overline{T}-t)+\frac{Y_t}{\alpha_2}} A(T,\overline{T})^v \textstyle\sum_{j=0}^{\infty} \frac{e^{-\lambda}\lambda^j}{j!\Gamma(v+j)} \gamma(v+j, \frac{a(t,T)}{D(t,T,\overline{T})}) D(t,T,\overline{T})^{v+j} \\ &= B_t(\overline{T}, \delta_t) \textstyle\sum_{j=0}^{\infty} \frac{e^{-D(t,T,\overline{T})\lambda}(D(t,T,\overline{T})\lambda)^j}{j!} \frac{\gamma(v+j, \frac{a(t,T)}{D(t,T,\overline{T})} Y(\rho K))}{\Gamma(v+j)}, \end{aligned}$$

where $D(t,T,\overline{T}) = \frac{A(t,\overline{T})}{A(T,\overline{T})}$. It has been shown in Appendix B

$$\begin{aligned} & ke^{-\rho(T-t)}\delta_t \textstyle\int_{\overline{k}}^{e^{-\beta_2/\alpha_2}} \delta_T^{-1} g(\delta_T \mid \delta_t) \mathrm{d}\delta_T \\ &= kB_t(T, \delta_t) \textstyle\sum_{j=0}^{\infty} \frac{e^{-A(t,T)\lambda}(A(t,T)\lambda)^j}{j!} \frac{\gamma(v+j, \frac{a(t,T)}{A(t,T)} Y(\overline{k}))}{\Gamma(v+j)}. \end{aligned}$$

Therefore

$$\begin{aligned} CB_t(k,T,\overline{T}) = B_t(\overline{T}, \delta_t) &\textstyle\sum_{j=0}^{\infty} \frac{e^{-D(t,T,\overline{T})\lambda}(D(t,T,\overline{T})\lambda)^j}{j!} \frac{\gamma(v+j, \frac{a(t,T)}{D(t,T,\overline{T})} Y(\overline{k}))}{\Gamma(v+j)} \\ -kB_t(T, \delta_t) &\textstyle\sum_{j=0}^{\infty} \frac{e^{-A(t,T)\lambda}(A(t,T)\lambda)^j}{j!} \frac{\gamma(v+j, \frac{a(t,T)}{A(t,T)} Y(\overline{k}))}{\Gamma(v+j)}. \end{aligned}$$

Similarly, we can show that, for $\alpha_2 < 0$, the value of a European put option on bond is

$$\begin{aligned} P_t(K,T,\overline{T}) = kB_t(T, \delta_t) &\textstyle\sum_{j=0}^{\infty} \frac{e^{-A(t,T)\lambda}(A(t,T)\lambda)^j}{j!} \frac{\Gamma(v+j, \frac{a(t,T)}{A(t,T)} Y(\overline{k}))}{\Gamma(v+j)} \\ -B_t(\overline{T}, \delta_t) &\textstyle\sum_{j=0}^{\infty} \frac{e^{-D(t,T,\overline{T})\lambda}(D(t,T,\overline{T})\lambda)^j}{j!} \frac{\Gamma(v+j, \frac{a(t,T)}{D(t,T,\overline{T})} Y(\overline{k}))}{\Gamma(v+j)}. \quad \blacksquare \end{aligned}$$

D. The Option Valuation with $\alpha_2 > 0$

D.1. Call and Put Options on Stock

The domain for δ under $\alpha_2 > 0$ is $\delta \in (e^{-\beta\ 2/\alpha_2}, \infty)$. We can compute the call and put on the risky stock in the similar way as we do in Appendix B.

$$\begin{aligned} C_t(K,T) &= e^{-\rho(T-t)} S_t E_t\left(\delta_T^{-1} \times \max(\delta_T - \rho K, 0)\right) \\ &= e^{-\rho(T-t)} S_t \mathrm{Prob}(\delta_T \geq \rho K) - e^{-\rho(T-t)} K \rho S_t \int_{\rho K}^{\infty} \delta_T^{-1} g(\delta_T \mid \delta_t) \mathrm{d}\delta_T . \end{aligned}$$

Tedious exercises show that

$$C_t(K,T) = S_t e^{-\rho(T-t)} \sum_{j=0}^{\infty} \frac{e^{-\lambda}\lambda^j}{j!} \frac{\Gamma(v+j, a(t,T)Y(\rho K))}{\Gamma(v+j)}$$

$$-KB_t(T,\delta_t) \sum_{j=0}^{\infty} \frac{e^{-A(t,T)\lambda}(A(t,T)\lambda)^j}{j!} \frac{\Gamma(v+j, \frac{a(t,T)}{A(t,T)}Y(\rho K))}{\Gamma(v+j)} .$$

Similarly, the European put option with a striking price K and maturity T at time $t \leq T$ is

$$P_t(K,T) = KB_t(T,\delta_t) \sum_{j=0}^{\infty} \frac{e^{-A(t,T)\lambda}(A(t,T)\lambda)^j}{j!} \frac{\gamma[v+j, \frac{a(t,T)}{A(t,T)}Y(\rho K)]}{\Gamma(v+j)}$$

$$-S_t e^{-\rho(T-t)} \sum_{j=0}^{\infty} \frac{e^{-\lambda}\lambda^j}{j!} \frac{\gamma(v+j, a(t,T)Y(\rho K))}{\Gamma(v+j)}$$

D.2. Call and Put Options on Bond

For $\alpha_2 > 0$, the value of a European call option on bond, $CB_t(k,T,\overline{T})$, is

$$CB_t(k,T,\overline{T}) = e^{-\rho(T-t)} \delta_t \int_{\bar{k}}^{\infty} \delta_T^{-1}(B_T(\overline{T},\delta_T) - k) g(\delta_T \mid \delta_t),$$

Tedious exercises give us

$$CB_t(k,T,\overline{T}) = B_t(\overline{T},\delta_t) \sum_{j=0}^{\infty} \frac{e^{-D(t,T,\overline{T})\lambda}(D(t,T,\overline{T})\lambda)^j}{j!} \frac{\Gamma(v+j, \frac{a(t,T)}{D(t,T,\overline{T})}Y(\bar{k}))}{\Gamma(v+j)}$$

$$-kB_t(T,\delta_t) \sum_{j=0}^{\infty} \frac{e^{-A(t,T)\lambda}(A(t,T)\lambda)^j}{j!} \frac{\Gamma(v+j, \frac{a(t,T)}{A(t,T)}Y(\bar{k}))}{\Gamma(v+j)} .$$

Similarly, we can show that, for $\alpha_2 > 0$, the value of a European put option on bond is

$$P_t(K,T,\overline{T}) = kB_t(T,\delta_t) \sum_{j=0}^{\infty} \frac{e^{-A(t,T)\lambda}(A(t,T)\lambda)^j}{j!} \frac{\gamma(v+j, \frac{a(t,T)}{A(t,T)}Y(\bar{k}))}{\Gamma(v+j)}$$

$$-B_t(\overline{T},\delta_t) \sum_{j=0}^{\infty} \frac{e^{-D(t,T,\overline{T})\lambda}(D(t,T,\overline{T})\lambda)^j}{j!} \frac{\gamma(v+j, \frac{a(t,T)}{D(t,T,\overline{T})}Y(\bar{k}))}{\Gamma(v+j)} .$$

Table 2

Comparison between the General Case SVSI and the Black-Scholes Model
Option Parameters: S = 100, $\alpha 1 = 0.25$, $\beta 1 = 0.3666$, $\alpha 2 = -0.1029$, $\beta 2 = 0.1827$, $\rho = 0.04$.

	T = 1 months			T = 6 months			T = 12 months		
K/S	SVSI c	B-S C	% Correction (c-C)/C	SVSI c	B-S C	% Correction (c-C)/C	SVSI c	B-S C	% Correction (c-C)/C
0.75	26.5608	24.9181	6.59%	28.2181	24.6373	14.53%	30.9969	24.6864	25.56%
0.80	21.2530	19.9349	6.61%	22.8966	19.9474	14.78%	25.6596	20.4626	25.40%
0.85	15.9525	14.9554	6.67%	17.8514	15.5362	14.90%	20.6686	16.5912	24.58%
0.90	10.7354	10.0413	6.91%	13.2393	11.5759	14.37%	16.1212	13.1513	22.58%
0.95	5.9667	5.5596	7.32%	9.2311	8.2194	12.31%	12.1061	10.1919	18.78%
1.00	2.4257	2.2960	5.65%	5.9703	5.5511	7.55%	8.6907	7.7263	12.48%
1.05	0.6268	0.6543	-4.20%	3.5284	3.5656	-1.04%	5.9106	5.7350	3.06%
1.10	0.0888	0.1243	-28.56%	1.8730	2.1810	-14.12%	3.7622	4.1730	-9.84%
1.15	0.0059	0.0158	-62.66%	0.8739	1.2735	-31.38%	2.2032	2.9806	-26.08%
1.20	0.0002	0.0014	-85.71%	0.3468	0.7118	-51.28%	1.1578	2.0928	-44.68%
1.25	0.0000	0.0001	-98.49%	0.1106	0.3821	-71.05%	0.5256	1.4465	-63.66%

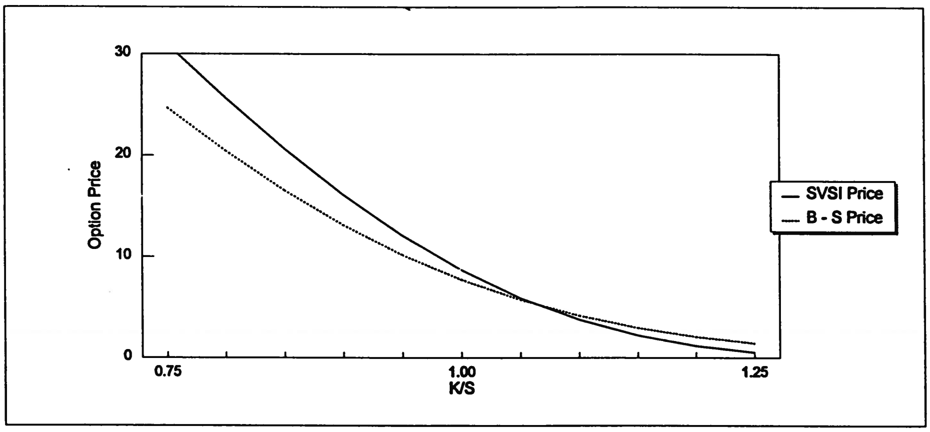

Figure 1: Price Correction, T=12 months

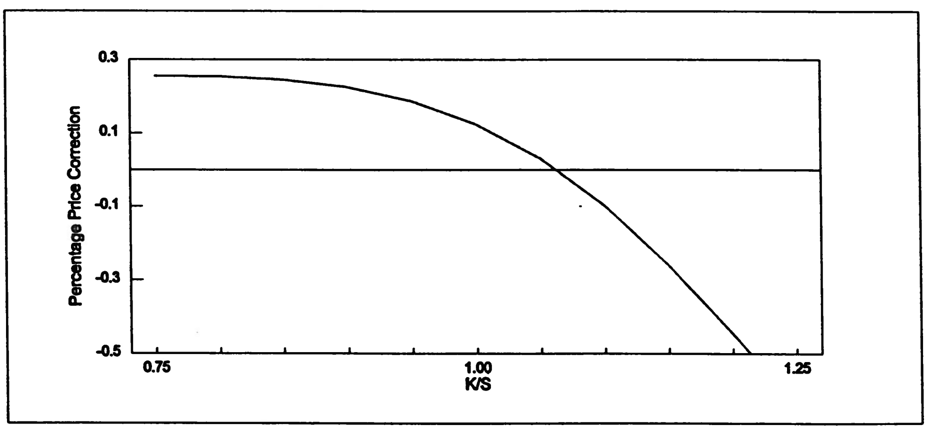

Figure 2: Percentage Price Correction, T = 12 months

Table 3

Comparison between the Special Cases and the Black-Scholes Model
Option Parameters: S = 100, $\alpha 1 = 0.25$, $\beta 1 = 0.3666$, $\rho = 0.04$.
For SVCI: $\alpha 2 = -2\alpha 1$, $\beta 2 = 2\beta 1$.
For CVSI: $\alpha 2 = 0$, $\beta 2 = 0.04$.

	T = 1 months			T = 6 months			T = 12 months		
	SVCI	CVSI	B-S	SVCI	CVSI	B-S	SVCI	CVSI	B-S
K/S	c	c	C	c	c	C	c		C
0.75	24.9183	24.9181	24.9181	25.2259	24.6061	24.6373	25.8826	24.4771	24.6864
0.80	19.9430	19.9349	19.9349	20.7169	19.8699	19.9474	21.6238	20.1049	20.4626
0.85	15.0003	14.9550	14.9554	16.3715	15.3873	15.5362	17.4756	16.0680	16.5912
0.90	10.1839	10.0372	10.0413	12.2398	11.3453	11.5759	13.4514	12.4747	13.1513
0.95	5.7568	5.5443	5.5596	8.3813	7.9223	8.2194	9.5657	9.4016	10.1919
1.00	2.2570	2.2724	2.2960	4.8663	5.2237	5.5511	5.8335	6.8793	7.7263
1.05	0.3402	0.6374	0.6543	1.7762	3.2502	3.5656	2.2703	4.8912	5.7350
1.10	0.0000	0.1181	0.1243	0.0000	1.9106	2.1810	0.0000	3.3838	4.1730
1.15	0.0000	0.0145	0.0158	0.0000	1.0637	1.2735	0.0000	2.2815	2.9806
1.20	0.0000	0.0012	0.0014	0.0000	0.5627	0.7118	0.0000	1.5020	2.0928
1.25	0.0000	0.0001	0.0001	0.0000	0.2839	0.3821	0.0000	0.9672	1.4465

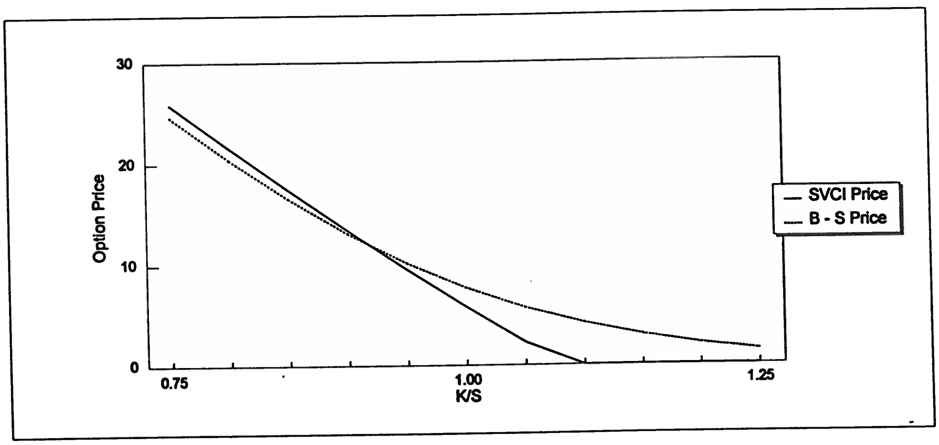

Figure 3: Price Correction for SVCI, T=12 months

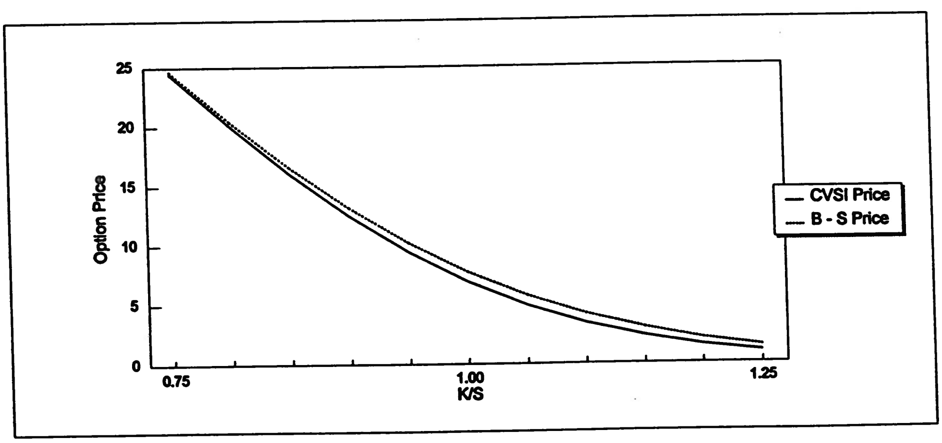

Figure 4: Price Correction for CVSI, T = 12 months

TAble 4

Implied Volatilities Calculated by Black-Scholes Formula from the Call Prices
Given by the General Case SVSI in Table 2

K/S	T = 1 month Implied Volatility	T = 6 month Implied Volatility	T = 12 month Implied Volatility
0.75	0.9245	0.5117	0.4897
0.80	0.7167	0.4135	0.4077
0.85	0.5236	0.3351	0.3417
0.90	0.3547	0.2781	0.2909
0.95	0.2491	0.2400	0.2531
1.00	0.2108	0.2153	0.2253
1.05	0.1965	0.1987	0.2046
1.10	0.1877	0.1866	0.1885
1.15	0.1812	0.1768	0.1751
1.20	0.1789	0.1679	0.1636
1.25	0.1701	0.1592	0.1520
Average Implied Volatiltiy	0.3540	0.2621	0.2629

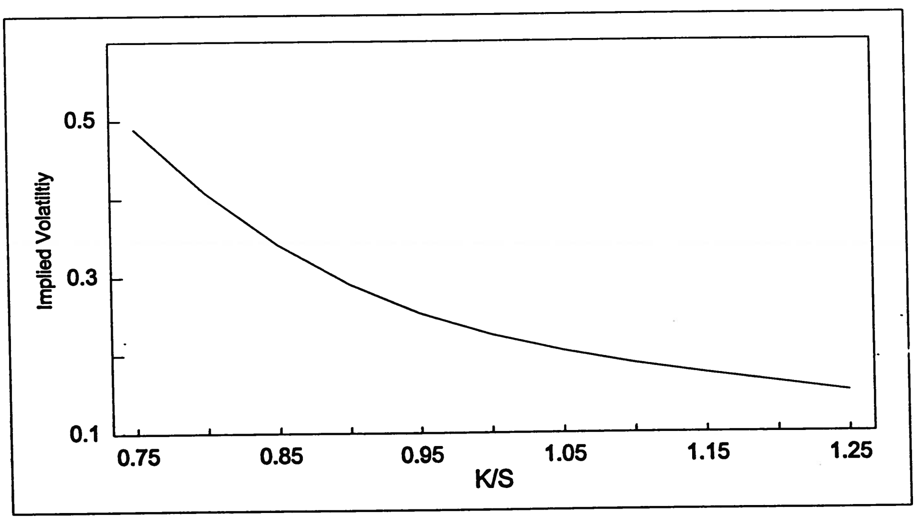

Figure 5: Impled Volatility for T = 12 month