Queen's Economics Department Working Paper No. 939

The Existence of Equilibrium in a Financial Market with General Personal and Corporate Tax Structures

Xing Jin Frank Milne Xing Jin Frank Milne

Department of Economics
Queen's University
94 University Avenue
Kingston, Ontario, Canada
K7L 3N6

11-1996

The Existence of Equilibrium in a Financial Market with General Personal and Corporate Tax Structures

Xing Jin[1]

Institute of Systems Science, Academia Sinica, Beijing 100080, China

Frank Milne[2]

Department of Economics, Queen's University, Kingston, Ontario, Canada

Abstract

This article concerns the existence of equilibrium in a two-period model with general personal and corporate tax structures. We show that an equilibrium exists if there is a price system under which no consumer and firm has an arbitrage opportunity. The model can be modified to handle non convex tax structures and capital gains.

1. Introduction

There is an extensive literature addressing the role of taxation in competitive financial markets. One of the difficult issues is to construct a model which is sufficiently general to deal with the complexities of the tax law, and yet remain tractable. There are many papers that consider the implications for asset prices or asset allocations in specialised models(for a small sample see:Constantinedes (1983), Dammon and Green(1987), Dybvig and Ross(1986), Green(1993), Zechner(1990)).

Before discussing the properties of an equilibrium it is important that there are sufficient restrictions on the economy to imply the existence of an equilibrium. This is important if agents face different tax functions that allow tax arbitrage possibilities. This issue has been addressed in a two period exchange economy by Dammon and Green(1987) and Jones and Milne(1992). Dammon and Green(1987) consider restrictions on tax functions to eliminate arbitrage and define a set of arbitrage-free asset prices. Their paper considered tax functions of considerable generality and exploited the theory of recession(asympototic) cones to determine arbitrage free prices. Jones and Milne(1992) argued that Dammon and Green(1987) did not include the government sector explicitly, ignoring the feasibility constraints implicit in the government budget constraint. Once these constraints were introduced and recognised by the agents, there were natural bounds on the possible asset trades consistent with tax arbitrage. These restrictions allowed more general tax functions to be consistent with equilibrium. Of course the differences in the two models could be resolved by assuming that Dammon and Green's tax functions included the implicit tax rules that come into play as soon as large tax arbitrages were

[01] The support of National Natural Science Foundation of China is gratefully acknowledged by Xing Jin.

[02] The support of SSHRC of Canada is gratefully acknowledged by Frank Milne.

claimed from the government by consumers.

In this paper we extend the ideas in Dammon and Green(1987), and Jones and Milne(1992) to a more general economy:we assume an explicit government sector, productive firms, spot commodity markets, and the possibility of non-convexities in the tax functions. The aim of the paper is to show how the basic ideas can be extended in a number of realistic directions. In particular the introduction of firms allows us to accomodate models(eg. Zechner(1990)) that discuss the interaction of personal and corporate taxation and its impact on corporate financial structure. The introduction of government is more general than the modelling in Jones and Milne (1992). We show that the introduction of spot commodity markets is easily accomodated(For simplicity we assume only one commodity is traded in the second period, but this is an expositional restriction that can be relaxed). Finally we show that tax law often allows for subsides or thresholds that imply non-convexities in the tax functions. In general this can imply the non-existence of competitive equilibrium. Exploiting ideas from General Equilibrium theory(see Heller and Starr (1976), or for an excellent text-book discussion see Ellickson(1993)) we prove the existence of an approximate equilibrium, where the approximation is bounded by the deviation from the convexified economy. We show that even though the tax functions may be non-convex we can obtain weak restrictions on the tax functions that will eliminate arbitrage opportunities. As most tax non-convexities occur at the lower income levels it should be reasonable to assume that the approximate equilibrium is a plausible representation of the true economy.

The plan of the paper is as follows:in Section 2 we set out the model; in Section 3 we state the assumptions and prove some preliminary lemmas on arbitrage free asset prices; in Section 4 we prove the existence of an equilibrium for the case of convex tax functions; in Section 5 we prove the existence of an approximate equilibrium for non-convex tax functions. Finally in the conclusion we indicate how the model may be extended in future research.

2. The Model

There are two dates, $t = 0, 1$; l physical commodities at time 0 and one commodity at time 1; and finite states of the world $\mathbf{S} = \{1, \cdots, S\}$ when uncertainty is resolved at $t = 1$. All economic agents in the economy observe the states, so there is symmetric information.

We assume a finite number of firms $\mathbf{J} = \{1, \cdots, J\}$. The firm $j \in \mathbf{J}$ has initial holding, $\bar{\beta}_j^0$, of some securities, which gives the firm a positive initial cash flow. The firm makes input of production and financial decisions at $t = 0$, facing competitive markets for commodities and assets and pays tax. At date $t = 1$, uncertainty is resolved and the firm must pay its security holders and taxation, from the revenue derived from assets and production. In general the firm's taxation function will discriminate across assets, combinations of assets and scale of the firm's portfolio.

In addition to firms, we have a finite set of consumers $\mathbf{I} = \{1, \cdots, I\}$. Each consumer has a consumption set and preferences over the commodities at time 0 and the contingent commodities at $t = 1$. At $t = 0$ the consumer is endowed with some of the first date commodities, $\bar{x}_i^0$ and initial holding of securities, $\bar{\alpha}_i^0$, and(possibly) pure profits from firms. The endowment is traded for assets on the competitive asset markets and used to consume and to pay the tax. At $t = 1$, uncertainty is resolved and the consumer receives asset returns net of personal tax. The consumer faces a general personal tax function over asset returns that discriminates over assets and combinations of assets.

The government is also included as part of the economy. Rather than provide a detailed analysis of the operations of the government, we will place weak restrictions on government activity. For simplicity assume that the government has net resources $\bar{x}_G^0 \in R_+^l$ at time 0 and $\bar{x}_G^s \in R_+$ at time 1 and state s, sets tax rates ex ante, and then consumes x_G at time 0. Since the economy ends at time 1, the government constrains the tax rebates, in other words, the government cannot promise infeasible tax rebates to consumers and firms. We call this a No Ponzi Game(NPG) condition[3]. We stress that this constraint is a weak bound and that more restrictive conditions could be introduced by appealing to tax laws or financial regulations.

There are N financial assets, indexed by n, which are characterized by their state-contingent payoffs $C_{n,s}, s \in \mathbf{S}$. We assume that these payoffs are in units of a single, numeraire commodity. And for asset $n(n = 1, \cdots, N)$, part of its payoff, $\gamma_{n,s}C_{n,s}$, contributes to taxable income, where $\gamma_{n,s} \in [0, 1]$. When $\gamma_{n,s} = 1, \forall s \in \mathbf{S}$ or $\gamma_{n,s} = 0, \forall s \in \mathbf{S}$, asset n is called fully taxable or tax exempt. We assume that one of the assets is a riskless bond.

We are now in a position to present the formal model.

The firm j has an underlying production technology $Y_j \subseteq (-R_+^l) \times R_+^S$. At time 0, firm j chooses a input plan $y_j^0 \in -R_+^l$ and an asset portfolio $\beta_j = (\beta_{j,1}, \cdots, \beta_{j,N})^T \in R^N$ (where T is the transpose transformation) with the usual sign convention for issuing of assets(positive) and holding of assets(negative). Then it pays tax $T_{j,0}^{(2)}[p^S(\bar{\beta}_j^0 + \beta_j) + p^C y_j^0]$, where p^C and p^S are prices of commodities and prices of assets respectively.

Now, at date 1 and state s, the firm obtains contingent revenue from its output $y_j^s \in R_+$ and net holdings of assets, $-C_s\beta_j$, where $C_s = (C_{1,s}, \cdots, C_{N,s})$. This revenue is taxed via a general tax function $T_{j,s}^{(2)}(y_j^s - \bar{C}_s\beta_j)$, where $\bar{C}_s = (\gamma_{1,s}C_{1,s}, \cdots, \gamma_{n,s}C_{n,s})$. We assume that $T_{j,0}^{(2)}(\cdot)$ and $T_{j,s}^{(2)}(\cdot)(s \in \mathbf{S})$ are continuous and convex with $T_{j,0}^{(1)}(0) = 0$ and $T_{j,s}^{(2)}(0) = 0$.

Finally, given price $p = (p^C, p^S) = (p_1^C, \cdots, p_l^C, p_1^S, \cdots, p_N^S) \subset R_+^{l+N}$ for the commodities and assets at $t = 0$, we can formulate firm j's value maximizing problem as:

[03] The No Ponzi Game condition has been introduced in macroeconomic models to eliminate unbounded borrowing positions by consumers and/or goverments. See Blanchard and Fischer(1989).

$$\max_{(y_j,\beta_j)\in F_j} \gamma_j(p,y_j,\beta_j) = p^S(\beta_j+\bar{\beta}_j^0)+p^C y_j^0 - T_{j,0}^{(2)}[p^S(\beta_j+\bar{\beta}_j^0)+p^C y_j^0], \qquad (*)$$

where $y_j = (y_j^0, y_j^1, \cdots, y_j^S)$ and F_j is set of all (y_j, β_j) satisfying

$$y_j \in Y_j,\ \ y_j^s - C_s\beta_j \geq T_{j,s}^{(2)}(y_j^s - \bar{C}_s\beta_j),\ \ \ s \in \mathbf{S}.$$

The above inequality is the condition of no unanticipated bankruptcy of of firm j.

We have assumed tax functions of similar form to Dammon and Green(1987) in allowing for different proportional state contingent rates on different asset returns. Our functions do not allow for capital gains taxation, but can be modified and extended in the same manner as Dammon and Green's extension(their Section V) to taxes that depend upon prices. For expositional clarity we omit this generality and refer the reader to Dammon and Green(1987) for further discussion.

Consider consumer $i \in \mathbf{I}$. the consumer has a contingent consumption set $X_i = X_i^0 \times X_i^1 \subseteq R_+^{l+S}$ ($X_i^0 \subseteq R_+^l, X_i^1 \subseteq R_+^S$), and preferences defined over $X_i^0 \times X_i^1$, represented by a utility function $U_i : X_i \longrightarrow R$. At time 0, the consumer chooses an asset portfolio $\alpha_i = (\alpha_{i,1}, \cdots, \alpha_{i,N})^T$(with sign convention for long position(positive) and short position(negative)) and a consumption plan x_i, paying taxes according to the tax function $T_{i,0}^{(1)}(\cdot)$, which is continuous and convex with $T_{i,0}^{(1)}(0) = 0$(It is trivial to add income from the commodity and asset endowments, and we omit them as arguments of the tax function.). When asset returns are realized at $t = 1$, the consumer pays personal tax on the proceeds and then, consumes. The consumer's tax function, at state s, is summarized by $T_{i,s}^{(1)}(\bar{C}_s\alpha_i)$, which is continuous and convex with $T_{i,s}^{(1)}(0) = 0$.

At date $t = 0$ the consumer has wealth:

$$W_i = p^C\bar{x}_i^0 + p^S\bar{\alpha}_i^0 + \sum_j \theta_{i,j}\gamma_j(p,y_j,\beta_j),$$

where $\theta_{i,j} \geq 0$ and $\sum_i \theta_{i,j} = 1$, for all j is consumer i's share in the present value of firm j. So i's consumption and portfolio are constrained by:

$$p^S\alpha_i + p^C x_i + T_{i,0}^{(1)}(p^S\alpha_i) \leq W_i. \qquad (2.1)$$

At date $t = 1$, the consumer's contingent consumption $x_i' = (x_i^1, \cdots, x_i^S)$ is constrained by:

$$x_i^s = C_s\alpha_i - T_{i,s}^{(1)}(\bar{C}_s\alpha_i),\ \ \ s \in \mathbf{S}. \qquad (2.2)$$

In summary, the consumer i's problem is:

$$\max_{(x_i,x_i')\in E_i} U_i(x_i, x_i'), \qquad (**)$$

where E_i is set of $(x_i, x_i') = (x_i, x_i^1, \cdots, x_i^S)$ which satisfies (2.1) and (2.2). Let $\tilde{U}_i(x_i, \alpha_i) = U_i[x_i, C_1\alpha_i - T_{i,1}^{(1)}(\bar{C}_1\alpha_i), \cdots, C_S\alpha_i - T_{i,S}^{(1)}(\bar{C}_S\alpha_i)]$, then $(**)$ is equivalent to the following problem:

$$\max_{(x_i,\alpha_i)\in E_i'} \tilde{U}_i(x_i, \alpha_i), \qquad (**)'$$

where E_i' is the set of feasible consumption-portfolio plans (x_i, α_i) of consumer i, which satisfies (2.1) and $C_s\alpha_i - T_{i,s}^{(1)}(\bar{C}_s\alpha_i) \geq 0, \forall s \in \mathbf{S}$.

Suppose that the government has preferences over its consumption set $X_G = R_+^l$, represented by a utility function U_G. At time 0, it spends its income which comes from its endowment $\bar{x}_G^0$ and tax revenue $\sum_i T_{i,0}^{(1)}(p^S\alpha_i) + \sum_j T_{j,0}^{(2)}[p^S(\bar{\beta}_j^0 + \beta_j) + p^C y_j^0]$; and, at time 1, the government has a future contingent budget constraint(a NPG) condition:

$$\bar{x}_G^s + \sum_i T_{i,s}^{(1)}(\bar{C}_s\alpha_i) + \sum_j T_{j,s}^{(2)}(y_j^s - \bar{C}_s\beta_j) > 0. \qquad (2.3)$$

That is, the government does not react to attempts to drain government resources through tax-avoidance measures. It can be seen later that (2.3) holds for sufficiently large $\bar{x}_G^s$.

The government's problem can be stated as:

$$\max_{x_G\in G} U_G(x_G), \qquad (***)$$

where G denote the set of government's consumption x_G which satisfies:

$$p^C x_G \leq p^C\bar{x}_G^0 + \sum_i T_{i,0}^{(1)}(p^S\alpha_i) + \sum_j T_{j,0}^{(2)}[p^S(\bar{\beta}_j^0 + \beta_j) + p^C y_j^0]. \qquad (2.4)$$

To conclude the section, we give the definition of competitive equilibrium.

Definition 2.1. A competitive equilibrium with asset taxation is a non-negative vector of price (p^{*C}, p^{*S}) and allocations $\{(x_i^*, \alpha_i^*)$ for all $i \in \mathbf{I}; (y_j^*, \beta_j^*)$ for all $j \in \mathbf{J}; x_G^*\}$ such that:
(i)(x_i^*, α_i^*) solves the consumer problem $(**)'$ for each $i \in \mathbf{I}$;
(ii)(y_j^*, β_j^*) solves $(*)$ for each $j \in \mathbf{J}$;
(iii)x_G^* solves $(***)$;
(iv)$\sum_j y_j^{*0} + \bar{x}_G^0 + \sum_i \bar{x}_i^0 = \sum_i x_i^* + x_G^*$;
(v)$\sum_i \alpha_i^* = \sum_j(\beta_j^* + \bar{\beta}_j^0) + \sum_i \bar{\alpha}_i^0$.

3. Tax Arbitrage and Assumptions

The purpose of this sectiion is to give a definition of no-tax-arbitrage prices, some basic assumptions and two basic propositions on no-tax-arbitrage.

Definition 3.1. The price vector p^S of assets is a "no-tax-arbitrage" price vector for consumer i(firm j)if and only if for each $\alpha_i^0(\beta_j^0)$ satisfying

$$C_s(\alpha_i + \alpha_i^0) - T_{i,s}^{(1)}[\bar{C}_s(\alpha_i + \alpha_i^0)] \geq C_s\alpha_i - T_{i,s}^{(1)}(\bar{C}_s\alpha_i) \tag{3.1}$$

for all α_i and all $s \in \mathbf{S}$ and there exists at least one α_i and one state s such that (3.1) holds with strict inequality;

$$(y_j^s - C_s(\beta_j + \beta_j^0) - T_{j,s}^{(2)}[y_j^s - \bar{C}_s(\beta_j + \beta_j^0)] \geq y_j^s - C_s\beta_j - T_{j,s}^{(2)}(y_j^s - \bar{C}_s\beta_j) \tag{3.2}$$

for some $y_j \in Y_j$ and all $\beta_j((y_j, \beta_j) \in F_j)$ and all $s \in \mathbf{S}$, and there exists at least one (y_j, β_j) and one state s such that (3.2) holds with strict inequality.) then $p^S\alpha_i^0 > 0(p^S\beta_j^0 < 0)$.

This definition captures the notion that, starting from α_i, there is a feasible direction of improvement that consumer i would wish to exploit. This is called, in Ross(1987), a local arbitrage opportunity at α_i of consumer i. Likewise, (3.2) is a local arbitrage opportunity of firm j. This arbitrage may depend on the investor's position; there may be positions held by no one from which this arbitrage would be available in equilibrium. And, moreover, this arbitrage depends upon, as shown in Dammon and Green(1987), investor's tax functions.

Regarding consumer i, if α_i satisfies (3.1) with strict inequality and if $T_{i,s}^{(1)}(\cdot)$ is differentiable, then

$$C_s\alpha_i^0 - \frac{dT^{(1)}(\bar{C}_s\alpha_i)}{dx}\bar{C}_s\alpha_i^0 > 0. \tag{3.3}$$

Therefore, from (3.3) and Assumption 2, it can be deduced that if all assets are fully taxable, that is, $\gamma_{n,s} = 1, \forall n, s$, or $\gamma_{n,s} = \gamma_s, \forall n, s$, then consumer i has an infinite arbitrage direction if and only if $C_s\alpha_i^0 > 0$. Thus, consummer i has an infinite arbitrage opportunity starting from any position. This arbitrage is inconsistenct with equilibrium, and the arbitrage opportunity is independent of the investor's tax functions.

Let $\mathbf{N}_i^{(1)}$ be the set of no-tax-arbitrage prices for consumer i, $\mathbf{N}_j^{(2)}$ denote the set of no-tax-arbitrage prices of firm j.

Assumption 1: The set of no-tax-arbitrage prices for each consumer and each firm is nonempty, that is, $\mathbf{N} = (\cap_{i\in\mathbf{I}}\mathbf{N}_i^{(1)}) \cap (\cap_{j\in\mathbf{J}}\mathbf{N}_j^{(2)}) \neq \emptyset$.

To guarantee the positivity of no-tax-arbitrage prices, we introduce assumptions on tax functions.

Assumption 2: For all $i \in \mathbf{I}$, all $j \in \mathbf{J}$ and all $s \in \mathbf{S}$, $T_{i,s}^{(1)}(\cdot)$, $T_{j,s}^{(2)}(\cdot)$ are differentiable and $0 \leq \inf_{x\in R}\frac{dT_{i,s}^{(1)}(x)}{dx} \leq \sup_{x\in R}\frac{dT_{i,s}^{(1)}(x)}{dx} < 1$, $0 \leq \inf_{x\in R}\frac{dT_{j,s}^{(2)}(x)}{dx} \leq \sup_{x\in R}\frac{dT_{j,s}^{(2)}(x)}{dx} < 1$.

Assumption 3: For each asset n and each state s, $C_{n,s} > 0$.

Proposition 3.1: Under Assumptions 2 and 3, each no-tax-arbitrage price $p^S = (p_1^S, \cdots, p_N^S) \in \mathbf{N}$ is positive, that is, $p_n^S > 0$, $n = 1, \cdots, N$.

Proof. From Assumption 3, $C_{n,s} > 0$ for all $s \in \mathbf{S}$ and all n. Let $\alpha^0 = (1, 0, \cdots, 0)$. We now show α^0 satisfies (3.1).

By convexity of $T^{(1)}_{1,s}$ and Assumption 2, for each α_1 and $s \in \mathbf{S}$,

$$\begin{aligned}
&C_s(\alpha_1 + \alpha^0) - T^{(1)}_{1,s}[\bar{C}_s(\alpha_1 + \alpha^0)] \\
&\geq C_s(\alpha_1 + \alpha^0) - T^{(1)}_{i,s}(\bar{C}_s\alpha_1 + C_s\alpha^0) \\
&\geq C_s\alpha_1 - T^{(1)}_{1,s}(\bar{C}_s\alpha_1) + C_s\alpha^0\Big(1 - \tfrac{dT^{(1)}_{1,s}(x)}{dx}\Big|_{x=\bar{C}_s\alpha_1 + C_s\alpha^0}\Big) \\
&> C_s\alpha_1 - T^{(1)}_{1,s}(\bar{C}_s\alpha_1).
\end{aligned}$$

Therefore, for each no-tax-arbitrage price $p^S \in \mathbf{N}$, $p^S\alpha^0_1 = p^S_1 > 0$. Similarly, we can show $p^S_n > 0 (n = 2, \cdots, N)$.□

For consumer i, let $W^{(1)}_i$ denote the set of α^0_i satisfying (3.1), $\overline{W^{(1)}_i}$ be the set of all $\alpha^{0'}_i$ which satisfies

$$C_s(\alpha_i + \alpha^{0'}_i) - T^{(1)}_{i,s}[\bar{C}_s(\alpha_i + \alpha^{0'}_i)] \equiv C_s\alpha_i - T^{(1)}_{i,s}(\bar{C}_s\alpha_i)$$

for all α_i and $s \in \mathbf{S}$.

Likewise, for firm j, let $W^{(2)}_j$ denote the set of β^0_j satisfying (3.2), $\overline{W^{(2)}_j}$ be the set of all $\beta^{0'}_j$ which satisfies

$$y^s_j - C_s(\beta_j + \beta^{0'}_j) - T^{(2)}_{j,s}[y^s_j - \bar{C}_s(\beta_j + \beta^{0'}_j)] \equiv y^s_j - C_s\beta_j - T^{(2)}_{j,s}(y^s_j - \bar{C}_s\beta_j)$$

for all $(y_j, \beta_j) \in F_j$ and $s \in \mathbf{S}$.

We can deduce, by Theorems 8.1, 8.7 and Corollary 8.3.3 of Rockafellar(1970), the sets $R^{(1)}_i = W^{(1)}_i \cup \overline{W^{(1)}_i}$ is the recession cone[4] of convex sets $\{\alpha_i | T^{(1)}_{i,s}(\bar{C}_s\alpha_i) - C_s\alpha_i \leq 0, \forall s \in \mathbf{S}\}$ and $R^{(2)}_j = \{(0, \beta_j) | \beta_j \in W^{(2)}_j \cup \overline{W^{(2)}_j}\}$ is a subset of the recession cone of $\{(y^s_j, \beta) | y^s_j - C_s\beta_j - T^{(2)}_{j,s}(y^s_j - \bar{C}_s\beta_j) \geq 0, \forall s \in \mathbf{S}$, *and* $\exists y^0_j$ *such that* $(y^0_j, y^1_j, \cdots, y^S_j) \in Y_j\}$.

Remark 3.1: When all assets are fully taxable, that is, $\gamma_{n,s} = 1, \forall n, s$, it is not difficult to deduce that $\alpha_i \in W^{(1)}_i$ and $\beta_j \in W^{(2)}_j$ if and only if $C_s\alpha_i \gneqq 0$ and $-C_s\beta_j \gneqq 0$. Therefore, it is deduced again that the arbitrage opportunity is independent of the tax function of any agent. Similarly, if $\gamma_{n,s} = \gamma_s$, for all n, the same conclusion holds.

The following proposition presents another characteristic of no-tax-arbitrage prices.

Proposition 3.2. Under Assumptions 2 and 3, each no-tax-arbitrage price $p^S \in \mathbf{N}$ is zero on the set $W = (\cup_{i \in \mathbf{I}} \overline{W^{(1)}_i}) \cup (\cup_{j \in \mathbf{J}} \overline{W^{(2)}_j})$, that is, $p^S\alpha = 0, \forall \alpha \in W$.

[4] The recession cone, denoted $O^+(C)$ of a subset C in R^m, is a subset of all points x satisfying $C + \lambda x \subseteq C, \forall \lambda \geq 0$.

Proof. Let $\alpha \in W$. Without loss of generality, suppose that $\alpha \in \overline{W_1^{(1)}}$. Then, by the definition of $\overline{W_1^{(1)}}$, for each α_1 and $s \in \mathbf{S}$,

$$C_s(\alpha_1 + \alpha) - T_{i,s}^{(1)}[\bar{C}_s(\alpha_1 + \alpha)] \equiv C_s\alpha_1 - T_{i,s}^{(1)}(\bar{C}_s\alpha_1)$$

Let $\bar{\alpha} = (\epsilon, \cdots, \epsilon)$, where $\epsilon(> 0)$ is sufficiently small.

As in the proof of Proposition 3.1 and by above equality,

$$\begin{aligned}
&C_s[\alpha_1 + (\alpha + \bar{\alpha})] - T_{1,s}^{(1)}\{\bar{C}_s[\alpha_1 + (\alpha + \bar{\alpha})]\} \\
&\geq C_s(\alpha_1 + \alpha) - T_{1,s}^{(1)}[\bar{C}_s(\alpha_1 + \alpha)] + C_s\bar{\alpha}\Big(1 - \frac{dT_{1,s}^{(1)}(x)}{dx}\Big|_{x=\bar{C}_s(\alpha_1+\alpha)+C_s\bar{\alpha}}\Big) \\
&> C_s\alpha_1 + T_{1,s}^{(1)}(\bar{C}_s\alpha_1)
\end{aligned}$$

for all α_1 and all $s \in \mathbf{S}$. Thus, $\alpha + \bar{\alpha} \in W_1^{(1)}$, which implies $p^S(\alpha + \bar{\alpha}) > 0$ and therefore, $p^S\alpha \geq 0$ by letting $\epsilon \longrightarrow 0$.

On the other hand, it follows from the definition of $\overline{W_1^{(1)}}$ that $-\alpha \in \overline{W_1^{(1)}}$ if $\alpha \in \overline{W_1^{(1)}}$. This implies that $p^S(-\alpha) \geq 0$ and therefore, $p^S\alpha = 0$. □

Before concluding this section, we make some standard assumptions about consumption and production.

For consumer i:

Assumption 4: $X_i^0 \in R_+^l$ is closed and convex set;

Assumption 5: U_i is a continuous, concave and strictly increasing function;

Assumption 6: $T_{i,0}^{(1)}$ is differentiable and $0 \leq \inf_{x\in R} \frac{dT_{i,0}^{(1)}(x)}{dx} \leq \sup_{x\in R} \frac{dT_{i,0}^{(1)}(x)}{dx} < 1$;

Assumption 7: $\bar{x}_i^0 > 0, \bar{\alpha}_i^0 \geq 0, \sum_i \bar{x}_i^0 \in R_{++}^l$ and $\sum_i \bar{\alpha}_i^0 \in R_{++}^N$.

For firm j:

Assumption 8: Y_j is a closed and convex set including zero;

Assumption 9: If $(\mathbf{0}, y_j^1, \cdots, y_j^S) \in Y_j$ then $y_j^s = 0, \forall s \in \mathbf{S}$, where $\mathbf{0}$ is a l-dimensional vector whose components are all zeros;

Assumption 10: $\bar{\beta}_j^0 > 0$;

Assumption 11: $T_{j,0}^{(2)}$ is differentiable and $0 \leq \inf_{x\in R} \frac{dT_{j,0}^{(2)}(x)}{dx} \leq \sup_{x\in R} \frac{dT_{j,0}^{(2)}(x)}{dx} < 1$.

For government;

Assumption 12: U_G is a continuous, concave and strictly increasing function;

Assumption 13: $\bar{x}_G^0$ and $\bar{x}_G^s$ are sufficiently large(these will be made explicit below).

4. Existence of Equilibrium

Section 4.1.1. The Main Theorem

This section is devoted to the proof of existence of equilibrium of our model. The technique of proof is very similar to that used in Werner(1987). The main theorem of this paper is as follows:

Theorem 4.1.1. Suppose that the Assumptions 1-13 hold, then there exists an equilibrium $e^* = ((x_i^*, \alpha_i^*)_{i\in\mathbf{I}}, (y_j^*, \beta_j^*)_{j\in\mathbf{J}}, x_G^*, p^*)$: that is, e^* satisfies (i)-(v) of Definition 2.1.

Section 4.1.2. The Proof

To show the existence of equilibrium, as in Arrow and Debreu(1954), we will construct a bounded commodity spot market.

For consumer i, define

$\bar{X}_i^0 = \{x_i \in X_i^0 |$there exists $x_{i'} \in X_{i'}^0$ for each $i' \neq i, y_j \in Y_j$ for each j and $x_G \in X_G$ such that $\sum_i x_i + x_G \leq \sum_j y_j^0 + \bar{x}_i^0 + \bar{x}_G^0\}$.

and, for firm j,

$\bar{Y}_j = \{y_j \in Y_j |$there exists $x_i \in X_i^0$ for each i $y_{j'} \in Y_{j'}$ for each $j' \neq j$ and $x_G \in X_G$ such that $\sum_i x_i + x_G \leq \sum_j y_j^0 + \bar{x}_i^0 + \bar{x}_G^0\}$.

It is not difficult to prove by Assumptions 8 and 9 that the set $\bar{Y}_j$ is bounded and therefore, the set $\bar{X}_i^0$ is bounded. So we can find a cubes C in R^l and C' in R^{l+S} such that C and C' include $\bar{X}_i^0$ and $\bar{Y}_j$ in their interiors respectively. And define $\hat{X}_i^0 = X_i^0 \cap C, \hat{Y}_j = Y_j \cap C'$ but $\hat{X}_G = X_G$, which will be used to prove Walras' law below.

Let $\overline{W_i^{(1)}}^{\perp}$ and $\overline{W_j^{(2)}}^{\perp}$ denote the orthogonal complements of $\overline{W_i^{(1)}}$ and $\overline{W_j^{(2)}}$ respectively. Let $\Delta = \{p = (p^C, p^S) | p^C \in R_{++}^l, p^S \in \mathbf{N}, \sum_l p_l^C + \sum_n p_n^S = 1\}$.

For consumer i, given $p = (p^C, p^S) \in \Delta$, define

$$\begin{aligned}\beta_i^{(1)}(p^C, p^S, e) &= \{(x_i, \alpha_i) | x_i \in X_i^0, C_s\alpha_i - T_i^{(1)}(\bar{C}_s\alpha_i) \geq 0, \quad s \in \mathbf{S}, \\ &\qquad p^S\alpha_i + p^C x_i + T_i^{(1)}(p^S\alpha_i) \leq p^C\bar{x}_i^0 + p^S\bar{\alpha}_i^0 + \textstyle\sum_j \theta_{i,j} \max(0, \gamma_j(p, y_j, \beta_j))\};\end{aligned}$$

and

$$\bar{\beta}_i^{(1)}(p^C, p^S, e) = \{(x_i, \alpha_i) | (x_i, \alpha_i) \in (\hat{X}_i \times \overline{W_i^{(1)}}^{\perp}) \cap \beta_i^{(1)}(p^C, p^S, e)\}.$$

where $e = (y_1, \beta_1, \cdots, y_J, \beta_J)$.

For firm j, given $p = (p^C, p^S) \in \Delta$, define

$$\beta_j^{(2)}(p^C, p^S) = \{(y_j, \beta_j) \in F_j | y_j^s - C_s\beta_j \geq T_{j,s}^{(2)}(y_j^s - \bar{C}_s\beta_j), \forall s \in \mathbf{S}\};$$

and

$$\bar{\beta}_j^{(2)}(p^C, p^S) = \{(y_j, \beta_j) \in \beta_j^{(2)}(p^C, p^S) | (y_j, \beta_j) \in \hat{Y}_j \times \overline{W_j^{(2)}}^{\perp}, \gamma_j(p, y_j, \beta_j) \geq 0\}.$$

For the government, given $p = (p^C, p^S) \in \triangle$ define

$$\beta_G(p^C, p^S, f) = \{x_G \in X_G | p^C x_G \leq p^C \bar{x}_G^0 + \sum_i T_{i,0}^{(1)}(p^S \alpha_i) + \sum_j T_{j,0}^{(2)}[p^S(\bar{\beta}_j^0 + \beta_j) + p^C y_j^0]\}.$$

where $f = (x_1, \alpha_1, \cdots, x_I, \alpha_I, y_1, \beta_1, \cdots, y_J, \beta_J)$.

Given $p = (p^C, p^S) \in \triangle$, set

$$\begin{aligned}
\Psi(p^C, p^S) &= \{(x_1, \alpha_1, \cdots, x_I, \alpha_I, y_1, \beta_1, \cdots, y_J, \beta_J, x_G) | (x_i, \alpha_i) \in \beta_i^{(1)}(p^C, p^S, e), i \in \mathbf{I}, \\
&\quad (y_j, \beta_j) \in \beta_j^{(2)}(p^C, p^S), j \in \mathbf{J}, x_G \in \beta_G(p^C, p^S, f); \\
&\quad \tilde{U}_i(x_i, \alpha_i) \geq \tilde{U}_i(x_i', \alpha_i'), \forall (x_i', \alpha_i') \in \beta_i^{(1)}(p^C, p^S, e); \\
&\quad \gamma_j(p, y_j, \beta_j) \geq \gamma_j(p, y_j', \beta_j'), \forall (y_j', \beta_j') \in \beta_j^{(2)}(p^C, p^S), j \in \mathbf{J}; \\
&\quad U_G(x_G) \geq U_G(x_G'), \forall x_G' \in \beta_G(p^C, p^S, f)\}
\end{aligned}$$

and

$$\begin{aligned}
\bar{\Psi}(p^C, p^S) &= \{(x_1, \alpha_1, \cdots, x_I, \alpha_I, y_1, \beta_1, \cdots, y_J, \beta_J, x_G) | (x_i, \alpha_i) \in \bar{\beta}_i^{(1)}(p^C, p^S, e), i \in \mathbf{I}, \\
&\quad (y_j, \beta_j) \in \bar{\beta}_j^{(2)}(p^C, p^S), j \in \mathbf{J}, x_G \in \beta_G(p^C, p^S, f); \\
&\quad \tilde{U}_i(x_i, \alpha_i) \geq \tilde{U}_i(x_i', \alpha_i'), \forall (x_i', \alpha_i') \in \bar{\beta}_i^{(1)}(p^C, p^S, e); \\
&\quad \gamma_j(p, y_j, \beta_j) \geq \gamma_j(p, y_j', \beta_j'), \forall (y_j', \beta_j') \in \bar{\beta}_j^{(2)}(p^C, p^S), j \in \mathbf{J}, \\
&\quad U_G(x_G) \geq U_G(x_G'), \forall x_G' \in \beta_G(p^C, p^S, f)\}
\end{aligned}$$

Finally, define

$$\begin{aligned}
Z(p^C, p^S) &= \{z = \textstyle\sum_{i \in \mathbf{I}}(x_i, \alpha_i) + (x_G, 0) - \sum_{j \in \mathbf{J}}(y_j^0, \beta_j) - \sum_{i \in \mathbf{I}}(\bar{x}_i^0, \bar{\alpha}_i^0) - \sum_{j \in \mathbf{J}}(0, \bar{\beta}_j^0) - (\bar{x}_G^0, 0) | \\
&\quad (x_1, \alpha_1, \cdots, x_I, \alpha_I, y_1, \beta_1, \cdots, y_J, \beta_J, x_G) \in \Psi(p^C, p^S)\},
\end{aligned}$$

$$\begin{aligned}
Z^1(p^C, p^S) &= \{z = \textstyle\sum_{i \in \mathbf{I}}(x_i, \alpha_i) + (x_G, 0) - \sum_{j \in \mathbf{J}}(y_j^0, \beta_j) - \sum_{i \in \mathbf{I}}(\bar{x}_i^0, \bar{\alpha}_i^0) - \sum_{j \in \mathbf{J}}(0, \bar{\beta}_j^0) - (\bar{x}_G^0, 0) | \\
&\quad (x_1, \alpha_1, \cdots, x_I, \alpha_I, y_1, \beta_1, \cdots, y_J, \beta_J, x_G) \in \bar{\Psi}(p^C, p^S)\}.
\end{aligned}$$

The approach to prove Theorem 4.1.1 is to use the following lemma similar to Lemma 1 and Remark 1 in Werner(1987).

Lemma 4.2.1. If (1)$co\triangle$, the cone generated by $\triangle$, is not a linear subspace and its relative interior is nonempty and convex:(2)Z^1 satisfies Walras' law on $\triangle$, that is $pZ^1(p) = 0, \forall p \in \triangle$; (3)the

set Z^1 is nonempty,convex- and compact-valued and upper hemi-continuous on $\triangle$; (4)if $p_k \longrightarrow p, z_k = \sum_i(x_i^k, \alpha_i^k)+(x_G^k,0)-\sum_j(y_j^k,\beta_j^k)-\sum_i(\bar{x}_i^0,\bar{\alpha}_i^0)-\sum_j(0,\bar{\beta}_j^0)-(\bar{x}_G^0,0) \in Z^1(p_k), p_k \in \triangle$, and $||z_k|| \longrightarrow \infty$, then for every cluster point $\hat{z}$ of $\{z_k/M^k\}$, $\hat{p}\hat{z} > 0, \forall \hat{p} \in \triangle$, where[5] $M^k = \sum_i ||(x_i^k, \alpha_i^k)||+\sum_j ||(y_j^k, \beta_j^k)|| +||x_G^k||$; (5)if $p_k \longrightarrow p^*, z_k \in Z^1(p_k)$ and $z_k \longrightarrow z^* = (x^*, \alpha^*)$ with $x^* \leq 0$, then $p^* \in \triangle$; (6)for $p \in \triangle$, if $z \in \triangle^0$(where $\triangle^0$ is polar of $\triangle$) and $pz = 0$, then $Z^1(p) - z \subseteq Z(p)$.

Then there exists $p^* \in \triangle$ such that $0 \in Z(p^*)$.

Proof. As in the first paragraph of proof of Lemma 1 in Werner(1987), by using conditions (1)-(4) of Lemma 4.2.1, it can be shown that there exist $p_k(\in \triangle) \longrightarrow p^*$, $z_k(\in Z^1(p_k)) \longrightarrow z^* = (x^*, \alpha^*) \in \triangle^0$. Then, $(p^C, p^S)z^* \leq 0, \forall (p^C, p^S) \in cl\triangle$(the smallest closed set including $\triangle$), particularly, $p^C x^* \leq 0, \forall p^C \in \triangle^l$(simplex of space R^l) and therefore, $x^* \leq 0$.

Hence, by (5), $p^* \in \triangle$ and moreover, by (2), $z^* \in Z^1(p^*)$. Finally, by Walras law and (6), $Z^1(p^*) - z^* \subseteq Z(p^*)$ and therefore, $0 \in Z(p^*)$.□

We will show that correspondences $Z(\cdot)$ and $Z^1(\cdot)$ satisfy the assumptions of Lemma 4.2.1. Before doing this, we will prove some auxiliary results.

Proposition 4.2.1. $\bar{\beta}_i^{(1)}(p^C, p^S, e)$ is nonempty, convex- and compact-valued and lower hemi-continuous on $\triangle \times \prod_{j\in \mathbf{J}} F_j$.

Proof. $\bar{\beta}_i^{(1)}(p^C, p^S, e)$ is nonempty since 0 belongs to it. The convexity and closedness are also obvious.

To show that $\bar{\beta}_i^{(1)}(p^C, p^S, e)$ is compact-valued it suffices to prove that it is bounded. Assume, by contrary, that there exists a sequence $\{(x_i^n, \alpha_i^n)\} \subseteq \bar{\beta}_i^{(1)}(p^C, p^S, e)$ such that $||(x_i^n, \alpha_i^n)|| \longrightarrow \infty$. This implies that $||\alpha_i^n|| \longrightarrow \infty$ since $x_i^n \in \hat{X}_i^0$ and therefore, is bounded.

Let $(\hat{x}, \hat{\alpha})$ be a cluster point of the bounded sequence $\{(x_i^n, \alpha_i^n)/||(x_i^n, \alpha_i^n)||\}$. It is not difficult to deduce that $\hat{x} = 0$,$\hat{\alpha} \neq 0$ and furthermore, $\hat{\alpha} \in \overline{W_i^{(1)}}^{\perp}$ since $\alpha_i^n \in \overline{W_i^{(1)}}^{\perp}$.

Note that $T_i^{(1)}(\bar{C}_s\alpha_i^n) - C_s\alpha_i^n \leq 0$ for each n and each $s \in \mathbf{S}$. Hence, by Theorem 8.2 of Rockafellar(1970), $\hat{\alpha} \in W_i^{(1)} \cup \overline{W_i^{(1)}}$. This implies that $\hat{\alpha} \in W_i^{(1)}$ and therefore, by Assumption 1,$p^S\hat{\alpha} > 0$.

On the other hand, as in the proof of Proposition 3.1, by Assumption 6, we have

$$p^C x_i^n + p^S \alpha_i^n \Big(1 + \frac{dT_{i,0}^{(1)}(x)}{dx}\Big|_{x=0}\Big) \leq p^C \bar{x}_i^0 + p^S \bar{\alpha}_i^0 + \sum_j \theta_{i,j} \max(0, \gamma_j(p, y_j, \beta_j)),$$

this implies $p^S\hat{\alpha} \leq 0$, a contradiction which proves the boundedness of set $\bar{\beta}_i^{(1)}(p^C, p^S, e)$.

It remains to show the lower hemi-continuity of $\bar{\beta}_i^{(1)}(p^C, p^S, e)$ at $(p^C, p^S, e) \in \triangle \times \prod_{j\in \mathbf{J}} F_j$. To this end, suppose $(p_n^C, p_n^S, e^n) = (p_n^C, p_n^S, y_1^n, \beta_1^n, \cdots, y_J^n, \beta_J^n) \longrightarrow (p^C, p^S, e) = (p^C, p^S, y_1, \beta_1, \cdots, y_J, \beta_J)$ and $(x_i, \alpha_i) \in \bar{\beta}_i^{(1)}(p^C, p^S, e)$.

[5] $||x|| = \sum_{i=1}^m |x_i|, \forall x = (x_1, ..., x_m) \in R^m$.

If

$$p^S\alpha_i + p^C x_i + T_{i,0}^{(1)}(p^S\alpha_i) = p^C\bar{x}_i^0 + p^S\bar{\alpha}_i^0 + \sum_j \theta_{i,j}\max(0, \gamma_j(p, y_j, \beta_j)), \tag{4.1}$$

then $p_n^S\alpha_i + p_n^C x_i + T_{i,0}^{(1)}(p_n^S\alpha_i) > 0$ for sufficiently large n since $p^C \in R_{++}^l$ and therefore, $p^C\bar{x}_i^0 > 0$.

Take, for sufficiently large n,

$$\lambda_n = [p_n^C\bar{x}_i^0 + p_n^S\bar{\alpha}_i^0 + \sum_{j\in J}\theta_{i,j}\max(0, \gamma_j(p_n, y_j^n, \beta_j^n))]/(p_n^S\alpha_i + p_n^C x_i + T_{i,0}^{(1)}(p_n^S\alpha_i)).$$

Clearly, $\lambda_n \longrightarrow 1$ as $n \longrightarrow \infty$. Let $\lambda_n' = \min(1, \lambda_n)$ and $(x_i^n, \alpha_i^n) = \lambda_n'(x_i, \alpha_i)$.

Hence, $\alpha_i^n \in \overline{W_i^{(1)}}^{\perp}$, $\alpha_i^n \longrightarrow \alpha_i$ and, by convexity of $T_{i,0}^{(1)}(\cdot)$,

$$p_n^S\alpha_i^n + p_n^C x_i^n + T_{i,0}^{(1)}(p_n^S\alpha_i^n) \le \lambda_n'(p_n^S\alpha_i + p_n^C x_i + T_{i,n}^{(1)}(p_n^S\alpha_i)) \le p_n^C\bar{x}_i^0 + p_n^S\bar{\alpha}_i^0 + \sum_j \theta_{i,j}\max(0, \gamma_j(p_n, y_j^n, \beta_j^n)).$$

Furthermore, by convexity of $T_{i,s}^{(1)}(\bar{C}_s\alpha_i) - C_s\alpha_i$, we have

$$T_{i,s}^{(1)}(\bar{C}_s\alpha_i^n) - C_s\alpha_i^n \le \lambda_n'(T_{i,s}^{(1)}(\bar{C}_s\alpha_i) - C_s\alpha_i) \le 0, \quad s \in \mathbf{S}.$$

Therefore, $(x_i^n, \alpha_i^n) \in \bar{\beta}_i^{(1)}(p_n^C, p_n^S, e^n)$, proving the lower hemi-continuity. And the proof for the case of strict inequality of (4.1) is standard. □

For $\bar{\beta}_j^{(2)}(p^C, p^S)$, we have the following similar result to Proposition 4.1.

Proposition 4.2.2. $\bar{\beta}_j^{(2)}(p^C, p^S)$ is nonempty, convex- and compact-valued and lower hemi-continuous on $\triangle$.

Proof. The nonempty of $\bar{\beta}_j^{(2)}(p^C, p^S)$ is obvious since it includes 0. We now show that $\bar{\beta}_j^{(2)}(p^C, p^S)$ is compact-valued. Suppose that $\{(y_j^n, \beta_j^n)\} \subseteq \bar{\beta}_j^{(2)}(p^C, p^S)$. Then

$$p^S(\bar{\beta}_j^0 + \beta_j^n) + p^C y_j^{0n} - T_{j,0}^{(2)}[p^S(\bar{\beta}_j^0 + \beta_j^n) + p^C y_j^{0n}] \ge 0. \tag{4.2}$$

It is obvious that $\bar{\beta}_j^{(2)}(p^C, p^S)$ is compact-valued if we can show the boundedness of sequence $\{x_n = (y_j^n, \beta_j^n) : n \ge 1\}$. Assume, by contrary, that there exists a subsequence of $\{x_n\}$ still denoted $\{x_n\}$ such that $\lim_{n\longrightarrow\infty}||x_n|| = \infty$. Let $\hat{x} = (\hat{y}_j, \hat{\beta}_j)$ be a cluster point of the bounded sequence $\{x_n/||x_n||\}$. As in the proof of Proposition 4.2.1, it can be shown that $\hat{y}_j = 0, \hat{\beta}_j \ne 0, \hat{\beta}_j \in \overline{W_j^{(2)}}^{\perp}$ and, by means of (4.2),

$$p^S\hat{\beta}_j \ge 0. \tag{4.3}$$

And, furthermore, as in the proof of Propositiion 4.2.1, it can be proved that

$$\hat{\beta}_j \in W_j^{(2)} \cup \overline{W_j^{(2)}}.$$

This implies $\hat{\beta}_j \in W_j^{(2)}$ since $\hat{\beta}_j \in \overline{W_j^{(2)}}^{\perp}$. Hence, by Assumption 1, $p^S\hat{\beta} < 0$, a contradiction to (4.3), proving the boundedness of $\{x_n\}$.

The convexity of $\bar{\beta}^{(2)}(p^C, p^S)$ is obvious. It remains to show its lower hemi-continuity. Suppose $(p_n^C, p_n^S) \longrightarrow (p^C, p^S)$ and $(y_j, , \beta_j) \in \bar{\beta}_j^{(2)}(p^C, p^S)$. Then $\gamma_j(p, y_j, \beta_j) \geq 0$ and

$$y_j^s - C_s\beta_j - T_{j,s}^{(2)}(y_j^s - \bar{C}_s\beta_j) \geq 0, \forall s \in \mathbf{S}.$$

Note that, $\gamma_j(p_n^C, p_n^S, 0, \frac{1}{2}\bar{\beta}_j^0) > 0$, and by Assumption 10, $C_s(\frac{1}{2}\bar{\beta}_j^0) - T_{j,s}^{(2)}[-\bar{C}_s(\frac{1}{2}\bar{\beta}_j^0)] > 0, \forall s \in \mathbf{S}$, that is, there exists an interior point in the set of feasible production-portfolio plan of firm j.

Hence, the proof of lower hemi-continuity of $\bar{\beta}_j^{(2)}(p^S, p^C)$ is standard. □

Proposition 4.2.3. Suppose that $\bar{x}_G^0$ is sufficiently large such that

$$p^C\bar{x}_G^0 + \sum_i T_{i,0}^{(1)}(p^S\alpha_i) + \sum_j T_{j,0}^{(2)}[p^S(\bar{\beta}_j^0 + \beta_j) + p^C y_j^0] > 0,$$

$\forall(x_i, \alpha_i) \in \bar{\beta}_i^{(1)}(p^C, p^S, e)$ and $(y_j, \beta_j) \in \bar{\beta}_j^{(2)}(p^C, p^S)$. Then $\beta_G(p^C, p^S, f)$ is nonempty, convex- and compact-valued and upper hemi-continuous at any $(p^C, p^S, x_1, \alpha_1, \cdots, x_I, \alpha_I, y_1, \beta_1, \cdots, y_J, \beta_J)$, where $(p^C, p^S) \in \triangle$, $(x_i, \alpha_i) \in \bar{\beta}_i^{(1)}(p^C, p^S, e)$ and $(y_j, \beta_j) \in \bar{\beta}_j^{(2)}(p^C, p^S)$.

Proof. The proof is standard.□

Proposition 4.2.4. $\bar{\Psi}(p^C, p^S)$ is nonempty, convex- and compact-valued and upper hemi-continuous on $\triangle$.

Proof. See Appendix.□

Proposition 4.2.5. For $(p^C, p^S) \in \triangle$, if $x = (x_1, \alpha_1, \cdots, x_I, \alpha_I, y_1, \beta_1, \cdots, y_J, \beta_J, x_G) \in \bar{\Psi}(p^C, p^S)$, $\alpha_i^0 \in W_i^{(1)} \cup \overline{W_i^{(1)}}$, $\beta_j^0 \in W_j^{(2)} \cup \overline{W_j^{(2)}}$ with $p^S\alpha_i^0 = p^S\beta_j^0 = 0$. Then $x + x^0 \in \Psi(p^C, p^S)$, where $x^0 = (0, \alpha_1^0, \cdots, 0, \alpha_I^0, 0, \beta_1^0, \cdots, 0, \beta_J^0, 0)$.

Proof. See Appendix. □

Proposition 4.2.6. If $(p_n^C, p_n^S)(\in \triangle) \longrightarrow (p^C, p^S)$, $z^n = \sum_{i\in\mathbf{I}}(x_i^n, \alpha_i^n) + (x_G^n, 0) - \sum_{j\in\mathbf{J}}(y_j^n, \beta_j^n) - \sum_{i\in\mathbf{I}}(\bar{x}_i^0, \bar{\alpha}_i^0) - \sum_{j\in\mathbf{J}}(0, \bar{\beta}_j^0) - (\bar{x}_G^0, 0) \in Z^1(p_n^C, p_n^S)$ and $||z^n|| \longrightarrow \infty$, then $(\bar{p}^C, \bar{p}^S)\hat{z} > 0$ for every cluster point $\hat{z}$ of $\{z^n/M^n\}$ and all $(\bar{p}^C, \bar{p}^S) \in \triangle$, where $M^n = \sum_{i\in\mathbf{I}} ||(x_i^n, \alpha_i^n)|| + \sum_{j\in\mathbf{J}} ||(y_j^n, \beta_j^n)|| + ||(x_G^n, 0)||$.

Proof. See Appendix.□

We now verify that correspondences $Z(p^C, p^S)$ and $Z^1(p^C, p^S)$ satisfy the assumptions of Lemma 4.2.1.

Proposition 4.2.7. Under the Assumptions 1-13, the conditions (1)-(6) of Lemma 4.2.1 holds.

Proof. (1)To show that the relative interior of $co(\triangle)$ is nonempty, it suffices to prove that $co(\mathbf{N})$ has nonempty interior. Suppose $co(\mathbf{N})$ is N_0-dimensional($N^0 \leq N$). Then there exists N^0 linearly

independent price vectors $(p_1^C, p_1^S), \cdots, (p_{N^0}^C, p_{N^0}^S) \in co(\mathbf{N})$. Clearly, for $\lambda_i \in (0,1), i = 1, \cdots, N^0$, with $\sum_{i=1}^{N^0} \lambda_i = 1$, $\sum_{i=1}^{N^0} \lambda_i (p_i^C, p_i^S) \in co(\mathbf{N})$. Hence, the relative interior of $co(\mathbf{N})$ is nonempty.

(2)Now we verify that $Z^1(p^C, p^S)$ satisfies Walras' law on $\triangle$. Consider the government. Since $\hat{X}_G = R_+^l$ and the utility function U_G is strictly increasing, (2.3) holds with equality for the optimal consumption of government.

Now we turn to the consumer $i(\in \mathbf{I})$. Note that, by Assumptions 2 and 3 and by Proposition 3.1, $p^S \beta > 0$ and $C_s \beta - T_{i,s}^{(1)}(\bar{C}_s \beta) > 0$, $\forall \beta \in R_{++}^N, p^S \in \mathbf{N}$.

Hence, it is not difficult to deduce that (2.1) holds with equality for any optimal consumption-portfolio plan of consumer i.

(3)(3) can be deduced from Proposition 4.2.4.

(4)(4) follows from Proposition 4.2.6.

(5)Since $x^* = \sum_i x_i^* + x_G^{*0} - \sum_j y_j^{*0} - \sum_i \bar{x}_i^0 - \bar{x}_G^0 \leq 0$, $x_i^* \in \hat{X}_i$ and $y_j^* \in \hat{Y}_j, \forall i \in \mathbf{I}, j \in \mathbf{J}$.

Note that $p^* = (p^{*C}, p^{*S}) > 0$. Hence, by Assumption 7, there exists at least one $i_0 \in \mathbf{I}$ such that the initial wealth, $p^{*C} \bar{x}_{i_0}^0 + p^{*S} \bar{\alpha}_{i_0}^0$, of consumer i_0 is positive. Therefore, if $p^{*C} \notin R_{++}^l$, then, by the strictly increasing utility of consumer i_0, it follows that $x_{i_0}^* \in C$ but $\notin \hat{X}_{i_0}$, a contradiction and therefore, $p^{*C} \in R_{++}^l$.

As in the proof of Walras' law, by Assumption 2 and strictly increasing utility of consumer i, it is not difficult to show $p^{*S} \in R_{++}^l$.

(6)The condition that $z = (x, \alpha) \in \triangle^0$ is equivalent to the following one.

$$\bar{p}^C x + \bar{p}^S \alpha \leq 0, \ \forall (\bar{p}^C, \bar{p}^S) \in cl\triangle. \tag{4.4}$$

By taking $\bar{p}^S = 0$ in (4.4), we have $\bar{p}^C x \leq 0$, $\forall p^C \in cl\triangle^l$. This implies $x \leq 0$. By taking $\bar{p}^C = 0$ in (4.4), we have $\bar{p}^S \alpha \leq 0$, $\forall \bar{p}^S \in cl\mathbf{N}$. This implies $\alpha \in (cl\mathbf{N})^0$. But, by Walras' law, $p^S \alpha + p^C x = 0$ and $p^C \in R_{++}^l$, thus $x = 0$ and $p^S \alpha = 0$.

Thus, we have arrived at the conclusion that $x = 0, \alpha \in (cl\mathbf{N})^0$. As in Werner[P.1413,(1987)], by Lemma A.1 in Appendix, there exist $\alpha_i \in W_i^{(1)} \cup \overline{W_i^{(1)}} (i \in \mathbf{I})$ and $\beta_j \in W_j^{(2)} \cup \overline{W_j^{(2)}} (j \in \mathbf{J})$ such that $-\alpha = \sum_i \alpha_i - \sum_j \beta_j$. It follows that $p^S \alpha_i \geq 0, \forall i \in \mathbf{I}$ and $p^S \beta_j \leq 0, \forall j \in \mathbf{J}$. Consequently, $p^S \alpha_i = p^S \beta_j = 0, \forall i \in \mathbf{I}, \forall j \in \mathbf{J}$ since $p^S \alpha = 0$.

By Proposition 4.2.5, $\bar{z} + z' \in \Psi(p^C, p^S), \forall \bar{z} \in \bar{\Psi}(p^C, p^S)$, where $z' = (0, \alpha_1, \cdots, 0, \alpha_I^0, 0, \beta_1, \cdots, 0, \beta_J, 0)$. Then it follows that $\bar{z} - z \in Z(p^C, p^S), \forall \bar{z} \in Z_1(p^C, p^S)$.

This completes the proof of this proposition.□

We have proved that there exists a general equilibrium for the economy with the bounded spot commodity market. It remains to show that this equilibrium is a equilibrium of the original economy.

Note that the feasible sets $\beta_i^{(1)}(p^C, p^S, e), \beta_j^{(2)}(p^C, p^S)$ and $\beta_G(p^C, p^S, f)$ of consumers, firms and government are all convex. Hence the remainder of the proof is standard.

5. Non-Convex Tax Functions

This section is devoted to an exchange economy with consumers and government, but no firms. We make this assumption for expositional convenience, the extension to an economy with production is straightforward.

In this model, each consumer pays tax according to a tax function which is not necessarily convex. Because of a number of exemption thresholds, subsidy schemes etc., taxation functions sometimes exhibit discontinuities and non-convexities. We will assume that discontinuities can be closely approximated by a continuous tax function. But there will remain a non-convexity that is the subject of this section.

By using the technique of Heller and Starr(1976), we will show the existence of an individual approximate equilibrium as defined by Heller and Starr(1976). An approximate equilibrium is generally defined as a price p^* and two allocations, a^* and $a^{*'}$: one, a^*, is the allocation desired by consumers and government at this price, which may not clear the market; the other, $a^{*'}$, is an allocation obeying the market clearance condition although it need not represent agents' optimizing behaviour. The equilibrium is approximate, of a modulus C, if some suitably chosen norm of the difference between these two allocations is no larger than C. The desired allocation represents an approximate equilibrium in the sense that the failure to clear the market at this price is bounded by C.

We have a finite set of consumer $\mathbf{I} = \{1, \cdots, I\}$. We omit the description of the behavior of consumers and the government since it is as same as in Section 2 except that the wealth W_i of consumer i does not include the firms' profit and the government's tax does not include taxes from firms. Assume that all tax functions $T_i^{(0)}$ in period 0 and $T_{i,s}^{(1)}$ in period 1 are all not necessarily convex. All assets are assumed to be fully taxable, that is, $\gamma_{n,s} = 1, \forall n, s$.

The main theorem of this section is as follows:

Theorem 5.1 Under the assumptions given below, there exists an individual approximate equilibrium of modulus C. That is, there exist a price p^* and two vectors $a^* = (x_1^*, \alpha_1^*, \cdots, x_I^*, \alpha_I^*, x_G^*)$ and $a^{*'} = (x_1^{*'}, \alpha_1^{*'}, \cdots, x_I^{*'}, \alpha_I^{*'}, x_G^{*'})$ such that

(i)$a^{*'}$ satisfies market clearance with respect to p^*, that is, $\sum_i (x^{*'}, \alpha^{*'}) + (x_G^{*'}, 0) = \sum_i (\bar{x}_i^0, \bar{\alpha}_i^0) + (\bar{x}_G^0, 0)$.

(ii)a^* is optimal with respect to p^*, that is, it solves problems $(**)'$ and $(***)$ without firms.

(iii)$(\sum_i ||a_i^* - a_i^{*'}||^2)^{1/2} \leq C$.

We will still adopt the Assumptions 3-5,7,12 and 13 in Section 3. We further assume the following.

Assumption 5.1: $T_i^{(q)}(0) = 0, T_i^{(q)}(x) \geq 0, \forall x \geq 0$ and $\sup_{x \in R} \left| \frac{T_i^{(q)}(x)}{x} \right| < 1, \forall x \neq 0, q = 0, 1$.

Assumption 5.2. $T^{(0)}(\cdot)$ is continuous function.

This assumption will be used to proved the upper hemi-continuity of excess demand correspon-

dence defined below(Proposition 5.6). But this assumption can be deleted if we do not consider the tax at time 0 as in Dammon and Green(1987).

Inspired by Remark 3.1, we now give the definition of no-tax-arbitrage price for the non-convex tax functions.

Definition 5.1. The price vector p^S of assets is a "no-tax-arbitrage" price vector if and only if for each α satisfying $C_s\alpha \geq 0$ for all $s \in \mathbf{S}$ and there exists at least one state s such that $C_s\alpha > 0$, then $p^S\alpha > 0$. And the set of no-tax-arbitrage prices is denoted $\mathbf{N}'$.

Remark 5.1: From Assumption 5.1, if $C_s\alpha > 0$, then $\lim_{n\longrightarrow\infty}[C_s(\alpha'+n\alpha)-T_i^{(1)}(C_s(\alpha'+n\alpha))] = \infty, \forall\alpha', i \in \mathbf{I}$. Thus, $\alpha'+n\alpha$ will be an opportunity every investor would wish to arbitrage ad infinitum starting from any position α'. And this opportunity is independent of investor's tax functions and is inconsistent with equilibrium.

Assumption 5.3: The set of no-tax-arbitrage prices is nonempty, that is, $\mathbf{N}' \neq \emptyset$.

As in Section 3, we have the following results similar to Propositions 3.1 and 3.2. We will omit their proofs since it is easy to prove:

Proposition 5.1: Under Assumptions 3, each no-tax-arbitrage price $p^S = (p_1^S, \cdots, p_N^S) \in \mathbf{N}'$ is positive, that is, $p_n^S > 0, \quad n = 1, \cdots, N$.

Proposition 5.2. Under Assumptions 3, each no-tax-arbitrage price $p^S \in \mathbf{N}'$ is zero on the set $\overline{W}$, that is, $p^S\alpha = 0, \forall\alpha \in \overline{W}$.

Here $\overline{W} = \{\alpha | C_s\alpha = 0, \forall s \in \mathbf{S}\}$ and let W denote the set of α which satisfies the condition of Definition 5.1.

We will truncate the economy and use Lemma 4.2.1 to prove the existence of individual approximate equilibrium.

Let $M(> \sum_i ||\bar{x}_i^0|| + ||\bar{x}_G^0)||)$ be sufficiently large and define

$$X^M = \{x = (x_1, \cdots, x_l) \in R_+^l | \sum_k x_k \leq M\}.$$

For consumer i, given $p = (p^C, p^S) \in \Delta$, define

$$\begin{aligned}\beta_i(p^C, p^S) \quad &= \{(x_i, \alpha_i) | x_i \in X_i^0 \cap X^M, C_s\alpha_i - T_{i,s}^{(1)}(\bar{C}_s\alpha_i) \geq 0, \quad s \in \mathbf{S}, \\ &\quad p^C x_i + p^S\alpha_i + T_i^{(0)}(p^S\alpha_i) \leq p^C\bar{x}_i^0 + p^S\bar{\alpha}_i^0\};\end{aligned}$$

and

$$\bar{\beta}_i(p^C, p^S) = \{(x_i, \alpha_i) \in \beta_i(p^C, p^S) | \alpha_i \in \overline{W}^{\perp}\}.$$

where $\overline{W}^{\perp}$ denote the orthogonal completements of $\overline{W}$.

For the government, given $p = (p^C, p^S) \in \Delta$ define

$$\beta_G(p^C, p^S, f) = \{x_G \in X_G | p^C x_G \leq p^C\bar{x}_G^0 + \sum_i T_i^{(0)}(p^S\alpha_i)\}.$$

where $f = (x_1, \alpha_1, \cdots, x_I, \alpha_I)$.

The following result is similar to Proposition 4.2.1.

Proposition 5.3. $\bar{\beta}_i(p^C, p^S)$ is a nonempty and bounded set on $\triangle$. And, hence, $cl\bar{\beta}_i(p^C, p^S)$ is compact-valued on $\triangle$.

Proof. It suffices to show that $\bar{\beta}_i(p^C, p^S)$ is bounded. Assume, by contrary, that there exists a sequence $\{(x_i^n, \alpha_i^n)\} \subseteq \bar{\beta}_i^{(1)}(p^C, p^S)$ such that $||(x_i^n, \alpha_i^n)|| \longrightarrow \infty$. And let $(\hat{x}, \hat{\alpha})$ be a cluster point of the bounded sequence $\{(x_i^n, \alpha_i^n)/||(x_i^n, \alpha_i^n)||\}$.

As in the proof of Proposition 4.2.1, we can deduce that $\hat{x} = 0, \hat{\alpha} \neq 0$ and furthermore, $\hat{\alpha} \in \overline{W}^{\perp}$.

Note that $T_i^{(1)}(\bar{C}_s \alpha_i^n) - C_s \alpha_i^n \leq 0$ for each n and each $s \in \mathbf{S}$. It follows from Assumption 5.1 that $C_s \alpha_i^n \geq 0, \forall s, n$. Hence, as in the proof of Proposition 4.2.1, by Theorem 8.2 of Rockaffellar(1970), $p^S \hat{\alpha} > 0$.

On the other hand, as in the proof of Proposition 4.2.1, by Assumption 5.1, it is easy to show that $p^S \hat{\alpha} \leq 0$, a contradiction proving the boundedness of set $\bar{\beta}_i(p^C, p^S)$. □

As in Heller and Starr(1976), in order to prove the continuity of $cl\bar{\beta}_i(p^C, p^S)$, we give the definition of local interior.

Definition 5.2. $\bar{\beta}_i$ is said to be locally interior if for each $(x, \alpha) \in \bar{\beta}(p^C, p^S)$ there is (x^*, α^*) so that

(i)$(x^*, \alpha^*) \in int\bar{\beta}_i(p^C, p^S)$.

(ii)There exists a continuous function $f : [0, 1] \to \bar{\beta}_i(p^C, p^S)$ so that $f(0) = (x^*, \alpha^*), f(1) = (x, \alpha)$ and for all $\sigma (\in [0, 1)), f(\sigma) \in int\bar{\beta}(p^C, p^S)$.

Assumption 5.4. $\bar{\beta}(p^C, p^S)$ is locally interior.

As in Heller and Starr(1976), we have the following result. We omit its proof since it can be shown in exactly the same manner as that of Heller and Starr(1976).

Proposition 5.4. $cl\bar{\beta}(p^C, p^S)$ is continuous under the Assumption 5.4.

For the government, we have the following result.

Proposition 5.5. Suppose that $\bar{x}_G^0$ is sufficiently large such that $p^C \bar{x}_G^0 + \sum_i T_{i,0}^{(1)}(p^S \alpha_i) > 0$, $\forall (x_i, \alpha_i) \in \bar{\beta}_i^{(1)}(p^C, p^S)$. Then $\beta_G(p^C, p^S, f)$ is nonempty, convex- and compact-valued and upper hemi-continuous at any $(p^C, p^S, x_1, \alpha_1, \cdots, x_I, \alpha_I)$, where $(p^C, p^S) \in \triangle$, $(x_i, \alpha_i) \in \bar{\beta}_i^{(1)}(p^C, p^S)$.

Given $p = (p^C, p^S) \in \triangle$, set

$$\begin{aligned}
\Psi(p^C, p^S) \quad &= \{(x_1, \alpha_1, \cdots, x_I, \alpha_I, x_G) | (x_i, \alpha_i) \in \beta_i(p^C, p^S), i \in \mathbf{I}, x_G \in \beta_G(p^C, p^S, f); \\
&\tilde{U}_i(x_i, \alpha_i) \geq \tilde{U}_i(x_i', \alpha_i'), \forall (x_i', \alpha_i') \in \beta_i(p^C, p^S); \\
&U_G(x_G) \geq U_G(x_G'), \forall x_G' \in \beta_G(p^C, p^S, f)\},
\end{aligned}$$

$$\begin{aligned}
\bar{\Psi}(p^C,p^S) &= \{(x_1,\alpha_1,\cdots,x_I,\alpha_I,x_G)|(x_i,\alpha_i)\in\bar{\beta}_i(p^C,p^S),i\in\mathbf{I},x_G\in\beta_G(p^C,p^S,f);\\
&\quad \tilde{U}_i(x_i,\alpha_i)\geq\tilde{U}_i(x_i',\alpha_i'),\forall(x_i',\alpha_i')\in\bar{\beta}_i(p^C,p^S);\\
&\quad U_G(x_G)\geq U_G(x_G'),\forall x_G'\in\beta_G(p^C,p^S,f)\},\\
cl[\Psi(p^C,p^S)] &= \{(x_1,\alpha_1,\cdots,x_I,\alpha_I,x_G)|(x_i,\alpha_i)\in cl[\beta_i(p^C,p^S)],i\in\mathbf{I},x_G\in\beta_G(p^C,p^S,f);\\
&\quad \tilde{U}_i(x_i,\alpha_i)\geq\tilde{U}_i(x_i',\alpha_i'),\forall(x_i',\alpha_i')\in cl[\beta_i(p^C,p^S)];\\
&\quad U_G(x_G)\geq U_G(x_G'),\forall x_G'\in\beta_G(p^C,p^S,f)\},\\
cl[\bar{\Psi}(p^C,p^S)] &= \{(x_1,\alpha_1,\cdots,x_I,\alpha_I,x_G)|(x_i,\alpha_i)\in cl[\bar{\beta}_i(p^C,p^S)],i\in\mathbf{I},x_G\in\beta_G(p^C,p^S,f);\\
&\quad \tilde{U}_i(x_i,\alpha_i)\geq\tilde{U}_i(x_i',\alpha_i'),\forall(x_i',\alpha_i')\in cl[\bar{\beta}_i(p^C,p^S)];\\
&\quad U_G(x_G)\geq U_G(x_G'),\forall x_G'\in\beta_G(p^C,p^S,f)\}.
\end{aligned}$$

Let $co\{cl[\Psi(p^C,p^S)]\}$ and $co\{cl[\bar{\Psi}(p^C,p^S)]\}$ denote the convex hull of sets $cl[\Psi(p^C,p^S)]$ and $cl[\bar{\Psi}(p^C,p^S)]$ respectively. We have following results.

Proposition 5.6. $co\{cl[\bar{\Psi}(p^C,p^S)]\}$ is nonempty, convex- and compact-valued and upper hemi-continuous on $\triangle$,.

Proof. The proof is standard.□

Proposition 5.7. For $(p^C,p^S)\in\triangle$, if $z=(x_1,\alpha_1,\cdots,x_I,\alpha_I,x_G)\in co\{cl[\bar{\Psi}(p^C,p^S)]\}$, $\alpha_i^0\in W\cup\overline{W}$ with $p^S\alpha_i^0=0$. Then $z+z^0\in co\{cl[\Psi(p^C,p^S)]\}$, where $z^0=(0,\alpha_1^0,\cdots,0,\alpha_I^0,0)$.

Proof. It suffices to show that $\bar{\Psi}(p^C,p^S)\in\Psi(p^C,p^S)$. The proof is same as that of Proposition 4.2.5.□

Finally, define

$$Z(p^C,p^S)=\{z=\sum_{i\in\mathbf{I}}(x_i,\alpha_i)+(x_G,0)-\sum_{i\in\mathbf{I}}(\bar{x}_i^0,\bar{\alpha}_i^0)-(\bar{x}_G^0,0)|(x_1,\alpha_1,\cdots,x_I,\alpha_I,x_G)\in co\{cl[\Psi(p^C,p^S)]\}\},$$

and

$$Z^1(p^C,p^S)=\{z=\sum_{i\in\mathbf{I}}(x_i,\alpha_i)+(x_G,0)-\sum_{i\in\mathbf{I}}(\bar{x}_i^0,\bar{\alpha}_i^0)-(\bar{x}_G^0,0)|(x_1,\alpha_1,\cdots,x_I,\alpha_I,x_G)\in co\{cl[\bar{\Psi}(p^C,p^S)]\}\}.$$

Proposition 5.8. If $(p_n^C,p_n^S)(\in\triangle)\longrightarrow(p^C,p^S)$, $z^n=\sum_{i\in\mathbf{I}}(x_i^n,\alpha_i^n)+(x_G^n,0)-\sum_{i\in\mathbf{I}}(\bar{x}_i^0,\bar{\alpha}_i^0)-(\bar{x}_G^0,0)\in Z^1(p_n^C,p_n^S)$ and $||z^n||\longrightarrow\infty$, then $(\bar{p}^C,\bar{p}^S)\hat{z}>0$ for every cluster point $\hat{z}$ of $\{z^n/M^n\}$ and all $(\bar{p}^C,\bar{p}^S)\in\triangle$, where $M^n=\sum_{i\in\mathbf{I}}||(x_i^n,\alpha_i^n)||+||x_G^n||$.

Proof. The proof of this proposition is as same as that of Proposition 4.2.6.□

Note that $cl\mathbf{N}'^0=-W\cup\overline{W}$. Hence, we have the following result similar to Proposition 4.2.7.

Proposition 5.9. The conditions of Lemma 4.2.1 hold for $Z(p^C, p^S)$ and $Z^1(p^C, p^S)$.

Proof. The proof of this proposition is as same as that of Proposition 4.2.7.□

Now we are in a position to prove the existence of an individual approximate equilibrium of the economy with non-convexity. But we omit its proof since it can be obtained in the same method as Heller and Starr(1976) using Shapley-Folkman Theorem. The constant C in Theorem 5.1 only depends on the bound of $\bar{\Psi}(p^C, p^S)$.

Now we have got the individual approximate equilibrium for the truncated economy. It is not difficult to show, by letting $M \longrightarrow \infty$, that this approximate equilibrium is also an approximate equilibrium of the original economy.

Conclusion

This paper has generalised the existing literature in a number of directions to provide a general equilibrium model of an asset economy with consumers, firms and goverment, and the taxation of assets. There are further directions that this research can proceed:(a)the model should be extended to many periods to capture the complexity and richness of the dynamics of tax planning; and (b)the equilibrium should be characterised to show the general structure of asset economy equilibrium prices and asset allocations with financial taxation.

Appendix

Proof of Proposition 4.2.4: $\bar{\Psi}(p^C, p^S)$ is clearly nonempty. The proof of being compact-valued is standard. From Propositions 4.2.1-4.2.3, to show the upper hemi-continuity it suffice to verify the condition of Proposition 4.2.3.

For consumer i, since the optimal consumption-portfolio plan satisfies Walras' Law, it can be shown from (2.1) that

$$p^S\alpha_i \geq \min\Big[\Big(1+\sup_x \frac{dT^{(1)}_{i,0}(x)}{dx}\Big)^{-1} p^C(\bar{x}^0_i - x_i), \Big(1+\inf_x \frac{dT^{(1)}_{i,0}(x)}{dx}\Big)^{-1} p^C(\bar{x}^0_i - x_i)\Big] = t_i(p^C, p^S, x_i, \alpha_i),$$

and therefore,

$$T^{(1)}_{i,0}(p^S\alpha_i) \geq \min\Big(\sup_x \frac{dT^{(1)}_{i,0}(x)}{dx} t_i(p^C, p^S, x_i, \alpha_i), \inf_x \frac{dT^{(1)}_{i,0}(x)}{dx} t_i(p^C, p^S, x_i, \alpha_i)\Big).$$

For firm j, since $\gamma_j(p, y_j, \beta_j) = p^S(\bar{\beta}^0_j + \beta_j) + p^C y^0_j - T^{(2)}_{j,0}[p^S(\bar{\beta}^0_j + \beta_j) + p^C y^0_j] \geq 0$, $p^S(\bar{\beta}^0_j + \beta_j) + p^C y^0_j \geq 0$ by Assumption 2 and therefore, $T^{(2)}_{j,0}[p^S(\bar{\beta}^0_j + \beta_j) + p^C y^0_j] \geq 0$.

Therefore, by combining the above, the condition holds when $\bar{x}^0_G$ is sufficiently large.

Note that α_i and β_j are all bounded. Hence $\bar{x}^s_G + \sum_i T^{(1)}_{i,s}(\bar{C}_s\alpha_i) + \sum_j T^{(2)}_{j,s}(y^s_j - \bar{C}_s\beta)$ is positive for any $s \in \mathbf{S}$ and sufficiently large $\bar{x}^s_G$.

We now turn to the proof of convexity.

Suppose $x = (x_1, \alpha_1, \cdots, x_I, \alpha_I, y_1, \beta_1, \cdots, y_J, \beta_J, x_G) \in \bar{\Psi}(p^C, p^S)$, and $x' = (x'_1, \alpha'_1, \cdots, x'_I, \alpha'_I, y'_1, \beta'_1, \cdots, y'_J, \beta'_J, x'_G) \in \bar{\Psi}(p^C, p^S)$ and $\lambda \in [0,1]$.

Note that

$$(y_j, \beta_j), (y'_j, \beta'_j) \in arg \max_{(\bar{y},\bar{\beta}) \in \bar{\beta}_j^{(2)}(p^C,p^S)} \gamma_j(p, \bar{y}, \bar{\beta}), j \in \mathbf{J}.$$

and

$$\lambda(y_j, \beta_j) + (1-\lambda)(y'_j, \beta'_j) \in \bar{\beta}_j^{(2)}(p^C, p^S).$$

Hence, by convexity of $T_{j,0}^{(2)}$,

$$\gamma_j(p, \lambda y_j + (1-\lambda)y'_j, \lambda\beta_j + (1-\lambda)\beta'_j) = \gamma_j(p, y_j, \beta_j) = \gamma_j(p, y'_j, \beta'_j) \geq 0, j \in \mathbf{J}, \qquad (A.1)$$

this implies that

$$\bar{\beta}_i^{(1)}(p^C, p^S, e) = \bar{\beta}_i^{(1)}(p^C, p^S, e') = \bar{\beta}_i^{(1)}[p^C, p^S, \lambda e + (1-\lambda)e'].$$

and therefore, by the concavity of $\tilde{U}_i(\cdot)$,

$$\tilde{U}_i[\lambda(x_i, \alpha_i) + (1-\lambda)(x'_i, \alpha'_i)] = \tilde{U}_i(x_i, \alpha_i) = \tilde{U}_i(x'_i, \alpha'_i).$$

It can be deduced from (A.1) that

$$\begin{aligned} &T_{j,0}^{(2)}\{\lambda[p^S(\bar{\beta}_j^0 + \beta_j) + p^C y_j^0] + (1-\lambda)[p^S(\bar{\beta}_j^0 + \beta'_j) + p^C y_j'^0]\} \\ &= \lambda T_{j,0}^{(2)}[p^S(\bar{\beta}_j^0 + \beta_j) + p^C y_j^0] + (1-\lambda)T_{j,0}^{(2)}[p^S(\bar{\beta}_j^0 + \beta'_j) + p^C y_j'^0]. \end{aligned}$$

On the other hand, since any optimal consumption-portfolio plan satisfies the Walras' Law, it is not difficult to prove from (2.1) and (A.1) that

$$T_{i,0}^{(1)}[\lambda p^S \alpha_i + (1-\lambda)p^S \alpha'_i] = \lambda T_{i,0}^{(1)}(p^S \alpha_i) + (1-\lambda)T_{i,0}^{(1)}(p^S \alpha'_i).$$

Hence, $\lambda x_G + (1-\lambda)x'_G \in \beta_G[p^C, p^S, \lambda f + (1-\lambda)f'] = \lambda\beta_G(p^C, p^S, f) + (1-\lambda)\beta_G(p^C, p^S, f')$ and therefore, $\lambda x_G + (1-\lambda)x'_G$ is a optimal solution of the government corresponding to $\lambda f + (1-\lambda)f'$.

Consequently, by combining above results, we arrive at the convexity.□

Proof of Proposition 4.2.5: As in Werner(1987), it suffices to show $\bar{\Psi}(p^C, p^S) \in \Psi(p^C, p^S)$. To this end, suppose $e = (x_1, \alpha_1, \cdots, x_I, \alpha_I, y_1, \beta_1, \cdots, y_J, \beta_J, x_G) \in \bar{\Psi}(p^C, p^S)$, we must show

$$\tilde{U}_i(x_i, \alpha_i) \geq \tilde{U}_i(x'_i, \alpha'_i), \forall (x'_i, \alpha'_i) \in \beta_i^{(1)}(p^C, p^S, e), \forall i \in \mathbf{I} \qquad (A.2)$$

and

$$\gamma_j(p, y_j, \beta_j) \geq \gamma_j(p, y'_j, \beta'_j), \forall (y'_j, \beta'_j) \in \beta_j^{(2)}(p^C, p^S), \forall j \in \mathbf{J}, \tag{A.3}$$

and

$$U_G(x_G) \geq U_G(\bar{x}_G), \ \ \forall \bar{x}_G \in \beta_G(p^C, p^S, f). \tag{A.4}$$

(A.4) holds automatically by the definitions of Ψ and $\bar{\Psi}$. We now prove (A.2) and (A.3).

Suppose that, for some j, there exists a $(y'_j, \beta'_j) \in F_j$ such that $y'_j \in \hat{Y}_j$ and (A.3) does not hold. Consider now the orthogonal projection $\tilde{\beta}'_j$ of β'_j into $\overline{W_j^{(2)}}^{\perp}$. Hence $p(\beta'_j - \tilde{\beta}'_j) = 0$ by Propositttion 3.2, and thus

$$\gamma_j(p, y'_j, \beta'_j) = \gamma_j(p, y'_j, \tilde{\beta}'_j), \ \ j \in \mathbf{J}, \tag{A.5}$$

Note that, by the definition of $\overline{W_j^{(2)}}$,

$$y_j^s - C_s\tilde{\beta}'_j - T_{j,s}^{(2)}(y_j^s - \bar{C}_s\tilde{\beta}'_j) = y_j^s - C_s\beta'_j - T_{j,s}^{(2)}(y_j^s - \bar{C}_s\beta'_j), \ \forall s \in \mathbf{S}.$$

Hence, $(y'_j, \tilde{\beta}'_j) \in \bar{\beta}^{(2)}(p^C, p^S)$, and by (A.5),$\gamma_j(p, y_j, \beta_j) < \gamma_j(p, y'_j, \tilde{\beta}'_j)$, which provides a contradiction and proves the asseration.

Suppose now that (A.2) does not hold for some (x'_i, α'_i). Consider now the orthogonal projection $\tilde{\alpha}'_i$ of α'_i into $\overline{W_i^{(1)}}^{\perp}$.

As in the first case, we have

$$p^S\tilde{\alpha}'_i = p^S\alpha'_i,$$

$$p^S\tilde{\alpha}'_i + p^Cx' + T_{i,0}^{(1)}(p^S\tilde{\alpha}) = p^S\alpha' + p^Cx' + T_{i,0}^{(1)}(p^S\tilde{\alpha}'),$$

and

$$C_s\tilde{\alpha}'_i - T_{i,s}^{(1)}(\bar{C}_s\tilde{\alpha}'_i) = C_s\alpha'_i - T_{i,s}^{(1)}(\bar{C}_s\alpha'_i) \ \forall s \in \mathbf{S}.$$

Thus, $(x'_i, \tilde{\alpha}'_i) \in \bar{\beta}_i^{(1)}(p^C, p^S, e)$, and

$$\tilde{U}_i(x'_i, \tilde{\alpha}'_i) > \tilde{U}_i(x_i, \alpha_i),$$

a contradiction proving the asseration.

Proof of Proposition 4.2.6: It is easy to see that $M^n \longrightarrow \infty$. Let $\hat{x} = (\hat{x}_1, \hat{\alpha}_1, \cdots, \hat{x}_I, \hat{\alpha}_I, \hat{y}_1, \hat{\beta}_1, \cdots, \hat{y}_{|J|}, \hat{\beta}_{|J|}, \hat{x}_G)$ denote any cluster point of sequence $\{x^n = (x_1^n, \alpha_1^n, \cdots, x_I^n, \alpha_I^n, y_1^n, \beta_1^n, \cdots, y_J^n, \beta_J^n, x_G^n) /M^n : n \geq 1\}$. Then the cluster point $\hat{z}$ of $\{z^n/M^n : n \geq 1\}$ can be expressed as $\hat{z} = \sum_{i\in\mathbf{I}}(\hat{x}_i, \hat{\alpha}_i) - \sum_{j\in\mathbf{J}}(\hat{y}_j, \hat{\beta}_j) + (\hat{x}_G, 0)$.

As in the proof of Proposition 3.2 , by Theorem 8.2 of Rockafellar(1970), we can show that $\hat{x}_i = \hat{y}_j = 0$, $\hat{\alpha}_i \in W_i^{(1)} \cup \overline{W_i^{(1)}}$, $\hat{\beta}_j \in W_j^{(2)} \cup \overline{W_j^{(2)}}$ and therefore, $p^S\hat{\alpha}_i \geq 0, \forall i \in \mathbf{I}$ and $p^S\hat{\beta}_j \leq 0, \forall j \in \mathbf{J}$. But $\hat{x}_G$ is not necessarily zero since $\hat{X}_G = R^l_+$. And, moreover, $\hat{\alpha}_i \in \overline{W_i^{(1)}}^{\perp}$ and $\hat{\beta}_j \in \overline{W_j^{(2)}}^{\perp}$. Therefore, or $\hat{x}_G > 0$ or there exists at least one $i \in \mathbf{I}$ or $j \in \mathbf{J}$ such that $p^S\hat{\alpha}_i > 0$, or $p^S\hat{\beta}_j < 0$, for every $p^S \in \mathbf{N}$.

Consequently, $p\hat{z} = p^C\hat{x}_G + p^S(\sum_i \hat{\alpha}_i - \sum_j \hat{\beta}_j) > 0$, proving the conclusion of Proposition 4.2.6.□

Lemma A. $(cl\mathbf{N})^0 = -[\sum_i(W_i^{(1)} \cup \overline{W_i^{(1)}}) - \sum_j(W_j^{(2)} \cup \overline{W_j^{(2)}})]$.

Proof. We will prove the conclusion in four steps.

Step 1: This step is to show $cl\mathbf{N} = (\cap_i cl\mathbf{N}_i^{(1)}) \cap (\cap_j cl\mathbf{N}_j^{(2)})$. It is obvious that the right side includes the left side. Now we prove the opposite inclusion. Let $p \in (\cap_i cl\mathbf{N}_i^{(1)}) \cap (\cap_j cl\mathbf{N}_j^{(2)})$ and $p^0 \in \mathbf{N}$. Then, by Definition 3.1, $\lambda p + (1-\lambda)p^0 \in \mathbf{N}, \forall \lambda \in (0,1)$. And $\lambda p + (1-\lambda)p^0 \longrightarrow p$ as $\lambda \longrightarrow 1$ and therefore, $p \in cl\mathbf{N}$.

Step 2: This step is to show $cl\mathbf{N}_i^{(1)} = -(W_i^{(1)} \cup \overline{W_i^{(1)}})^0$. It follows from Definition 3.1 and Proposition 3.2 that $cl\mathbf{N}_i^{(1)} \subseteq -(W_i^{(1)} \cup \overline{W_i^{(1)}})^0$. It remains to show the opposite inclusion. Suppose that $p \in -(W_i^{(1)} \cup \overline{W_i^{(1)}})^0$, that is $p\alpha \geq 0, \forall \alpha \in W_i^{(1)} \cup \overline{W_i^{(1)}}$ and therefore, $\lambda p + (1-\lambda)p' \in \mathbf{N}_i^{(1)}$ for all $p' \in \mathbf{N}_i^{(1)}$ and $\lambda \in (0,1)$. Hence, by taking $\lambda \longrightarrow 1$, $p \in cl\mathbf{N}_i^{(1)}$.

Likewise, it can be shown that $cl\mathbf{N}_j^{(2)} = (W_j^{(2)} \cup \overline{W_j^{(2)}})^0$.

Step 3: This step is to show that the set $\sum_i(W_i^{(1)} \cup \overline{W_i^{(1)}}) - \sum_j(W_j^{(2)} \cup \overline{W_j^{(2)}})$ is closed. By Corollary 9.1.1 in Rockafellar(1970), it suffice to show that if $z_i \in O^+(W^{(1)} \cup \overline{W_i^{(1)}})$ and $z'_j \in O^+(W_j^{(2)} \cup \overline{W_j^{(2)}})$ with $\sum_i z_i - \sum_j z'_j = 0$, then $-z_i \in O^+(W_i^{(1)} \cup \overline{W_i^{(1)}})$ and $-z'_j \in O^+(W_j^{(2)} \cup \overline{W_j^{(2)}})$.

Since $0 \in W_i^{(1)} \cup \overline{W_i^{(1)}}$ and $0 \in W_j^{(2)} \cup \overline{W_j^{(2)}}$, $z_i \in W_i^{(1)} \cup \overline{W_i^{(1)}}$ and $z'_j \in W_j^{(2)} \cup \overline{W_j^{(2)}}$ by definition of recession cone. This implies $pz_i \geq 0$ and $pz'_j \leq 0, \forall p \in \mathbf{N}'$ and therefore, $pz_i = pz'_j = 0$ since $\sum_i z_i - \sum_j z'_j = 0$.

Consequently, $z_i \in \overline{W_i^{(1)}}$ and $z'_j \in \overline{W_j^{(2)}}$ and therefore, $-z_i \in \overline{W_i^{(1)}} \subseteq O^+(W_i^{(1)} \cup \overline{W_i^{(1)}})$ and $-z'_j \in \overline{W_j^{(2)}} \subseteq O^+(W_j^{(2)} \cup \overline{W_j^{(2)}})$..

Step 4: As in Werner(1987), by Corollary 16.4.2 of Rockafeller(1970) and by combining Step 1-Step 3, we finish the proof of this lemma.

References

[1] Arrow,K.J. and G.Debreu(1954):"Existence of an Equilibrium for a Competitive Economy," *Econometrica*,22,265-290.

[2] Blanchard, O. and S. Fischer(1989):*Lectures in Macroeconomics*. Cambridge:MIT Press.

[3] Constantinides, G.(1983):"Capital Market Equilibrium with Personal Tax," *Econometrica*, 51,611-636.

[4] Dammon, R.M. and R.C. Green(1987):"Tax Arbitrage and the Existence of Equilibrium Prices for Financial Assets," *Journal of Finance*, 42,1143-1166.

[5] Dybvig, P.H. and S.A. Ross(1986):"Tax Clienteles and Asset Pricing," *Journal of Finance*, 39,751-762.

[6] Ellickson, B.(1993):*Competitive Equilibrium*, Cambridge University Press.

[7] Green, R.C.(1993):"A Simple Model of the Taxable and Tax-Exempt Yield Curves," *The Review of Financial Studies*, 6,233-264.

[8] Heller,W. and Starr,R.(1976):"Equilibrium with Non-Convex Transaction Costs:Monetary and Non-Monetary Economics," *Review of Economic Studies*, 43,195-215.

[9] Jones, C. and F. Milne(1992):"Tax Arbitrage, Existence of Equilibrium, and Bounded Tax Rebates," *Mathematical Finance*, 2,189-196.

[10] Rockafeller, R.T.(1970):*Convex Analysis*, N.J,:Princeton University Press.

[11] Ross,S.(1987):"Arbitrage and Martingales with Taxation," *Journal of Political Economy*, 95,371-393.

[12] Werner,J.(1987):"Arbitrage and the Existence of Competitive Equilibrium", *Econometrica*, 55,1403-1418.

[13] Zechner, J.(1990):"Tax Clienteles and Optimal Capital Structure under Uncertainty," *Journal of Business*, 63,465-491.