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The Existence of Equilibrium in a Financial Market with Transaction Costs

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Abstract

This paper proves the existence of a general equilibrium in a financial model with transaction costs. The general equilibrium is shown to exist in a model with convex trading technology, in which the agents include consumers, production firms, brokers and dealers. When the trading technology is non-convex, an individual approximate equilibrium, introduced by Heller and Starr(1976), is proved in the above model. And, moreover, under a further assumption of finite p -convexity on the commodity excess demand correspondence, the general equilibrium for a non-convex exchange economy is obtained for an economy with consumers, brokers and dealers.

Keywords: Arbitrage, general equilibrium, transaction cost, individual approximate equilibrium, finite p -convexity.

1. Introduction

A number of authors have considered financial markets with transaction costs, particularly the impact of transaction costs on optimal portfolio selection, (e.g., Magill and Constantinides(1976), Kandell and Ross(1983), Taksar, Klass and Assaf(1988), Duffie and Sun(1990), Fleming et.al.(1989), Davis and Norman(1990)); and the pricing and hedging of derivative securities using the underlying stock and bond(e.g.,Leland(1985), Boyle and Vorst(1992), Bensaid, et al.(1992), Edirisinghe, Naik and Uppal(1993), Constantinedes and Zariphopoulou(1995)).

More recently, some authors(e.g., Jouini and Kallal(1995), Ortu(1995), Milne and Neave(1996)) have investigated economies with transaction costs and the implications for asset prices and allocations. Jouini and Kallal(1995) use arbitrage methods as introduced by Harrison and Kreps(1979) to obtain a set of equivalent Martingale measures that are deduced from an economy with transaction costs and an absence of arbitrage. Ortu(1995) uses duality and linear programming methods in finite dimensions extending Jouini and Kallal's results. Milne and

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Neave(1996) formulate a competitive economy with multiple dates, uncertainty, a single physical commodity and a set of consumers trading assets through a set of broker/intermediaries, who have explicit transaction technologies. The formulation draws upon an older literature in General Equilibrium theory that characterises transaction costs by discriminating between bought and sold commodities or assets(see Ostroy and Starr(1990) for a survey of this literature). Assuming the existence of an equilibrium, Milne and Neave characterise pricing and asset allocations for a number of special cases of their model, emphasising the model's flexibility in encompassing many cases discussed separately in the literature(e.g., broking, personal transaction costs, fixed and variable transaction costs, inventory-type models, incomplete markets). The Milne-Neave model is consistent with the formulations of Jouini-Kallal and Ortu, in providing a general primal formulation. The latter papers exploit no arbitrage/duality methods to obtain similar, or complementary results.

This paper constructs a more general version of Milne and Neave(1996) including many physical commodities, producers/firms and general assumptions on feasible consumption, production sets, and transaction technologies. We provide conditions that guarantee the existence of an equilibrium in this economy when the transaction technology is convex. To incorporate non-convexities(or fixed costs) in transactions, the model considers a modification introduced in an earlier general equilibrium literature(see Heller and Starr(1976)) that allows us to prove the existence of an approximate equilibrium. In addition we introduce a different method for proving existence in a version of our economy with non-convex transaction technologies. We suppose a condition of finite p-convexity on the commodity excess demand correspondence. This assumption allows a limited degree of non-convexity in the asset trade technology, and yet a well-defined element of convexity in commodity demand to generalise the existence of an equilibrium.

The rest of this paper is organized as follows. In Section 2, we outline the basic model and introduce the concept of no arbitrage and its equivalent condition. In Section 3, we will prove some preliminary results and, finally, show the existence of equilibrium for the model with convex trading technology. Section 4 is devoted to individual approximate equilibrium in the model with non-convex trading technology. In Section 5, a concept of finite p-convexity is introduced and the general equilibrium is proved in an exchange economy. The appendix includes a proof of one preliminary result.

2. The Model and No Arbitrage

Consider an economy with a finite time horizon $\mathbf{T} = \{1, \dots, T\}$ and uncertainty character-

ized by a finite states set $\mathbf{S} = \{1, \dots, S\}$, each state represents a realization of the economy. As discussed in Duffie(1987) and DeMarzo(1988), in an event tree, each state here corresponds to one arc on the tree from the root to one of the terminal nodes. Suppose there are M commodities and N securities at each date $t \in \mathbf{T} = \{1, \dots, T\}$ and each state $s \in \mathbf{S}$.

There are J production firms(indexed by j) with objective function $V_j(\cdot)$, each of whom chooses a production plan and a trade plan. A production plan of firm j is an array of numbers $y_{j,t,m}^s$, one for each $m \in \mathbf{M} = \{1, \dots, M\}$, $s \in \mathbf{S}$ and $t \in \mathbf{T}$ with the usual sign convention for inputs(non-positive) and outputs(non-negative). Thus a contingent production plan $y_j^s = (y_{j,1,1}^s, \dots, y_{j,1,M}^s, \dots, y_{j,T,1}^s, \dots, y_{j,T,M}^s)$ of firm j is a point of vector space $\mathbf{R}^{T \times M}$. The set of all contingent production plans that are technologically feasible for firm j will be denoted by $Y_j \subseteq \mathbf{R}^{T \times M \times S}$.

A trade plan of firm j is a vector $\theta_j = (\theta_j(s))_{s \in \mathbf{S}} = (\theta_j^B(s), \theta_j^S(s))_{s \in \mathbf{S}} = (\theta_{j,0,1}^B, \dots, \theta_{j,0,N}^B, \theta_{j,0,1}^S, \dots, \theta_{j,0,N}^S, \theta_{j,1,1}^B(s), \dots, \theta_{j,1,N}^B(s), \theta_{j,1,1}^S(s), \dots, \theta_{j,1,N}^S(s), \dots, \theta_{j,T,1}^B(s), \dots, \theta_{j,T,N}^B(s), \theta_{j,T,1}^S(s), \dots, \theta_{j,T,N}^S(s))_{s \in \mathbf{S}}$ in $\mathbf{R}_+^{2[(T+1) \times N \times S]}$, where $\theta_{j,t,n}^B(s)(\theta_{j,t,n}^S(s))$ represents the accumulated purchase(sale) of asset n by firm j after trading at time $t(\in \mathbf{T})$ and at state s ; $\theta_{j,0,n} = (\theta_{j,0,n}^B, \theta_{j,0,n}^S)$ denotes the initial trading of asset n by firm j .

The variables below, such as commodity price, buying price and selling price of asset, portfolio selection and consumption etc., all depend upon states. For the sake of simplicity, we will omit the state variable in these symbols.

There are H brokers(indexed by h) with objective function $W_h(\cdot)$. They are intermediaries specializing in the transaction technology that transforms bought and sold assets. Let $\phi_{h,t,n}^B(\phi_{h,t,n}^S)$, depending on states, be the accumulated number of bought(sold) asset n supplied by intermediary h after trading at time $t(t = 0, \dots, T)$; and $z_{h,t} = (z_{h,t,1}, \dots, z_{h,t,M})$, depending on states, is the vector of contingent commodities used up in the activity of intermediation at date t and denote $(z_{h,1}, \dots, z_{h,M})$ by z_h . For intermediary h , let $T_{h,t} \subseteq \mathbf{R}_+^N \times \mathbf{R}_+^N \times \mathbf{R}^M$ denote its technology at time t and state s .

There are I consumers, indexed by i , with endowment $\omega_{i,t}$ and utility function $U_i(\cdot)$ and consumption set $X_i = \mathbf{R}_+^{T \times M \times S}$. The consumer i chooses a consumption plan $x_i = (x_{i,1,1}, \dots, x_{i,1,M}, \dots, x_{i,T,1}, \dots, x_{i,T,M})_{s \in \mathbf{S}} \in X_i$ and a portfolio plan $\psi_i = (\psi_i^B, \psi_i^S) \in \mathbf{R}_+^{2[(T+1) \times N]}$ which can be explained analogously to θ_j .

Now we turn to assets. Suppose there are N assets at each date and each state. At each date t and each state s , asset $n(n = 1, \dots, N)$ has a buying price B_t^n and selling price S_t^n and dividend d_t^n denominated in the first commodity(numeraire). Suppose that at each date t and

each state s , the asset n pays its dividend d_t^n and is then available for trade at prices B_t^n and S_t^n .

Suppose these N assets are defined by the \mathbf{R}^N -valued process $d = \{d_t = (d_t^1, \dots, d_t^N) : t = 0, \dots, T, s \in \mathbf{S}\}$, buying price process $B = \{B_t = (B_t^1, \dots, B_t^N) : t = 0, \dots, T, s \in \mathbf{S}\}$ and selling price process $S = \{S_t = (S_t^1, \dots, S_t^N) : t = 0, \dots, T, s \in \mathbf{S}\}$. A dividend process d^θ generated by a generic strategy $\theta = (\theta^B, \theta^S)$ is defined by

$$d_t^\theta(s) = d_t^\theta = (\theta_{t-1}^B - \theta_{t-1}^S)dt + \Delta\theta_t^S S_t - \Delta\theta_t^B B_t, t = 0, \dots, T, s \in \mathbf{S},$$

with " $\theta_{-1} = (\theta_{-1}^B, \theta_{-1}^S)$ " taken to be zero by convention. Define $\Delta\theta_t^B = \theta_t^B - \theta_{t-1}^B$; and $\Delta\theta_t^S = \theta_t^S - \theta_{t-1}^S$; and $\Delta\theta_t = \theta_t - \theta_{t-1}$.

Let $p = (p)_{s \in \mathbf{S}} = (p_1, \dots, p_T)_{s \in \mathbf{S}} = (p_{1,1}, \dots, p_{1,M}, \dots, p_{T,1}, \dots, p_{T,M}) \in (\Delta_0)_{s \in \mathbf{S}}$ denote the spot price of commodities, where Δ_0 is the unit simplex of $\mathbf{R}^{T \times M \times S}$.

Now the problem of firm j can be specified as:

$$\sup_{(\theta_j, y_j) \in \Gamma_j^1(p)} V_j(d^{\theta_j} + p y_j), \quad (*)$$

where $\Gamma_j^1(p)$ denote the set of feasible production-trade plans (θ_j, y_j) of firm j given price p , which satisfies:

$$(2.1) y_j \text{ is in } Y_j;$$

$$(2.2) d_t^{\theta_j} + p_t y_{j,t} \geq 0, t = 1, \dots, T, s \in \mathbf{S}.$$

The maximization problem of broker h can be stated as:

$$\sup_{(\phi_h, z_h) \in \Gamma_h^2(\gamma_h)} W_h(-d^{\phi_h} - p z_h), \quad (**)$$

where $\Gamma_h^2(\gamma_h)$ ($\gamma_h = ((\psi_i), (\theta_j), (\phi_{h' \neq h}), p)$) is the space of feasible trade-production plans $(\phi_h, z_h) = (\phi_h^B, \phi_h^S, z_h)$ given γ_h , which satisfies: at each state,

$$(2.3) (\Delta\phi_{h,t}^B, \Delta\phi_{h,t}^S, z_{h,t}) \in T_{h,t} \text{ and } z_{h,t} \geq 0;$$

$$(2.4) -d_t^{\phi_h} - p_t z_{h,t} \geq 0, t = 1, \dots, T;$$

$$(2.5) \sum_h \Delta\phi_{h,t} \geq \sum_i \Delta\psi_{i,t} + \sum_j \Delta\theta_{j,t}, t = 0, \dots, T.$$

The condition (2.5) requires that all consumers and all production firms buy and sell securities through brokers.

The productive firms and intermediaries firms are treated similarly to consumers. Because of transaction costs, it is well-known that the Fisher separation theorem fails. Therefore we assume that each firm has an objective function (utility function) derived in some fashion from

the preference of the owners. For example, we can either assume one-owner firms or draw on DeMarzo(1988) and argue that the objective is derived from a more complicated composition of owner preferences.

The problem of consumer i is as follows:

$$\sup_{(x_i, \psi_i) \in \Gamma_i^3(\tau)} U_i(x_i), \quad (***)$$

where $\tau = ((\phi_h, z_h), (\theta_j, y_j), p)$ and $\Gamma_i^3(\tau)$ is the set of feasible portfolio-consumption plans (ψ_i, x_i) given τ , which satisfies:

(2.6) x_i is in X_i ;

(2.7) $p_t x_{i,t} \leq p \omega_{i,t} + d_t^{\psi_i} + \sum_j \alpha_{i,j} \bar{y}_{j,t} + \sum_h \beta_{i,h} \bar{z}_{h,t}, t = 1, \dots, T, s \in \mathbf{S}$;

Here $\bar{y}_{j,t} = d_t^{\theta_j} + p_t y_{j,t}, \bar{z}_{h,t} = -d_t^{\phi_h} - p_t z_{h,t}; \alpha_{i,j} \geq 0 (\sum_i \alpha_{i,j} = 1)$ is consumer i 's initial share of the net cash flow of firm j ; and $\beta_{i,h} \geq 0 (\sum_i \beta_{i,h} = 1)$ is consumer i 's initial share of net cash flow of broker h .

Now we can define the abstract economy: $E = (X_1 \times \mathbf{R}^{2[(T+1) \times N \times S]}, \dots, X_I \times \mathbf{R}^{2[(T+1) \times N \times S]}, Y_1 \times \mathbf{R}^{2[(T+1) \times N \times S]}, Y_J \times \mathbf{R}^{2[(T+1) \times N \times S]}, T'_1, \dots, T'_H, \Delta_0, U_1(x_1), \dots, U_I(x_I), V_1(d^{\theta_1} + p y_1), \dots, V_J(d\theta_J + p y_J), W_1(-d^{\phi_1} - p z_1), \dots, W_H(-d^{\phi_H} - p z_H), \sum_{s \in \mathbf{S}} \sum_{t=1}^T \pi_t^s p_t w_t, \Gamma_1^3(\tau), \dots, \Gamma_I^3(\tau), \Gamma_1^1(p), \dots, \Gamma_J^1(p), \Gamma_1^2(\gamma_1), \dots, \Gamma_H^2(\gamma_H), \Delta_0)$, where $\pi_t^s (t = 0, \dots, T, s \in \mathbf{S})$ will be defined in Lemma 2.1 below, $w_t = \sum_i x_{i,t} + \sum_h z_{h,t} - \sum_j y_{j,t} - \sum_i \omega_{i,t}$ and

$$T'_h = \{(\phi_h^B, \phi_h^S, z_h) : (\Delta \phi_{h,t}^B, \Delta \phi_{h,t}^S, z_{h,t}) \in T_{h,t}, t \in \mathbf{T}, s \in \mathbf{S}\}.$$

A point $e^* = ((\psi_i^*, x_i^*), (\theta_j^*, y_j^*), (\phi_h^*, z_h^*), p^*)$ is called an equilibrium solution of economy E given the market system (B, S, d) if e^* solves problems (*), (**) and (***) and

$$\sum_i x_i^* + \sum_h z_h^* = \sum_j y_j^* + \sum_i \omega_i,$$

$$\sum_h \Delta \phi_h^* = \sum_i \Delta \psi_i^* + \sum_j \Delta \theta_j^*.$$

where $\omega_i = (\omega_{i,1}, \dots, \omega_{i,T})$.

The following assumptions are made in the remainder of this paper.

For consumer i :

(A.1) $U_i(\cdot)$ is continuous, concave and strictly increasing function.

For firm j :

(A.2) Y_j is a closed convex subset of $\mathbf{R}^{T \times M \times S}$ containing $-\mathbf{R}_+^{T \times M \times S}$;

$$(A.3) Y_j \cap \mathbf{R}_+^{T \times M \times S} = 0;$$

$$(A.4) (\sum_j Y_j) \cap (-\sum_j Y_j) = 0.$$

(A.5) $V_j(\cdot)$ is continuous, concave and strictly increasing function.

For broker h :

(A.6) For each t , $T_{h,t}$ is closed and convex set with $0 \in T_{h,t}$.

(A.7) For any t and given $x = (x_1, \dots, x_{2N}, z_1, \dots, z_M) \in T_{h,t}$, If $y = \sum_{n=1}^{2N} x_n \rightarrow \infty$, then $|(z_1, \dots, z_M)| = \sum_{m=1}^M z_m \rightarrow \infty$.

(A.8) For each t , if $(\psi, z) \in T_{h,t}$ and $z' \geq z$, then $(\psi, z') \in T_{h,t}$ (free disposal).

(A.9) $W_h(\cdot)$ is continuous, concave and strictly increasing function;

(A.10) The initial holdings $\psi_{i,0}, \theta_{j,0}, \phi_{h,0}$ of securities of all consumers, all firms and all brokers are taken as given and satisfy:

$$\sum_i \psi_{i,0} + \sum_j \theta_{j,0} = \sum_h \phi_{h,0}.$$

The assumptions (A.1)–(A.6) are standard. (A.7) says that transactions must consume resource. (A.8) and (A.9) are standard. And (A.10) says that the initial net trading is zero.

Now we conclude this section by introducing the concept of no-arbitrage.

Given a price-dividend triple (B, S, d) for N securities, a trading strategy θ is an arbitrage if for each state $s \in S$ and $t = 0, \dots, T$, $d_t^\theta(s) \geq 0$ but $d_t^\theta(s) \neq 0$.

(A.11) The price-dividend triple (B, S, d) admit no-arbitrage.

The following equivalent condition of no arbitrage is similar to Proposition 2C of Duffie(1996) and will play an important role in the proof of market clearing later.

Lemma 2.1 There is no arbitrage if and only if there is a strictly increasing linear function $F: \mathbf{R}^{(T+1) \times S} \rightarrow \mathbf{R}^1$ such that $F(d^\theta) \leq 0$ for any trading strategy $\theta \in \Theta$, where Θ denotes the space of trading strategy and is a closed and convex set.

Proof. There is no arbitrage if and only if the cones $\mathbf{R}_+^{(T+1) \times S}$ and $M^0 = \{d^\theta : \theta \in \Theta\}$ intersect precisely at zero. If there is no arbitrage, the theorem "Linear Separation of Cones" in Appendix B of Duffie(1996) implies the existence of nonzero linear functional F such that $F(x) < F(y)$ for each x in M^0 and each nonzero y in $\mathbf{R}_+^{(T+1) \times S}$. Since M^0 is a cone, this implies $F(x) \leq 0$ for each x in M^0 . And, moreover, $0 \in M^0$, thus $F(y) > F(0) = 0$ for each nonzero $y \in \mathbf{R}_+^{(T+1) \times S}$, that is, F is strictly increasing. The converse is immediate. \square

In comparison with Proposition 2C in Duffie(1992), the next result shows a difference between the model without transaction costs and the model with transaction costs caused by the bid-ask spread at some date.

Lemma 2.2 Suppose there is no arbitrage and $B_t^n > S_t^n$ for security n at certain date t and state s . Then $F(d^\theta) \neq 0$ over Θ .

Proof. Suppose not. Then $F(d^\theta) = 0$ for each $\theta \in \Theta$. Since $F(\cdot)$ is a strictly increasing linear functional on $\mathbf{R}^{(T+1) \times S}$, this implies that there exists a vector $\pi = (\pi^s)_{s \in S} = (\pi_0^s, \dots, \pi_T^s)_{s \in S} \in \text{int}(\mathbf{R}_+^{(T+1) \times S})$ such that $F(x) = \sum_{s \in S} \sum_{t=0}^T \pi_t^s x_t$ for each $x = (x_0, \dots, x_T)_{s \in S} \in \mathbf{R}^{(T+1) \times S}$. Without loss of generality, assume $S = 1$ and $n = 1$. For date t , set

$$\theta_q^B = (0, \dots, 0), q = 0, \dots, t-1, \theta_q^B = (1, 0, \dots, 0), q = t, \dots, T:$$

$$\theta_q^S = (0, \dots, 0), q = 0, \dots, T.$$

Then $d_q^\theta = 0, q = 0, \dots, t-1, d_t^\theta = -B_t^1, d_q^\theta = d_q^1, q = t+1, \dots, T$.

Hence

$$\pi_t^s B_t^1 = \sum_{q=t+1}^T \pi_q^s d_q^1,$$

Likewise, we can show

$$\pi_t^s S_t^1 = \sum_{q=t+1}^T \pi_q^s d_q^1.$$

Thus $B_t^1 = S_t^1$, which provides a contradiction and proves the conclusion of the lemma. \square

3. Proof of Existence of Equilibrium

To simplify the proofs in this paper, we will prove the existence of equilibrium of economy with only one state. The multi-state case can be shown in exactly the same manner.

We will adopt the technique of proof used in Arrow-Debreu(1954). First of all, we will show that the set of attainable plans for economy E is bounded, and replace the original economy E by a bounded one. Secondly, we will show the continuity of the constrained correspondences.

For broker h , define

$$Z_h = \{z_h : \text{there exists } (\phi_h^B, \phi_h^S) \geq 0 \text{ such that } (\phi_h^B, \phi_h^S, z_h) \in T_h'\}.$$

$\hat{Z}_h = \{z_h \in Z_h : \text{there exist } z_{h'} \in Z_{h'} \text{ for each } h' \neq h, x_i \in X_i \text{ for each } i \text{ and } y_j \in Y_j \text{ for each } j \text{ such that } w = \sum_i x_i + \sum_h z_h - \sum_j y_j - \sum_i \omega_i \leq 0\};$

$$\Phi_h = \{\phi_h = (\phi_h^B, \phi_h^S) : \text{there exists } z_h \in \hat{Z}_h \text{ such that } (\phi_h, z_h) \in T_h'\};$$

$\hat{\Phi}_h = \{\phi_h = (\phi_h^B, \phi_h^S) \in \Phi_h : \text{there exist } \phi_{h'} \in \Phi_{h'} \text{ for each } h' \neq h, \Delta\theta_j \geq 0 \text{ for each } j \text{ and } \Delta\psi_i \geq 0 \text{ for each } i \text{ such that } \sum_i \Delta\psi_i + \sum_j \Delta\theta_j \leq \sum_h \Delta\phi_h\}.$

For consumer i , define

$\hat{X}_i = \{x_i \in X_i : \text{there exist } x_{i'} \in X_{i'} \text{ for each } i' \neq i, z_h \in Z_h \text{ for each } h \text{ and } y_j \in Y_j \text{ for each } j \text{ such that } w = \sum_i x_i + \sum_h z_h - \sum_j y_j - \sum_i \omega_i \leq 0\}$;

$\hat{\Psi}_i = \{\psi_i = (\psi_i^B, \psi_i^S) : \text{there exist } \Delta\psi_{i'} \geq 0 \text{ for each } i' \neq i, \phi_h \in \Phi_h \text{ for each } h, \Delta\theta_j \geq 0 \text{ for each } j \text{ such that } \sum_i \Delta\psi_i + \sum_j \Delta\theta_j \leq \sum_h \Delta\phi_h\}$.

Likewise, we can define \hat{Y}_j and $\hat{\Theta}_j$ for firm j .

By use of the technique of 3.3.1 of Arrow-Debreu(1954) and by Lemma A1. we can show the following result.

Lemma 3.1 The sets \hat{X}_i, \hat{Y}_j and \hat{Z}_h are all compact and convex.

In exactly the same method as Lemma A1, we can show the following boundedness result of trade plan for broker h .

Lemma 3.2 The set Φ_h is a compact and convex subset of $\mathbf{R}^{(T+1) \times N}$. And so is $\hat{\Phi}_h, \hat{\Psi}_i$ and $\hat{\Theta}_j$.

Thus there exist cubes $C^1(\subseteq \mathbf{R}^{T \times M})$ and $C^2(\subseteq \mathbf{R}_+^{2[(T+1) \times N]})$ so that C^1 contains in its interior all \hat{X}_i , all \hat{Y}_j and all \hat{Z}_h ; C^2 contains all $\hat{\Psi}_i$, all $\hat{\Theta}_j$ and all $\hat{\Psi}_h$. Define $\tilde{X}_i = C^1 \cap X_i, \tilde{\Psi}_i = C^2, \tilde{Y}_j = C^1 \cap Y_j, \tilde{\Theta}_j = C^2, \tilde{Z}_h = C^1 \cap Z_h$ and $\tilde{\Phi}_h = C^2$. And let $\tilde{\Gamma}_j^1(p), \tilde{\Gamma}_h^2(\gamma_h)$ and $\tilde{\Gamma}_i^3(\tau)$ be the resultant modification of $\Gamma_j^1(p), \Gamma_h^2(\gamma_h)$ and $\Gamma_i^3(\tau)$ respectively.

We now turn to the proof of continuity of $\tilde{\Gamma}_j^1(p), \tilde{\Gamma}_h^2(\gamma_h)$ and $\tilde{\Gamma}_i^3(\tau)$. We only investigate the continuity of $\tilde{\Gamma}_h^2$, the continuity of the others can be shown similarly.

Lemma 3.3 Given p , all ψ_i , all θ_j and all $\phi_{h'} (h' \neq h)$, and there exists $(\phi_h, z_h) \in T'_h$ such that $\sum_i \Delta\psi_i + \sum_j \Delta\theta_j \ll \sum_h \Delta\phi_h$ and $0 \ll -d^{\phi_h} - pz_h$. Then $\tilde{\Gamma}_h^2(\gamma_h)$ is continuous.

Proof. Without loss of generality, we show the continuity of $\tilde{\Gamma}_1^2(\gamma_1)$. Let $\gamma_1^k = (\psi_1^k, \dots, \psi_I^k, \theta_1^k, \dots, \theta_J^k, \phi_2^k, \dots, \phi_H^k, p^k) \rightarrow \gamma_1 = (\psi_1, \dots, \psi_I, \theta_1, \dots, \theta_J, \phi_2, \dots, \phi_H, p)$. Consider a point $(\phi_1, z_1) \in \tilde{\Gamma}_1^2(\gamma_1)$, then

$$0 \leq -d^{\phi_1} - pz_1, \quad \sum_i \Delta\psi_i + \sum_j \Delta\theta_j \leq \sum_h \Delta\phi_h.$$

If $0 \ll -d^{\phi_1} - pz_1, \sum_i \Delta\psi_i + \sum_j \Delta\theta_j \ll \sum_h \Delta\phi_h$, then for k sufficiently large. $0 \ll -d^{\phi_1} - p^k z_1$, and $\sum_i \Delta\psi_i^k + \sum_j \Delta\theta_j^k \ll \sum_{h \neq 1} \Delta\phi_h^k + \Delta\phi_1$. By taking $(\phi_1^k, z_1^k) = (\phi_1, z_1)$, we prove the conclusion of the lemma.

If the above case does not hold, then there exist $T_0 \subseteq \mathbf{T} = \{1, \dots, T\}$ and $T'_0 \subseteq \mathbf{T}' = \mathbf{T} \times \{1, \dots, N\}$ (there is at least one nonempty set among T_0 and T'_0) such that

$$\begin{aligned} -d_t^{\phi_1} - p_t z_{1,t} &= 0, \quad t \in T_0; \\ -d_t^{\phi_1} - p_t z_{1,t} &> 0, \quad t \in \mathbf{T} - T_0; \end{aligned}$$

and

$$\sum_i \Delta \psi_{i,t,q} + \sum_j \Delta \theta_{j,t,q} = \sum_h \Delta \phi_{h,t,q}, \quad (t, q) \in T'_0;$$

$$\sum_i \Delta \psi_{i,t,q} + \sum_j \Delta \theta_{j,t,q} < \sum_h \Delta \phi_{h,t,q}, \quad (t, q) \in \mathbf{T}' - T'_0;$$

By assumption, we can choose $(\phi'_1, z'_1) \in T'_1$ such that

$$0 \ll -d^{\phi'_1} - pz'_1,$$

and

$$\sum_i \Delta \psi_i + \sum_j \Delta \theta_j \ll \sum_{h \neq 1} \Delta \phi_h + \Delta \phi'_1.$$

Clearly

$$\Delta \phi'_{1,t,q} > \Delta \phi_{1,t,q}. \quad (t, q) \in T'_0.$$

Let

$$\lambda_k^1 = \min \left\{ 1, \frac{\Delta \phi'_{1,t,q} - (\sum_i \Delta \psi_{i,t,q}^k + \sum_j \Delta \theta_{j,t,q}^k - \sum_{h \neq 1} \Delta \phi_{h,t,q}^k)}{\Delta \phi'_{1,t,q} - \Delta \phi_{1,t,q}} : (t, q) \in T'_0 \right\};$$

and

$$\lambda_k^2 = \min \left\{ 1, \frac{-d_t^{\phi'_1} - p_t^k z'_{1,t}}{-d_t^{\phi'_1} - p_t^k z'_{1,t} - (-d_t^{\phi_1} - p_t^k z_{1,t})} : t \in T_0 \right\}$$

Let $\lambda_k = \min(\lambda_k^1, \lambda_k^2)$ and $(\phi_1^k, z_1^k) = \lambda_k(\phi_1, z_1) + (1 - \lambda_k)(\phi'_1, z'_1)$.

It is easy to verify that $\lambda_k^i \rightarrow 1 (i = 1, 2)$ and

$$-d_t^{\phi_1^k} - p_t^k z_{1,t}^k \geq 0 : t \in T_0; \quad (3.1)$$

and

$$\sum_i \Delta \psi_{i,t,q}^k + \sum_j \Delta \theta_{j,t,q}^k \leq \sum_{h \neq 1} \Delta \phi_{h,t,q}^k + \Delta \phi_{1,t,q}^k, \quad (t, q) \in T'_0; \quad (3.2)$$

for k sufficiently large.

On the other hand, since $\alpha = \min_{t \in \mathbf{T} - \mathbf{T}_0} \{-d_t^{\phi_1} - p_t z_{1,t}\} > 0$ and

$$\lim_{k \rightarrow \infty} (-d_t^{\phi_1^k} - p_t^k z_{1,t}^k) = -d_t^{\phi_1} - p_t z_{1,t}.$$

Hence, for k sufficiently large,

$$-d_t^{\phi_1^k} - p_t^k z_{1,t}^k \geq 0 : t \in \mathbf{T} - \mathbf{T}_0,$$

which, combining with (3.1), implies that for k sufficiently large,

$$-d^{\phi_1^k} - p^k z_1^k \geq 0.$$

Likewise, we can show that for k sufficiently large,

$$\sum_i \Delta \psi_i^k + \sum_j \Delta \theta_j^k \leq \sum_h \Delta \phi_h^k.$$

Consequently, $(\phi_1^k, z_1^k) \in \tilde{\Gamma}_1^2(\gamma_1)$ and converges to (ϕ_1, z_1) , proving the continuity of $\tilde{\Gamma}_1^2(\cdot)$. \square
For firms and consumers, we have the following similar results.

Lemma 3.4 For firm j , given any price $p \in \Delta_0$, and there exist $\theta_j^p \geq 0$ and $y_j^p \in Y_j$ such that $0 \ll d^{\theta_j^p} + p y_j^p$. Then $\tilde{\Gamma}_j^1(\cdot)$ is continuous.

Lemma 3.5 For consumer i , given any price $p \in \Delta_0$, and there exist $\psi_i^p \geq 0$ and $x_i^p \in X_i$ such that

$$p x_i^p \ll p \omega_i + d^{\psi_i^p} + \max \left\{ 0, \sum_j \alpha_{i,j} \bar{y}_j + \sum \beta_{i,h} \bar{z}_h \right\}.$$

Then $\tilde{\Gamma}_i^3(\cdot)$ is continuous.

Remark 3.1: The conditions in Lemmas 3.4 and 3.5 will be satisfied if there exists a portfolio θ such that $d_t^\theta > 0, t = 1, \dots, T$, which is the Assumption 2.1 of Ortu(1995) called the "Internality Condition."

Let

$$\mu_i = \mu_i(\tau) = \{(\psi_i, x_i) : U_i(x_i) = \sup_{(\bar{\psi}_i, \bar{x}_i) \in \tilde{\Gamma}_i^3(\tau)} U_i(\bar{x}_i)\};$$

$$v_j = v_j(p) = \{(\theta_j, y_j) : V_j(d^{\theta_j} + p y_j) = \sup_{(\bar{\theta}_j, \bar{y}_j) \in \tilde{\Gamma}_j^1(p)} V_j(d^{\bar{\theta}_j} + p \bar{y}_j)\};$$

$$\tau_h = \tau_h(\gamma_h) = \{(\phi_h, z_h) : W_h(-d^{\phi_h} - p z_h) = \sup_{(\bar{\phi}_h, \bar{z}_h) \in \tilde{\Gamma}_h^2(\gamma_h)} W_h(-d^{\bar{\phi}_h} - p \bar{z}_h)\};$$

$$\bar{p} = \bar{p}(w) = \{p : \sum_{t=1}^T \pi_t p_t w_t = \sup_{p' \in \Delta_0} \sum_{t=1}^T \pi_t p'_t w_t\},$$

and

$$\Psi = \prod_{i=1}^I \mu_i \times \prod_{j=1}^J v_j \times \prod_{h=1}^H \tau_h \times \bar{p}.$$

By Berge's Maximum Theorem and standard methods, we can prove that the correspondences μ_i, v_j, τ_h and \bar{p} are upper hemi-continuous and convex valued. This implies Ψ is also upper hemi-continuous and convex valued.

The correspondence Ψ has been shown to satisfy the hypotheses of the Kakutani fixed point theorem, and therefore to have a fixed point, say $e^* = ((\psi_i^*, x_i^*), (\theta_j^*, y_j^*), (\phi_h^*, z_h^*), p^*)$. Especially, this fixed point satisfies:

$$\sum_i \Delta \psi_{i,t}^* + \sum_j \Delta \theta_{j,t}^* \leq \sum_h \Delta \phi_{h,t}^*, \quad t = 1, \dots, T, \quad (3.3)$$

$$\sum_{t=1}^T \pi_t p_t^* w_t^* \geq \sum_{t=1}^T \pi_t p_t w_t^*, \quad \forall p \in \Delta_0. \quad (3.4)$$

By assumption (A.10) and Lemma 2.1,

$$\begin{aligned} \sum_{t=1}^T \pi_t p_t^* w_t^* &\leq \sum_{t=1}^T \pi_t d_t \sum_i \psi_i^* + \sum_j \theta_j^* - \sum_h \phi_h^* \\ &= \sum_{t=0}^T \pi_t d_t \sum_i \psi_i^* + \sum_j \theta_j^* - \sum_h \phi_h^* \\ &= F(d \sum_i \psi_i^* + \sum_j \theta_j^* - \sum_h \phi_h^*) \leq 0 \end{aligned}$$

Hence, from (3.4), $w^* \leq 0$.

Let $\Delta y_{J,t}^* = -w_t^* \geq 0$, $\Delta \theta_{J,t}^* = \sum_h \Delta \phi_{h,t}^* - \sum_i \Delta \psi_{i,t}^* - \sum_j \Delta \theta_{j,t}^* \geq 0$, $t = 1, \dots, T$. And set

$$\bar{y}_J^* = y_J^* - \Delta y_J^*, \quad \Delta \bar{\theta}_J^* = \Delta \theta_J^* + \Delta \theta_{J'}^*.$$

Clearly,

$$\bar{y}_J^* \in Y_J, \quad \sum_i x_i^* = \sum_{j \neq J} y_j^* + \bar{y}_J^* + \sum_h z_h^*,$$

and

$$\sum_i \Delta \psi_i^* + \sum_{j \neq J} \Delta \theta_j^* + \Delta \bar{\theta}_J^* = \sum_h \Delta \phi_h^*.$$

Moreover

$$\begin{aligned} p^*(\Delta y_J^*) &= -p^* w^* = -d \sum_i \psi_i^* + \sum_j \theta_j^* - \sum_h \phi_h^* \\ d^{\theta^*} &= -d \sum_i \psi_i^* + \sum_j \theta_j^* - \sum_h \phi_h^*, \end{aligned}$$

this implies that

$$d^{\bar{\theta}_J^*} + p^* \bar{y}_J^* = d^{\theta_J^*} + p^* y_J^*.$$

Finally, in exactly the same method as Arrow and Debreu(1954), it is not difficult to show $\bar{e}^* = ((x_i^*, \psi_i^*), (y_{j \neq J}^*, \theta_{j \neq J}^*), (\bar{y}_J^*, \bar{\theta}_J^*), (\phi_h^*, z_h^*), p^*)$ is an equilibrium point of the original Economy E.

4. Non-Convex Production Economy

This section is devoted to a economy in which the trading technology of each broker is not necessarily convex so that we allow for fixed costs in trading assets. By using the technique of Heller and Starr(1976), we will show the existence of an individual approximate equilibrium defined by Heller and Starr(1976). An approximate equilibrium is generally defined as a price p^* and two allocations, a^* and $a^{*'}$. One, a^* , is the allocation desired by households, firms and brokers at this price, which may not clear the market. The other, $a^{*'}$, is an allocation obeying the market clearance condition although it need not represent agents' optimizing behaviour. The equilibrium is approximate of a modulus C if some suitably chosen norm of the difference between these two allocations is no larger than C . The desired allocation represents an approximate equilibrium in the sense that the failure to clear the market at this price is bounded by C . And, furthermore, the bound of the approximation improves as the number of the agents in the economy increases.

We will still make all the assumptions in Section 2 except the convexity of broker's technology. We further assume the following.

(A.12) $B_\omega = Y \cap (X + Z - \omega)$ is bounded, where $X = \sum_{i \in I} X_i$, $\omega = \sum_{i \in I} \omega_i$, $Y = \sum_{j \in J} Y_j$ and $Z = \sum_{h \in H} Z_h$.

Since the assumptions of Theorem 1 of Hurwicz and Reiter(1973) can be easily verified through Assumptions A.4 and A.12, we can show the boundedness of $\hat{Z}_h, \hat{X}_i, \hat{Y}_j$. And, hence, $\hat{\Phi}_h, \hat{\Psi}_i$ and $\hat{\Theta}_j$ are all bounded.

In order to show that the equilibrium of the bounded economy is the equilibrium of the original economy, a additional assumption is required.

(A.13) There is a positive number L_0 such that $|z_h| \leq L_0, \forall z_h \in Z_h$.

That is, the quantity of commodities used in transaction of assets is limited. This is reasonable since a quantity larger than the the total supply of the world is not feasible. So the feasible plan of broker should satisfies the additional assumption (A.13). And the cubes C^1 and C^2 used in defining the bounded economy can be chosen to be large enough to contain the feasible plan of any broker.

As in Heller and Starr(1976), in order to prove the contiuity of $\tilde{\Gamma}_h^2(\gamma_h)$, we give the definition of local interior.

Definition 4.1 $\tilde{\Gamma}_h^2$ is said to be locally interior if for each $(\phi, z) \neq 0, (\phi, z) \in \tilde{\Gamma}_h^2(\gamma_h)$ there is (ϕ^*, z^*) so that

$$(i) (\phi^*, z^*) \in \tilde{\Gamma}_h^2(\gamma_h).$$

(ii) $0 \ll -d\phi^* - pz^*$.

(iii) There exists a continuous function $f : [0, 1] \rightarrow \tilde{\Gamma}_h^2(\gamma_h)$ so that $f(0) = (\phi^*, z^*)$, $f(1) = (\phi, z)$ and for all $\sigma \in [0, 1)$, $f(\sigma)$ satisfies the strict inequality in (ii).

(A.14) $\tilde{\Gamma}_h^2(\gamma_h)$ is locally interior.

Now we are in a position to prove the existence of an individual approximate equilibrium of the economy with non-convexity. But we omit its proof since it can be obtained in the same method as Heller and Starr(1976). In the proof, we use the correspondence Ψ (defined in Section 3) instead of $\gamma(p)$ defined in Heller and Starr(1976). The boundedness of $R(\Psi)$ defined in Heller and Starr(1976) can be clearly guaranteed by the assumption (A.13).

Theorem 4.1 Under the assumptions (A.1)–(A.14), there exists an individual approximate equilibrium of modulus C which only depends on M, N, T and $R(p^*)$, where p^* is an approximate equilibrium price. That is, there exist two vectors $a^* = (\phi_1^*, z_1^*, \dots, \phi_H^*, z_H^*, \psi_1^*, x_1^*, \dots, \psi_I^*, x_I^*, \theta_1^*, y_1^*, \dots, \theta_J^*, y_J^*)$ and $a^{*'} = (\phi_1^{*'}, z_1^{*'}, \dots, \phi_H^{*'}, z_H^{*'}, \psi_1^{*'}, x_1^{*'}, \dots, \psi_I^{*'}, x_I^{*'}, \theta_1^{*'}, y_1^{*'}, \dots, \theta_J^{*'}, y_J^{*'})$ such

(i) $a^{*'}$ satisfies market clearance with respect to p^* .

(ii) a^* solves problems (*), (**) and (***) with respect to p^* .

(iii) $(\phi_i^*, x_i^*) = (\phi_i^{*'}, x_i^{*'}), (\theta_j^*, y_j^*) = (\theta_j^{*'}, y_j^{*'}), i \in I, j \in J$.

(iv) $(\sum_h |(\psi_h^*, z_h^*) - (\psi_h^{*'}, z_h^{*'})|^2)^{1/2} \leq C$.

5. Non-Convex Exchange Economy

In this section, an exchange economy is investigated, which only includes consumers, brokers and dealers. We will retain all the assumptions in Section 4 except (A.12) and (A.13). It is not difficult to show the boundedness of sets $\hat{X}_i, \hat{Z}_h, \hat{\Psi}_i$ and $\hat{\Phi}_h$. And, moreover, we will introduce another assumption called finite p-convexity. Finally, the existence of general equilibrium is proved. To this end, we give the following definition of finite p-convexity.

Definition 5.1 Let X be a subset of R^n and $\Delta^{(n-1)}$ be the simplex of R^n . Then X is called finitely p-convex if for any $x_1, x_2 \in X$ and $p_1, \dots, p_m \in \text{int}(\Delta^{(n-1)})$ there is $\bar{x} \in X$ such that $p_i \bar{x} \leq p_i (\frac{x_1 + x_2}{2})$ for all $i = 1, 2, \dots, m$. (see Fig.1 and Fig.2)

Let $(\phi, z) = (\phi_1, z_1, \dots, \phi_h, z_h)$ and define the feasible set $\tilde{\Gamma}_i(p, \psi, z)$ (given price p and broker's plan (ϕ, z)) of consumer i analogously to $\tilde{\Gamma}_i^3(\tau)$; and the feasible set $\tilde{\Gamma}_h(\gamma_h)$ of broker h analogously to $\tilde{\Gamma}_h^2(\gamma_h)$ in Section 3. Define the demand function $\mu_i(p, \phi, z)$ of consumer i as $\mu_i(\tau)$ and the demand function $\tau_h(\gamma_h)$ of broker h as $\tau_h(\gamma_h)$ in Section 3.

Let

$$\xi(p) = \{ \sum_h(\phi_h, z_h) + \sum_i(\psi_i, x_i) - \sum_I(0, \omega_i) | (\phi_h, z_h) \in \tau_h(\gamma_h),$$

$$\forall h \in H, (\psi_i, x_i) \in \mu_i(p, \phi, z), \forall i \in I \}.$$

As shown in Heller and Starr(1976), it can be shown that the set $\tilde{\Gamma}_h(\gamma_h)$ and $\tilde{\Gamma}_i(p, \phi, z)$ are all continuous. And, thus, the correspondences $\mu_i(p, \phi, z)$ and $\tau_h(\gamma_h)$ are upper hemi-continuous and also compact valued. Now it is not difficult to show that the correspondence $\xi(p)$ is upper hemi-continuous. And, moreover, the projection $\xi_0(p)$ of $\xi(p)$ onto the commodity space is also upper hemi-continuous.

Before the proof of the main result of this section, we introduce two lemmas.

Lemma 5.1 Let $P \subseteq R^l$ be a compact set and let $\phi : P \rightarrow R^m$ be an upper hemi-continuous correspondence. If $\forall p \in P$,

$$\Phi(p) = \{ z \in R^m : z\mu > 0, \forall \mu \in \phi(p) \} \neq \emptyset,$$

then there exists a continuous function, $W : P \rightarrow R^m$, such that $W(p) \in \Phi(p), \forall p \in P$ (cf. McCabe(1981)).

Lemma 5.2 Suppose that X and Y are two non-empty compact spaces and that $f : X \times Y \rightarrow R$ is a real-valued function such that

(i) $x \rightarrow f(x, y)$ is lower hemi-continuous on X for each $y \in Y$; $y \rightarrow f(x, y)$ is upper hemi-continuous for each $x \in X$.

(ii) X is finitely f-convex; i.e., for any $x_1, x_2 \in X$ and $y_1, \dots, y_n \in Y$ there is $\bar{x} \in X$ such that $f(\bar{x}, y_i) \leq \frac{1}{2}[f(x_1, y_i) + f(x_2, y_i)]$ for all $i = 1, \dots, n$;

(iii) Y is finitely f-concave; i.e., for any $y_1, y_2 \in Y$ and $x_1, \dots, x_m \in X$ there exist $\bar{y} \in Y$ such that $f(x_j, \bar{y}) \geq \frac{1}{2}[f(x_j, y_1) + f(x_j, y_2)]$ for all $j = 1, \dots, m$.

Then

$$\min_X \max_Y f(x, y) = \max_Y \min_X f(x, y).$$

(cf. Granas and Fon-Che Liu(1987)).

We now turn to the main result of this section.

Theorem 5.1 Suppose that $\xi_0(p)$ is finitely p-convex and all assumptions in previous sections except that about producers hold. Then there exists a general equilibrium $e^* = ((\psi_i^*, x_i^*)_{i \in I}, (\phi_h^*, z_h^*)_{h \in H}, p^*)$ in the non-convex exchange economy, that is, e^* satisfies the following condition.

- (i) (ψ_i^*, x_i^*) solves problem (***) for each $i \in I$;
- (ii) (ϕ_h^*, z_h^*) solves problem (**) for each $h \in H$;
- (iii) e^* satisfies market clearance, that is,

$$\sum_h z_h^* + \sum_i x_i^* - \sum_i \omega_i = 0;$$

and

$$\sum_h \psi_h^* = \sum_i \phi_i^*.$$

Remark 5.1: In fact, we only need the finite p -convexity of commodity excess demand correspondence $\xi_0(p)$ of the truncated economy (defined below). If the utility function of the broker is assumed to be strictly concave, the finitely p -convex commodity excess demand correspondence $\xi_0(p)$ of the truncated economy is shown in Fig.3.

Proof of Theorem 5.1. We first truncate by a natural number n the set Z_h (defined in Section 3) and prove the existence of general equilibrium in the truncated economy E^n . And then by taking limits, the existence of equilibrium can be obtained as in Geanakoplos and Polemarchakis (1986).

Furthermore, the cubes C^1 and C^2 are also chosen large enough to include the truncated feasible sets of all brokers.

Note that the consumption sets of all consumers are $R_+^{T \times M}$. Hence, by the definition of $\xi(p)$, to prove the existence of general equilibrium it suffice to show that there exists $p_0 \in \Delta^{(T \times M - 1)}$ such that $\xi_0(p_0) \cap (-R_+^{T \times M}) \neq \emptyset$.

It is equivalent to that there exist $z_0 \in \pi \xi_0(p_0)$ such that $\max_{p \in \Delta^{(T \times M - 1)}} p z^0 \leq 0$, where $\pi = (\pi_1, \dots, \pi_T)$ as defined in Section 2 and

$$\pi \xi_0(p_0) = \{(\pi_1 z_{1,1}, \dots, \pi_1 z_{1,M}, \dots, \pi_T z_{T,1}, \dots, \pi_T z_{T,M}) \mid z = (z_{1,1}, \dots, z_{T,M}) \in \xi_0(p_0)\}.$$

And it is easy to show that $\pi \xi_0(p)$ is upper hemi-continuous and finitely p -convex.

We will prove the conclusion of this theorem by a contradiction. To this end, let, for each $k \geq T \times M$,

$$\Delta_k^{(T \times M - 1)} = \{p = (p_1, \dots, p_{T \times M}) \in \Delta^{(T \times M - 1)} \mid p_i \geq \frac{1}{k}, i = 1, \dots, T \times M\}.$$

For each $p \in \Delta_k^{(T \times M-1)}$ and each $z \in \pi \xi_0(p)$, suppose that there exists a $p' \in \Delta_k^{(T \times M-1)}$ such that $p'z > 0$. Hence, $\max_{\Delta_k^{(T \times M-1)}} p'z > 0$. By the continuity of the function $\max_{\Delta_k^{(T \times M-1)}} p'z$, we have

$$\min_{\pi \xi_0(p)} \max_{\Delta_k^{(T \times M-1)}} p'z > 0.$$

In Lemma 5.2, by taking $X = \pi \xi_0(p)$, $Y = \Delta_k^{(T \times M-1)}$ and $f(z, p') = p'z$, it is easy to verify that all conditions in this lemma are satisfied. And, particularly, the condition finite f-convexity of X corresponds to the finite p-convexity of $\pi \xi_0(p)$.

Therefore,

$$\max_{\Delta_k^{(T \times M-1)}} \min_{\pi \xi_0(p)} p'z > 0,$$

which implies that there exist $p'_k \in \Delta_k^{(T \times M-1)}$, such that $\min_{\pi \xi_0(p)} p'_k z > 0$ and moreover, $p'_k z > 0, \forall z \in \pi \xi_0(p)$. This is equivalent to that

$$\Phi(p) = \{p' \in \Delta_k^{(T \times M-1)} | p'z > 0, \forall z \in \pi \xi_0(p)\} \neq \emptyset$$

for each $p \in \Delta_k^{(T \times M-1)}$.

Thus, by Lemma 5.1, there exists a continuous function $W(p) : \Delta_k^{(T \times M-1)} \rightarrow \Delta_k^{(T \times M-1)}$ such that $W(p) \in \Phi(p), \forall p \in \Delta_k^{(T \times M-1)}$. Then, by the Brouwer fixed point theorem, there is a $p_k^0 \in \Delta_k^{(T \times M-1)}$ such that $p_k^0 = W(p_k^0)$. This means that $p_k^0 z > 0, \forall z \in \pi \xi_0(p_k^0)$, which contradicts the Walras' Law which has been established in Section 3. Therefore, for each $p_k \in \Delta_k^{(T \times M-1)}$, there exists $z_k \in \pi \xi_0(p_k)$ such that $\max_{\Delta_k^{(T \times M-1)}} p z_k \leq 0$.

Since $\{\pi \xi_0(p_k) | p_k \in \Delta_k^{(T \times M-1)}, k = 1, 2, \dots\}$ are compact subsets of a compact set, there is a convergent subsequence of (p_k, z_k) with limit (p^0, z^0) .

Note that the set $\xi_0(p)$ is empty for each $p \in \partial \Delta^{(T \times M-1)}$ since $\pi \xi$ is upper hemi-continuous and the utility function of consumer 1 is strictly increasing. It is not difficult to show that $p^0 \in \text{int} \Delta^{(T \times M-1)}$ and $z^0 \in \pi \xi_0(p^0)$ $\max_{\Delta^{(T \times M-1)}} p z^0 \leq 0$, proving the existence of equilibrium of the truncated economy.

In exactly the same method as that of Geanakoplos and Polemarchakis(1986), it can be shown that there exists a equilibrium in the original economy. \square

6. Conclusion

This paper has attempted to attain three objectives:

1. Prove the existence of equilibrium of an asset economy with transaction costs. The model is sufficiently general to cover most cases (finite states, -time horizon) in the literature.
2. The method of proof proves some new results (see Ortu(1995)) extending arbitrage pricing dual results to cover transaction costs and different buying and selling prices.
3. In addition, two proofs are provided of existence of an equilibrium with nonconvex transaction technologies. These proofs are important for addressing economies with fixed costs in transacting.

One final comment: in Milne-Neave(1996), it is shown that the basic model can be adapted easily to accommodate a number of variations common in the literature. For example, by considering $I = J = \emptyset$, and brokers are considered as ordinary consumers with a "transaction technology" representing short-sales constraints on trading, the proofs can be interpreted as proving the existence of an equilibrium with trading constraints.

Appendix

Lemma A1 If the assumption (A.7) holds, then the set Z_h is a closed convex set.

Proof. The convexity of Z_h is obvious. It remains to show its closedness. Suppose $z_h^k \in Z_h$ and $z_h^k \rightarrow z_h$. For each k , there exists $\phi_h^{(k)}$ such that $(\Delta\phi_h^{(k)}, z_h^k) \in T_{h,t}$, and, in particular, by (A.8), $(\Delta\phi_h^{(k)}, z_h^k) \in T_{h,t}$, where $z_h^k = (\max_k z_{h,1}^k, \dots, \max_k z_{h,M}^k)$. If $\{\Delta\phi_h^{(k)}\}$ is unbounded, we may suppose $\Delta(\phi_h^{(k)})_{h,1}^B \rightarrow \infty$ without loss of generality.

But, by assumption (A.7),

$$\lim_{k \rightarrow \infty} |z_h^k| = \infty$$

which provides a contradiction and proves the boundedness of $\{\Delta\phi_h^{(k)}\}$. Hence, we can choose a subsequence $\{\Delta\phi_h^{(k_n)}\}$ from $\{\Delta\phi_h^{(k)}\}$ such that

$$\lim_{n \rightarrow \infty} \Delta\phi_h^{(k_n)} = \Delta\phi_h,$$

this implies, by closedness of $T_{h,t}$, the closedness of Z_h . \square

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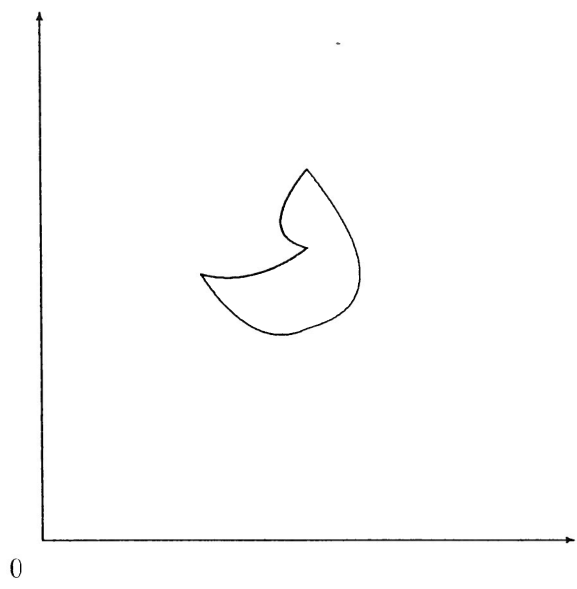


Fig.1 Finite p -convex set

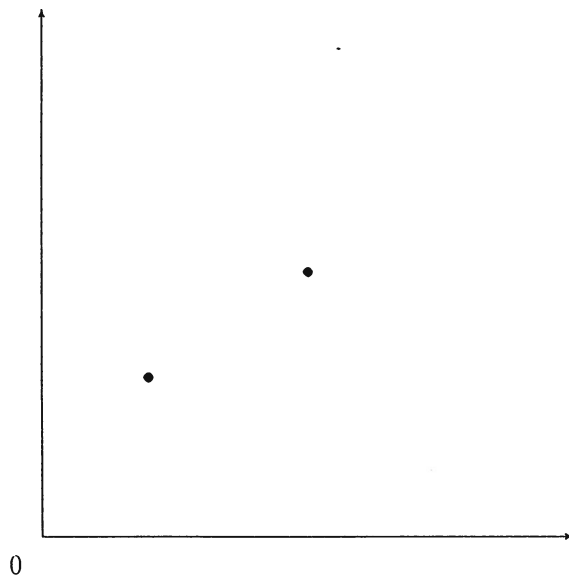


Fig.2 Finite p -convex set

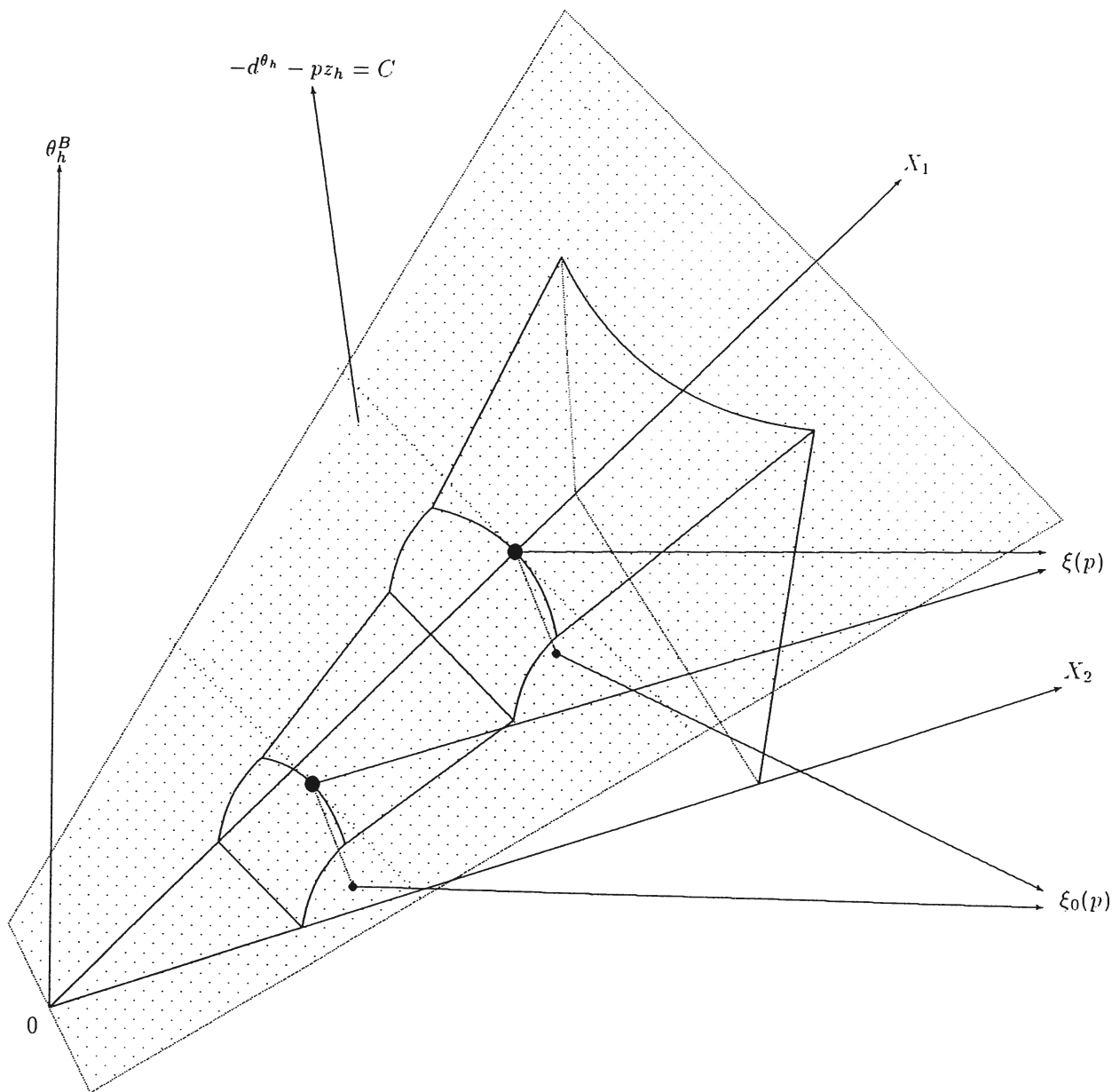


Fig.3 Finite p -convex $\xi_0(p)$