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Exhaustible Resources in an Overlapping Generations Economy

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OVERLAPPING GENERATIONS ECONOMY

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ABSTRACT

This paper explores the natural resource consumption behaviour of a competitively determined economy relative to a socially planned benchmark when agents are characterized as having finite lifespans which overlap. A general equilibrium model of a production economy which uses inputs from a finite stock of an aggregate natural resource is formulated and solved for the rates of resource extraction associated with the competitive outcome and the socially planned one. It is shown that resource extraction in the competitive economy can exceed that of the socially planned optimum and that intergenerational inequites result.

Keywords: overlapping generations, essential non-renewable resources, social planning problem, competitive agents, extraction rates, dynamic programming, stability and convergence.

JEL classification numbers: 721, 023, 911.

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Introduction

The study of the use of so called "natural resources", and particularly of "exhaustible" natural resources, has a long history rife with debate along both normative and positive lines. In 1931, Harold Hotelling published a classic paper which examined a social planner's problem of maximizing welfare from the production of a minerals industry. Hotelling's results at this industry level where extended to a perfectly competitive environment where he showed that the same rule for the individual firms optimal extraction path resulted, given the same extraction costs and if the participating firms had perfect foresight about minerals prices. This rule, subsequently called Hotelling's rule, stated simply that a finite stock of a homogeneous resource be depleted at a rate such that the rate of price increase of the resource over time be exactly equal to the appropriate rate of discount. In other words, the capital gain, or instantaneous return from leaving a dollars worth of the resource in the ground is equal to the instantaneous return available by extracting and selling the resource on the margin and investing that dollar in the alternative asset.

This result is very well known and indeed reappears in many subsequent analyses with appropriate modifications. Stiglitz (1974), for example derives a variant of Hotellings rule as a social optimum in a growth economy, where the comparative rate of return is given by the marginal product of capital. Hotelling himself investigated the case of a monopolistic firm where equalization of the rate of change of marginal revenue and the rate of return on an alternative asset define the optimal extraction path.

Indeed this result may be robust even to the optimization criterion chosen. Solow (1974) shows that a social planner using the Max-Min criterion (i.e. maximizing the minimum level of consumption, C_t, for any "generation" for $t \in [0,\infty)$) also will choose an extraction path characterized by equalization of returns (measured in production of consumption units), on alternative assets.

The choice of an optimization criterion and the labelling of a particular extraction path as optimal are of course based upon normative arguments which in turn are mostly concerned, in the exhaustible resources literature, as we are here, with the concept of intergenerational equity.

John Rawls (1971) has argued strongly against the conventional utilitarian approach to these problems suggesting that in the context of an economy accumulating capital "the utilitarian doctrine may direct us to demand heavy sacrifices of the poorer generations for the sake of greater advantages for latter ones..." Rawls then goes on to sometimes advocate a Max-Min principle and other times a more vague principle aiming at intergenerational equity by balancing what an individual would feel was appropriate to ask of his parents with what he would be willing to provide for his children.

Other normative arguments center around the use of a social discount factor (Ω) of less than one, to discount the utility of future generations in the formulation of the social optimand. It has been argued, as by Ramsey (1928), that certainly any current generation (and thus any planner) does not have the "right" to treat different generations asymmetrically. Further it can be argued that for *individuals* to have private rates of discount (β) of less than one would be irrational. Rational agents, it is argued, would recognize that when tomorrow arrives an additional unit of consumption will very likely be valued as highly as it is today or was yesterday.

On the other hand there is a rational interpretation of a social (or private) discount factor of less than one as reflecting the positive probability of extinction (or death) in the next period. In the case of society of course one might argue that we cannot derive an objective probability to reflect the likelihood of extinction. This does not suggest that we set $\Omega = 1$, however, only that a best subjective probability be applied. The later argument will be adopted here.

As mentioned already, Solow shows that the Max-Min principle results in the same optimal extraction path as the utilitarian (at least with zero technical progress and zero population growth) while the difficulty in deriving an objective and tractable formulation for the vaguer criterion proposed by Rawls leaves us to set it aside for future normative discussion.

Thus, this paper adopts the utilitarian criterion for optimization in a social planning context and sets Ω equal to something less than one. These choices coincide with the vast majority of the existing literature examining the problems arising from the existence of a finite stock of resources which are required in production. This literature also usually characterizes different generations as identical and existing atemporally. In this paper however, a simple resource-based economy is developed in which agents exhibit the fundamentally realistic and important heterogeneity that arises from the existence of overlapping-generations.[1]

Both a competitively determined version of the economy and a socially-planned one are formulated as dynamic optimization problems. The solutions to these problems are examined with an aim towards characterizing the implied resource extraction paths of each economy and the implications of these paths on intergenerational equity and efficiency. It is shown that although both economic systems produce Pareto-efficient extraction paths, the one determined by the competitive system often results in the depletion of the finite resource at rates which are far in excess of those determined from the socially-planned system. Furthermore, the descrepencies that exist between the rates of extraction determined in the two economies are not dependent upon the existence of a gap between the social and private rates of discount.

The paper proceeds as follows:

Section 2 describes the economy to be analyzed and sets up appropriate notation.

Section 3 examines the socially optimal solution to the model of Section 2 while Section 4 discusses the competitive solution.

Section 5 compares the results from the competitive and social solutions and Section 6 is a conclusion drawing suggestions for further work.

[1] Other resource models incorporating the OLG concept are; Ivor Pearce (1975) and Kemp & Long (1980).

Section 2: The Model

The model presented here is a variant of the Samuelson (1958) 2-period lived overlapping generations model in discrete time. The competitive version of the economy is discussed first as it provides the most insight into the assumptions and thus the characteristics of the model.

There is no uncertainty in the model, all agents have perfect foresight. At any time t there exists both an old and a young generation where the members of any generation live for at most two periods. Population is constant and equal to 2N with N new agents being born at the beginning of each period. Each individual (i) has utility described by the constant relative risk aversion (CRRA) function;

$$U_t(C_{i1},C_{i2}) = (1-\sigma)^{-1}(C_{i1}^{(1-\sigma)} + \beta C_{i2}^{(1-\sigma)}).$$

$$\sigma\in(0,\infty), \qquad \beta\in(0,1)$$

Where C_{i1} and C_{i2} are consumption by individual i of the t^{th} generation when young (period t) and then old (t + 1), respectively. The parameter σ, describes the individuals risk aversion/intertemporal substitution characteristics as is well known for this class of preference functions. β is the individuals private discount factor derived from a pure rate of time preference.

The non-storable consumption good is produced in this economy by combining labour with a homogeneous resource which is non-renewable has well defined property rights and exists as an aggregate stock R_t, at time t. The production function for any individual (i), at time t, will be written as;

$$F(l_{it},r_{it}) = Al_{it}^{1-\alpha}r_{it}^{\alpha} .$$

This is standard Cobb-Douglas with constant returns to scale where l_{it} is the quantity of labour used in production and r_{it} is the quantity of the resource used by the i^{th} individual. Note that both factors are "essential" to the production process. We also have; $R_t = R_{t-1} - \Sigma_{i=1}^{N} r_{it-1}$, which implies that the resources used in production in any period, are completely dissipated.

In this environment labour demand will be determined competitively. Labour supply is assumed to be inelastic with each member of any generation providing 1 unit of labour when young and zero when old.

It will be assumed that the current old generation has claim to the entire stock of the resource and thus own the rights to production. The old generation may extract some (or all) of the resource at constant marginal cost g, and produce output by combining it with the young generation's labour in production. In exchange for their labour the young are paid consumption wages which they may then use for current consumption or to purchase rights to the remaining resource stock from the old. These rights are the only form of savings.

In the competitive environment then we may write individual i's problem as;

$$\max_{\{C_{i1},C_{i2}\}} U_t(C_{i1},C_{i2}) = (1-\sigma)^{-1}(C_{i1}^{(1-\sigma)} + \beta C_{i2}^{(1-\sigma)})$$

$$\text{s.t. } C_{i1} = \omega_t - \gamma_t S_{it}$$

$$(2.1) \qquad C_{i2} = F(l_{it}, r_{it}) - gr_{it} - \omega_{t+1} l_{it+1} + \gamma_{t+1}(S_{it} - r_{it+1})$$

$$r_{it+1} \le S_{it} \quad \forall\, i,t$$

$$\sum_{i=1}^{N} S_{it} = R_t \quad \forall\, t$$

Where ω_t is the consumption wage paid for the supply of 1 unit of labour and S_{it} is the quantity of resource purchased as savings at consumption price γ_t per unit.

The social planner's problem shares the same production and resource extraction/stock characteristics as the competitive problem by virtue of the constant extraction costs and CRS nature of the production function. The ownership rights of the resource stock however and thus the rights to produce are controlled by the central planner, whose interests are described by a social welfare function.

This planners function will be based upon the function;

$$W = \sum_{t=-\infty}^{-1} \Omega^t (1-\sigma)^{-1}[C_{1t}^{(1-\sigma)} + \beta C_{2t+1}^{(1-\sigma)}] + \sum_{t=0}^{\infty} \Omega^t (1-\sigma)^{-1}[C_{1t}^{(1-\sigma)} + \beta C_{2t+1}^{(1-\sigma)}].$$

Where σ, and β, are as in the competitive problem. $\Omega \in (0,1)$ is taken as the social planners subjective probability that the society will exist next period. C_{1t} and C_{2t+1} are the aggregate consumption levels of generation t when young and then old respectively. The social planner is interested in maximizing W with respect to the aggregate consumption levels of each generation in each of their two periods of life subject to the appropriate resource and production constraints, from t=0 on. The reverse discounting of the lifetime utilities of those currently alive and those past may seem strange when the planner is concerned with welfare only from the present on, however it is necessary in this context in order to ensure the time consistency of the optimal intertemporal allocations. As pointed out by Calvo and Obtsfeld (1988), "unless those alive and those to be born are treated symmetrically, the planner has an incentive to change the consumption previously planned for unborn generations once they come into existence."

This planner's problem can be decomposed into 3 separate parts. The first is the optimal distribution of consumption among the members of any one generation. Implicit in the functional form given by W is the equal weighting of all individuals of any generation and thus the solution $C^i_{1t} = C^j_{1t}$ and $C^i_{2t} = C^j_{2t}$ $\forall$ i, j$\in$(1,N) and $\forall$t is implied for this first problem. The second problem is the optimal allocation of consumption between generations while the third is the optimal depletion of the resource stock.

Writing out the social planner's problem then for a finite time interval and noting that the social planner has a total of N units of labour available to use in production in any period, we have;

$$\max_{\{C_{1t}, C_{2t}\}_{t=0}^{t=\infty}} (1-\sigma)^{-1}\left[\Omega^{-1}\beta C_{2t}^{(1-\sigma)} + C_{1t}^{(1-\sigma)} + \beta C_{2t+1}^{(1-\sigma)} + \Omega C_{1t+1}^{(1-\sigma)} + \Omega\beta C_{2t+2}^{(1-\sigma)} + \cdots \right]$$

$$\text{s.t.} \quad C_{1t} + C_{2t} \le F(N, r_t) - gr_t$$

$$(2.2) \qquad C_{1t+1} + C_{2t+1} \le F(N, r_{t+1}) - gr_{t+1}$$

$$\vdots$$

$$\sum_{t=0}^{\infty} r_t \le R_0$$

Where r_t is aggregate extraction of the resource in period t.

Section 3: The Socially Optimal Solution.

Rewriting the problem (2.2) by substituting for C_{1t}, C_{1t+1}, etc., from the aggregate budget constraints and using a Lagrangian to impose the aggregate resource constraint we have;

$$\max_{\{C_t, r_t\}_{t=0}^{t=\infty}} (1-\sigma)^{-1} \Big[\Omega^{-1}\beta C_{2t}^{(1-\sigma)} + (F(N,r_t) - gr_t - C_{2t})^{(1-\sigma)}$$

$$(3.1) \qquad + \beta C_{2t+1}^{(1-\sigma)} + \Omega(F(N,r_{t+1}) - gr_{t+1} - C_{2t+1})^{(1-\sigma)} + \ . \ . \ . \Big]$$

$$+ \lambda(R_0 - \textstyle\sum_{t=0}^{\infty} r_t).$$

The following five equations are some of the first order conditions from this problem.

$$(3.2) \qquad C_{2t} \ ; \qquad \Omega^{-1}\beta C_{2t}^{-\sigma} = C_{1t}^{-\sigma}$$

$$(3.3) \qquad r_t \ ; \qquad C_{1t}^{-\sigma} = \lambda[F_r(N,r_t) - g]^{-1}$$

$$(3.4) \qquad C_{2t+1} ; \qquad \Omega^{-1}\beta C_{2t+1}^{-\sigma} = C_{1t+1}^{-\sigma}$$

$$(3.5) \qquad r_{t+1} \ ; \qquad \Omega C_{1t+1}^{-\sigma} = \lambda[F_r(N,r_{t+1}) - g]^{-1}$$

$$(3.6) \qquad \lambda \ ; \qquad R_0 = \textstyle\sum_{t=0}^{\infty} r_t$$

Equations (3.2) and (3.4) are familiar expressions giving the conditions for optimal intergenerational allocations. Equations (3.3) and (3.5) are expressions relating the marginal costs of extraction of an additional unit of the resource with their utility benefits. Equation (3.6) is of course just our ultimate resource constraint repeated.

As mentioned in the introduction, the extraction path implied by the solution to these first order conditions is of major interest to us in this paper. In following with this then, using equation (3.2) and a rewritten budget constraint we write;

$$(3.7) \qquad C_{1t} = \Omega^{1/\sigma}(\beta^{1/\sigma} + \Omega^{1/\sigma})^{-1}[F(N,r_t) - gr_t] \qquad \forall t.$$

$$(3.8) \qquad C_{2t} = \beta^{1/\sigma}(\beta^{1/\sigma} + \Omega^{1/\sigma})^{-1}[F(N,r_t) - gr_t] \qquad \forall t.$$

Equations (3.7) and (3.8) give the social planner's optimal division of net output for consumption between the current young and current old.

Now, from equations (3.3) and (3.5) we can eliminate the Lagrange multiplier λ to yield;

$$(3.9) \qquad C_{1t}^{-\sigma}[F_r(N,r_t) - g] = \Omega C_{1t+1}^{-\sigma}[F_r(N,r_{t+1}) - g].$$

Substituting into equation (3.9) with equation (3.7) written for times t and t+1 we derive the following expression;

$$(3.10) \quad [F(N,r_t) - gr_t]^{-\sigma}[F_r(N,r_t) - g] = \Omega[F(N,r_{t+1}) - gr_{t+1}]^{-\sigma}[F_r(N,r_{t+1}) - g].$$

Now, substituting into equation (3.10) with $F(N,r_t) = AN^{(1-\alpha)}r_t^{\alpha}$ our CRS Cobb-Douglas production function and letting $g = 0$ we can derive a closed form expression for the socially optimal rate of extraction of the resource. We have;

$$[AN^{(1-\alpha)}r_t^{\alpha}]^{-\sigma}[\alpha AN^{(1-\alpha)}r_t^{(\alpha-1)}] = \Omega[AN^{(1-\alpha)}r_{t+1}^{\alpha}]^{-\sigma}[\alpha AN^{(1-\alpha)}r_{t+1}^{(\alpha-1)}]$$

which yields $\rho_t \equiv r_{t+1}/r_t = \Omega^{1/(1-\alpha+\alpha\sigma)}$.

By definition we have $r_{t+1} = R_{t+1} - R_{t+2}$ $\forall t$, and therefore using the above we may write; $R_{t+1} - R_{t+2} = (R_t - R_{t+1})\varphi$, where $\varphi = \Omega^{1/(1-\alpha+\alpha\sigma)} < 1$. This last expression may be rewritten to yield;

$$(3.11) \qquad 1 - \frac{R_{t+2}}{R_{t+1}} = \left(\frac{R_t}{R_{t+1}} - 1\right)\varphi.$$

Defining $\mu_t \equiv R_{t+1}/R_t$, we can write equation (3.11) as;

$$(3.12) \qquad \mu_{t+1} = \varphi + 1 - \varphi/\mu_t.$$

Now, μ_t is identically equal to $1 - \phi_t$, where $\phi_t \equiv r_t/R_t$ is the rate of extraction of the resource, so a steady state in μ_t implies a steady state in ϕ_t and vice-versa. Letting $\mu_{t+1} = \mu_t = \mu$ $\forall t$, then, we can derive the following quadratic expression for a steady state in the optimal social rate of extraction of the resource.

$$(3.13) \qquad \mu^2 - (1+\varphi)\mu + \varphi = 0.$$

Equation (3.13) has two solutions; $\mu = 1$ or $\mu = \varphi$. The first of these solutions is ruled out as it would imply $\phi = 1 - \mu = 0$. That is, a zero rate of extraction of the resource which implies zero consumption and thus can clearly not yield a maximizing solution to the social planning problem. Thus the optimal steady state social rate of extraction will be given by the second solution to equation (3.13), $\mu = \varphi$ which yields an optimal social rate of extraction given by;

$$\phi_s = 1 - \Omega^{1/(1-\alpha+\alpha\sigma)}.$$

Clearly for $\Omega \in (0,1)$, we will have $\phi_s \in (0,1)$, and the extraction path will be characterized by declining quantities of extracted resources in each period. That is, in each period we will have $r_{t+1} < r_t$. This of course is an intuitive result which implies that the social optimum be characterized by an asymptotic depletion of the initial stock of the resource at the constant rate ϕ_s.

Note that if $\Omega = 1$ then $r_t = r_{t+1}$ $\forall t$ is the path implied. But clearly this path will violate the resource constraint of equation (3.6) and thus cannot represent a solution to the problem. It is also interesting to note that the extraction path implied by ϕ_s is relatively unchanged for even large changes in production and/or preference parameters but is very closely related to the social rate of discount, Ω. That is, the elasticity of the rate of extraction with respect to either α or σ is less than that with respect to Ω, in absolute value terms.

The simplicity of the above result (which of course depends upon zero marginal extraction costs) unfortunately will not be repeated in the case of the competitive economy.

Section 4: The Competitive Solution.

Rewriting the competitive problem (2.1) by substituting for the appropriate constraints we have individual i's problem, at time t.

$$(4.1) \quad \underset{\{S_{it}, r_{it+1}, l_{it+1}\}}{\text{Max}} \quad (1-\sigma)^{-1}\{(\omega_t - \gamma_t S_{it})^{(1-\sigma)} + \beta\big(F(l_{it+1}, r_{it+1}) - gr_{it+1} - \omega_{t+1} l_{it+1} + \gamma_{t+1}(s_{it} - r_{it+1})\big)^{(1-\sigma)}\}.$$

The first-order conditions from problem (4.1) can be written as;

$$(4.2) \qquad S_{it} \; ; \qquad (C_{i1})^{-\sigma} = \beta(\gamma_{t+1}/\gamma_t)(C_{i2})^{-\sigma}$$

$$(4.3) \qquad r_{it+1}; \qquad F_r(l_{it+1}, r_{it+1}) = g + \gamma_{t+1}$$

$$(4.4) \qquad l_{it+1}; \qquad F_l(l_{it+1}, r_{it+1}) = \omega_{t+1}$$

Equations (4.3) and (4.4) are written having assumed that the marginal utility of consumption by the old is not identically zero, or alternatively, given the structure of preferences, that $C_{i2} \neq \infty$, a rather perspicuous assumption.

Equation (4.2) is a familiar expression describing the optimal intertemporal relationship between consumption in the two periods of the agents life. Clearly he will desire to equate his discounted marginal utilities in each period where in this case the agent also takes into account the potential for capital gains in holding (or purchasing) savings, as given by the ratio of resource prices γ_{t+1}/γ_t. Recall that there is no uncertainty in the model.

Equation (4.3) is an expression characterizing the agents optimal resource use and is easily recognized as a variant of Hotellings rule. The equation simply states that resources will be extracted by the agent up to the point at which marginal benefits from extraction (given by the marginal product of the resource) are equal to marginal costs. In this case costs include both the direct extraction cost g and the opportunity cost in forgone sales to the young generation at price γ_{t+1} per unit. In terms of Hotellings rule again we have the instantaneous return from leaving a unit of the resource in the ground (i.e. $g + \gamma_{t+1}$) set equal to the instantaneous return from extracting the resource and investing in its alternative use, that is, using it to produce consumption goods with a return of $F_r(l_{it+1}, r_{it+1})$, the marginal product of resources in period t+1.

Equation (4.4) is the standard competitive result determining labour demand.

Now, substituting for $F(l_{it}, r_{it})$ with our CRS Cobb-Douglas functional form and noting that with the inelastic labour supply $l_{it} = 1 \quad \forall\ i,t$ we can write the system of first-order conditions and budget constraints as follows.

(4.5) $$C_{i1}^{-\sigma} = \beta(\gamma_{t+1}/\gamma_t)C_{i2}^{-\sigma}$$

(4.6) $$\alpha A r_{it+1}^{(\alpha-1)} = g + \gamma_{t+1}$$

(4.7) $$(1-\alpha)A r_{it+1}^{\alpha} = \omega_{t+1}$$

(4.8) $$C_{i1} = \omega_t - \gamma_t S_{it}$$

(4.9) $$C_{i2} = A r_{t+1}^{\alpha} - g r_{t+1} - \omega_{t+1} + \gamma_{t+1}(S_{it} - r_{t+1})$$

Equations (4.6) and (4.7) substituted into the two budget constraints (4.8) and (4.9) give the following expressions for consumption when young and old respectively.

(4.10) $$C_{i1} = (1-\alpha)A r_t^{\alpha} - \gamma_t S_{it}$$

(4.11) $$C_{i2} = \gamma_{t+1} S_{it}.$$

Equation (4.11) is a result of zero profits in the competitive environment and thus indicates that consumption in the second period of life for any agent will be out of savings only.

Now using equations (4.10) and (4.11) in equation (4.5) we can derive the following expression characterizing these savings.

(4.12) $$S_{it} = (\omega_t/\gamma_t)[\beta^{(-1/\sigma)}(\gamma_{t+1}/\gamma_t)^{(\sigma-1)/\sigma} + 1]^{-1}$$

Clearly savings are a positive function of the marginal product of labour (wages) the usual pure income effect. Further, noting that γ_{t+1}/γ_t is the gross rate of return on savings we differentiate equation (4.12) with respect to this rate to derive;

$$\frac{\partial S_{it}}{\partial(\gamma_{t+1}/\gamma_t)} = (-\omega_t/\gamma_t) \left[(\beta)^{\frac{-1}{\sigma}} (\gamma_{t+1}/\gamma_t)^{\frac{\sigma-1}{\sigma}} + 1\right]^{-2} (\beta)^{\frac{-1}{\sigma}} (\tfrac{\sigma-1}{\sigma}) (\gamma_{t+1}/\gamma_t)^{\frac{-1}{\sigma}}$$

Which is greater or less than zero as $\sigma \lesseqgtr 1$ and is equal to zero at $\sigma = 1$. This is also a standard result reflecting the relative strength of income and substitution effects under the CRRA preferences.

Equilibrium in the savings market requires that the sum of all savings equal the quantity of the resource remaining to carry over to the next period. That is:

$$\textstyle\sum_{i=1}^{N} S_{it} = R_t - \sum_{i=1}^{N} r_{it} = R_{t+1}.$$

Since all agents are assumed to be identical this can be rewritten as;

$$NS_t = R_t - Nr_t,$$

or as;

$$S_t = \mathcal{R}_t - r_t. \tag{4.13}$$

Where $\mathcal{R}_t$ is the stock per-capita of the resource at time t, S_t and r_t are the representative agents level of savings and resource extraction respectively.

Now equating (4.13) and (4.12), letting g=0 and using equation (4.6) written at time t and time t+1 we may write;

$$\mathcal{R}_t - r_t = \left[(\beta)^{\frac{-1}{\sigma}} \alpha A r_{t+1}^{((\sigma-1)(\alpha-1))/\sigma} r_t^{(\alpha-1)/\sigma} + \alpha A r_t^{\alpha-1}\right]^{-1} (1-\alpha) A r_t^{\alpha} \tag{4.14}$$

Solving for r_{t+1} we have;

$$r_{t+1} = \left[\frac{\beta^{(1/\sigma)} (1-\alpha) r_t^{(\tau+1)}}{\alpha(\mathcal{R}_t - r_t)} - \beta^{(1/\sigma)} r_t^{\tau} \right]^{\frac{1}{\tau}} \tag{4.15}$$

where $\tau = \frac{(\sigma-1)(\alpha-1)}{\sigma} < 0.$

Rearranging the identity $\mathcal{R}_t \equiv \mathcal{R}_{t-1} - r_{t-1}$ and substituting into equation (4.15) we arrive at the following second-order difference equation in terms of resource stocks per capita.

$$(4.16) \qquad \mathcal{R}_{t+2} = \mathcal{R}_{t+1} - \left[\frac{\beta^{(1/\sigma)}(1-\alpha)(\mathcal{R}_t - \mathcal{R}_{t+1})^{\tau+1}}{\alpha\mathcal{R}_{t+1}} - \beta^{(1/\sigma)}(\mathcal{R}_t - \mathcal{R}_{t+1})^{\tau} \right]^{\frac{1}{\tau}}$$

As with our analysis of the social optimum here we are interested in the *rates* of extraction implied by this result. Accordingly we make the definition; $\mu_{t+1} = \frac{\mathcal{R}_{t+1}}{\mathcal{R}t} = 1 - \phi_{tc}$. Rewritting equation (4.16) in terms of this ratio we obtain a first-order difference equation which characterizes the extraction rate of the competitive economy.

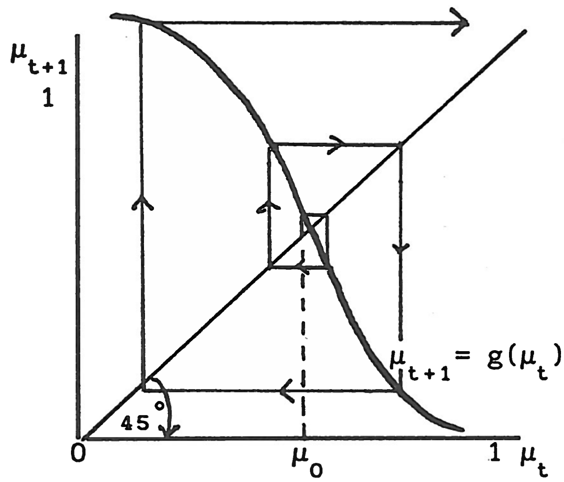
$$(4.17) \qquad \mu_{t+1} = 1 - \beta^{(1/\tau\sigma)} \left[\frac{1 - \alpha - \mu_t}{\alpha\mu_t} \right]^{\frac{1}{\tau}} \left(\frac{1 - \mu_t}{\mu_t} \right)$$

It is unfortunately the case that equation (4.17) does not yield a closed form solution for the competitive extraction rate despite the number of simplifying assumptions already made. However one immediate implication of equation (4.17) is that we must have $\mu_t \leq (1 - \alpha)$ $\forall t$ in order for μ_{t+1} to be defined. By the definition of μ_t then this immediately implies that the *smallest* extraction rate possible in order for this economy to have a well-defined equilibrium path, is $\phi_c = \alpha$. This result together with the well-known properties of the Cobb-Douglas functional form suggests that the competitive version of this economy will always extract the resource at a rate greater than or equal to the share that said resource represents in production.

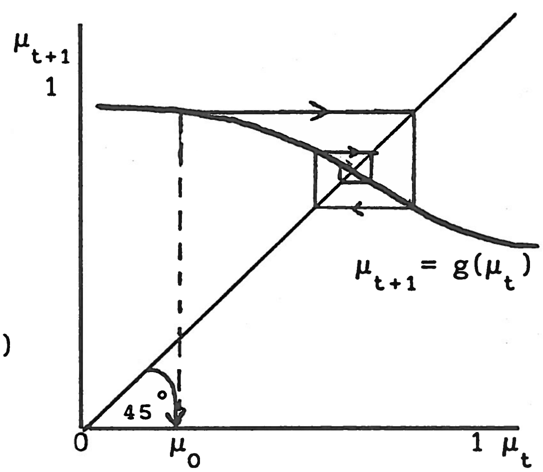
We turn now to a discussion of the dynamics of the extraction rate paths implied in the above competitive economy solution. We will be able to show that a steady state extraction rate does exist and this will allow us to make some comparisons of this rate with that implied by the social optimum of Section 3 when we come to Section 5.

The time path of the extraction rate implied by equation (4.17) will display either cyclical or non-cyclical behaviour depending upon the sign of $\partial\mu_{t+1}/\partial\mu_t$. It is easy to show that for the case of $\sigma > 1$ this derivative is less than zero for all t, while for the case of $\sigma < 1$ this derivative is everywhere positive, and not surprisingly we have $\partial\mu_{t+1}/\partial\mu_t = 0$ for $\sigma = 1$.[2]

Under the assumption of $\sigma > 1$ then, the model will display cyclical behaviour as in either of the two cases diagramed below.



A: $\left|\dfrac{\partial\mu_{t+1}}{\partial\mu_t}\right| > 1$; explosive case



B: $\left|\dfrac{\partial\mu_{t+1}}{\partial\mu_t}\right| < 1$; convergent case

As is obvious from a quick examination of the above diagrams a further implication of the fact that $\partial\mu_{t+1}/\partial\mu_t < 0 \ \forall t$ is that a steady state equilibrium must exist for extraction rates $\phi_c \in (0,1)$. These steady states occur at the intersection of the 45° line and the line labelled $\mu_{t+1} = g(\mu_t)$, in both case A and case B. Clearly at these intersections we have $\mu_t = \mu_{t+1}$ $\forall t$. The difference between these cases is the out-of-steady-state behaviour of the extraction rate variable which is determined by the magnitude of the absolute value of the derivative $\partial\mu_{t+1}/\partial\mu_t$.

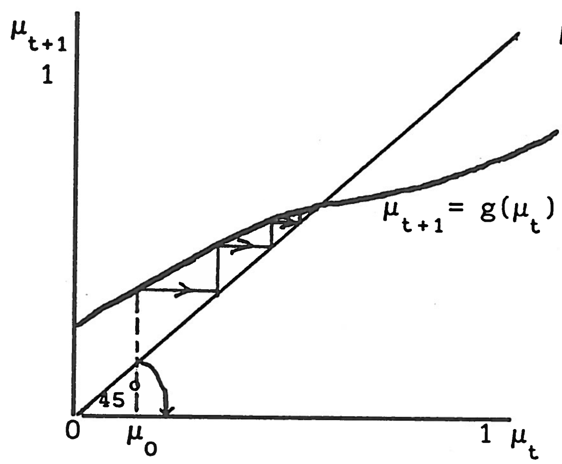
Case A represents the explosive case where any deviation from the steady

[2] A proof of these results may be found in Appendix A.

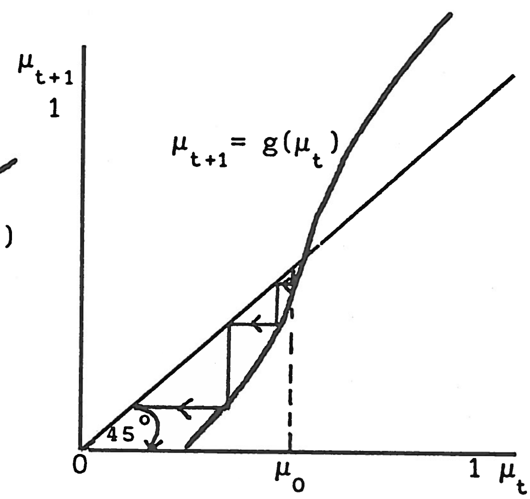
state results in extraction rates over time varying about the steady state in larger and larger increments until eventually a viability constraint must be violated.

Case B gives the stable case where variation around the steady state decreases steadily as the extraction rate converges to this steady state.

Under the assumption of $\sigma < 1$ the model's dynamics can be conceptualized as in the next two diagrams.



$$\text{C: } \left|\frac{\partial \mu_{t+1}}{\partial \mu_t}\right| < 1; \quad \text{stable case}$$



$$\text{D: } \left|\frac{\partial \mu_{t+1}}{\partial \mu_t}\right| > 1; \quad \text{explosive case}$$

Case C represents the stable case for which any starting value of the extraction rate between zero and one will converge over time to the steady state rate at the intersection of the line $\mu_{t+1} = g(\mu_t)$ and the 45° line.

Case D is the explosive case analogous to case A above, where only one possible value for the extraction rate will lead to a steady state, that value being exactly the one that occurs at the intersection of the two lines.

Note that for either of the latter two cases above the existence of a steady-state value for μ is not guaranteed. In particular a steady state does not exist for case C if the line $\mu_{t+1} = g(\mu_t)$ should intersect the vertical axis below the origin and a steady state will not exist for case D if the same line should intersect the vertical axis above the origin. An examination of equation (4.17) shows that $\lim_{\mu_t\ o} \mu_{t+1} < 0$ and thus it is clear that an equilibrium will exist if the model, for $\sigma < 1$ is of the type given by Case D but will not exist if the model for $\sigma < 1$ is of the type given by Case C.

What becomes of critical importance for the model then is the magnitude of the absolute value of the derivative $\partial\mu_{t+1}/\partial\mu_t$.

Our model is very nonlinear and indeed for arbitrary parameter values it is not possible to derive the magnitude of the derivative $\partial\mu_{t+1}/\partial\mu_t$, analytically, even when evaluated at the steady state. Thus another approach must be taken.

Differentiating equation (4.17) with respect to μ_t we obtain;

$$\text{(4.18)} \qquad \frac{\partial\mu_{t+1}}{\partial\mu_t} = \beta^{\frac{1}{\tau\sigma}} \left[\frac{1-\alpha-\mu_t}{\alpha\mu_t}\right]^{\frac{1}{\tau}} \left(\frac{1}{\mu_t^2}\right) \left[\frac{\alpha(1-\sigma)+(\mu_t-1)}{(\sigma-1)(1-\alpha-\mu_t)}\right].$$

Now consider the case of $\sigma > 1$ where a steady state, $\mu_{t+1} = \mu_t = \mu$, must exist. Imposing this steady state on equation (4.17) gives us;

$$\text{(4.19)} \qquad \mu = 1 - \beta^{1/\tau\sigma}\left[\frac{(1-\alpha-\mu)}{\alpha\mu}\right]^{\frac{1}{\tau}} \left(\frac{1-\mu}{\mu}\right)$$

which yields;

$$\text{(4.20)} \qquad \mu = 1 - \alpha - \alpha(\beta)^{-1/\sigma}\mu^{(1-\alpha+\alpha\sigma)/\sigma}.$$

Equation (4.20) shows very clearly what has already been noted about this competitive solution, namely that it will always be characterized by rates $\mu \le 1 - \alpha$, which implies extraction rates $\phi > \alpha$. Beyond this however equation (4.20) makes it a relatively simple matter to use numerical techniques to calculate the steady state value of μ for a range of parameter values α and σ. This gives us a series $\{\mu, \alpha, \sigma\}$ with which we may then calculate values of the derivative $\partial\mu_{t+1}/\partial\mu_t$, using equation (4.18) and thus obtain some information about the local stability properties of the model, for $\sigma > 1$, in the neighbourhood of the steady state.

Applying the above technique for the case of $\sigma < 1$ might be construed as pulling oneself up by one's bootstraps, however, it is much less than that. If using equation (4.20) with $\sigma < 1$ yields a converged solution for μ then it must be the case that the absolute value of $\partial\mu_{t+1}/\partial\mu_t > 1$ and we have case D above since clearly a steady state does exist. Indeed, calculating this derivative after having found a μ for $\sigma < 1$ would be redundant since we know that such a μ exists only in the explosive case and knowing the magnitude of that derivative provides us with no more information.

Nevertheless the above mentioned procedure was implemented in its entirety using a variant of the Gauss-Seidel solution technique for values of the parameters as follows. α from 0.005 to 0.505 in steps of 0.005, σ from 0.255 to 0.955 in steps of 0.1 and from 1.005 to 15.005 in steps or 0.5, and for Ω and β = 0.95 or 0.5. Thus some 15200 different solutions to the steady state competitive rate of extraction were obtained. There was difficulty in obtaining convergence only for those parameter combinations which yield very low rates of extraction or fairly high ones. In all cases the calculated derivatives at the converged values for the steady state were of magnitude greater than one[3]. Some samples of these data are given in Appendix B.

It is apparent then that the competitive model with $\sigma > 1$ exhibits dynamics similar to Case A above in the neighbourhood of the steady state. Furthermore, it is clear that a steady state does exist for the model with $\sigma < 1$ indicating that the dynamics in this case while not being of the cyclical type are also explosive, as in case D above.

Indeed these dynamics are very interesting. Initially one might be concerned about these predictions of a lack of stability in the resource extraction paths implied by this model in the competitive environment. However, under a rational expectations interpretation the existence of a single steady state point in this one variable model is perfectly analogous to the well known "saddlepath" equilibriums of many other familiar models. Under such an interpretation the rate of extraction of the resource is a "jump" variable which will be determined exactly every period dependent upon the

[3] Note that as the parameter σ increases to higher and higher values the magnitude of the derivative for small values of the parameter α becomes quite close to one. Extra computational work around these values never yielded a value for the derivative of less than one. This work provides a fair level of confidence that the approach of this derivative to the value of one is asymptotic, rather than convergent.

current state(s) of the economy.[4] Thus the model will exhibit a rational expectations equilibrium at the unique steady state with an extraction rate given by $1-\mu$, in the competitive environment.

In the process of computing the derivatives $\partial\mu_{t+1}/\partial\mu_t$ it was a simple matter to also compute the values of ϕ_c and ϕ_s over the same range of parameters and some of these values also are included in Appendix B. We now turn to a comparative discussion of these two series which represent extraction rates from the competitive and socially optimal economies respectively.

[4]For a discussion and example of this "saddlepath" property see Begg (1982), chpt3.

Section 5: The Socially Planned Versus The Competitive Economy.

As discussed already an immediate implication of equation (4.17) is that $\mu_{tc} \leq 1 - \alpha \;\; \forall t$. This result suggests that the competitive economy will almost always extract at a rate greater than that of the socially planned economy, and indeed a quick look at the data in Appendix B or the plots in Appendix C will show that this is the case. The solution to the competitive economy is such that the economy extracts the resource at a rate proportional to the resources productivity as measured by the coefficient α from the Cobb-Douglas production function. This makes sense as the higher the value of α the higher is the marginal product of a unit of the resource in immediate production and thus more of it will be extracted.

But what about its value in future use? Clearly this must also be higher the greater is the resource's productivity. As a consequence of this the young generation seeking to purchase the remaining resource stocks as savings should be willing to pay a higher price γ for each unit, not only because of its higher productivity when they are old but because they should recognize that the next young generation will also value the resource highly and so will be willing to pay a high price for each unit as well. By backwards induction then, one might be led to suggest that, given frictionless markets etc., as are present here, that the current old generation would not extract the resource at too fast a rate as its value in all future production would be conveyed to them through the price γ. Thus the old generation would be led to extract less in exchange for a higher price from the young generations.

But clearly this is not the case in the model at hand. Where does the above argument break down? The price γ is denominated in units of the single consumption good which are in turn produced only with the inputs of the resource. Thus in order for the younger generation to be able to pay a higher price for the rights to a quantity of the resource in aggregate, a higher consumption wage would have to be paid to them which implies the use of *more* inputs in production. But labour inputs are fixed and so clearly additional output can be had only through the extraction and use of *more* of the stock of the resource. Indeed higher wages will be paid to the young workers when the productivity of the resource is higher and this is how a higher price for the resource is sustained. Because future generations are not present to offer additional labour of their own or some other form of payment to augment the incomes of the current generations the backwards induction argument breaks down. Higher *real* prices cannot be sustained without the production of more and thus the use of more. As can be seen in the plots of Appendix C the resource is used up at too fast a rate (relative to the social optimum) whenever the share of the resource in production is anywhere above about five percent.

What can we say about this result in terms of Pareto optimality and/or social welfare?. Since the consumption good is non-storable it is clear that, given optimality and thus efficiency in every period, total consumption will equal total output. This will be true for both the socially planned economy and the competitive economy and is simply proven. Thus for any period t we may write;

(5.1) $$C_t = F(N, r_t)$$

where C_t is total consumption by both young and old at time t and r_t is total resource extraction in period t. From equation (5.1) we may write;

(5.2) $$dC_t = F_N(N, r_t)dN + F_r(N, r_t)dr_t$$

which for zero population growth (dN = 0) becomes;

(5.3) $$dC_t = F_r(N, r_t)dr_t.$$

Clearly then (given fixed technology and labour supplies) the only way to increase consumption (and thus utility) in any period t is to increase the quantity of the resource extracted and used in production of the consumption good in that period. This fact together with the inherent finiteness of the resource stock leads to the following proposition.

Proposition: The constant (steady state) extraction rates derived from the competitive problem (4.1) and the socially planned problem (3.1) imply sequences of quantities of extracted resources $\{r_t^c\}$ and $\{r_t^s\}$ which satisfy the finite resource constraint and yield Pareto optimal outcomes.

Proof: Since it has been shown that ϕ_c and ϕ_s are both less than one and we define $r^c_t = \phi_c R_{t-1}$ and $r^s_t = \phi_s R_{t-1}$ $\forall t$, it is clear that the finite resource constraint will be satisfied by asymptotic depletion for the sequences $\{r^c_t\}$ and $\{r^s_t\}$ generated by ϕ_c, ϕ_s and any R_0.

Now, assume other than Pareto optimality.

Then there exists a sequence $\{r^*_t\}$ of extraction quantities different from the sequences $\{r^c_t\}$ and $\{r^s_t\}$ which satisfies the resource constraint and allows some individuals to be better off while leaving none worse off. In order for this criterion to be satisfied equation (5.3) would imply that (using the competitive case as an example);

$$r^*_t \geq r^c_t \quad \forall t \quad \text{and} \quad r^*_t > r^c_t \quad \text{for at least one t.}$$

Clearly however this implies that;

$$\sum_{t=0}^{\infty} r^*_t > \sum_{t=0}^{\infty} r^c_t = R_0$$

and thus the sequence $\{r^*_t\}$ must violate the resource constraint and this is a contradiction. Q.E.D.

The implications of the larger than socially optimal extraction rates found in the competitive environment for social welfare are fairly obvious. Clearly the competitive economy extracting at too fast a rate will leave future generations (perhaps only three or four periods off if the resource is particularly productive) with a much smaller stock from which to produce consumption goods. Indeed it is possible that in a very short time the stock may have been depleted to almost nothing leaving all future generations with extremely low levels of consumption. Given any positive weighting in the social welfare function towards these future generations then it is obvious that a welfare improvement could be obtained via a transfer of resource consumption from the earlier generations to the later.

Section 6: Conclusion.

A model of a resource economy with overlapping generations is developed and solutions for extraction rates of the finite stock of the resource are derived in both a competitive environment and a socially planned one. It is shown that the competitive environment results in the extraction of the resource at rates far outstripping those derived for the socially planned economy. While the socially-determined rate of extraction is highly sensitive to the rate of discount used, it is relatively constant across values of the resource share in production. The competitively determined rate, on the other hand is strongly increasing in the value of the resource share in production. The discrepancies which arise between the two sets of extraction rates determined are not a function of the existence of a gap between the social and private rates of discount.

The excess initial consumption implied by the competitive solution would lead to low levels of the productive resource stock in the future and thus low future consumption and utility. A loss of social welfare can easily occur in the competitive environment due to intergenerational inequities in the extraction of the finite resource stock.

Future work on the model could include, the addition of labour in the utility function and thus the introduction of a labour supply decision by consumers. The addition of a production augmenting capital stock to the model.[5] The addition of population growth and/or technological change. The examination of the possible policy options of a government attempting to alter the competitive result to alleviate the intergenerational inequities previously mentioned.

[5] This work is currently in progress, preliminary papers are available from the auther upon request.

Appendix A.

We have;

$$\text{(A.1)} \qquad \frac{\partial\mu_{t+1}}{\partial\mu_t} = \beta^{\frac{1}{\tau\sigma}} \left[\frac{1-\alpha-\mu_t}{\alpha\mu_t} \right]^{\frac{1}{\tau}} \left(\frac{1}{\mu_t^2} \right) \left[\frac{\alpha(1-\sigma)+(\mu_t-1)}{(\sigma-1)(1-\alpha-\mu_t)} \right] .$$

Clearly since we must have $\mu_t < \alpha - 1$, and $\alpha < 1$ then for $\sigma > 1$ equation (A.1) must be less than zero.

Now with $\sigma < 1$ it is no longer clear what the sign of the numerator in the far right hand expression is. However, in order for this numerator to be positive it must be the case that $(\alpha + \mu_t - 1) > \alpha\sigma$. But again we must have $\alpha + \mu_t - 1 < 0$ and clearly $\alpha\sigma > 0$. Therefore it is not possible for said numerator to be positive and we must have equation (A.1) greater than zero with $\sigma < 1$.

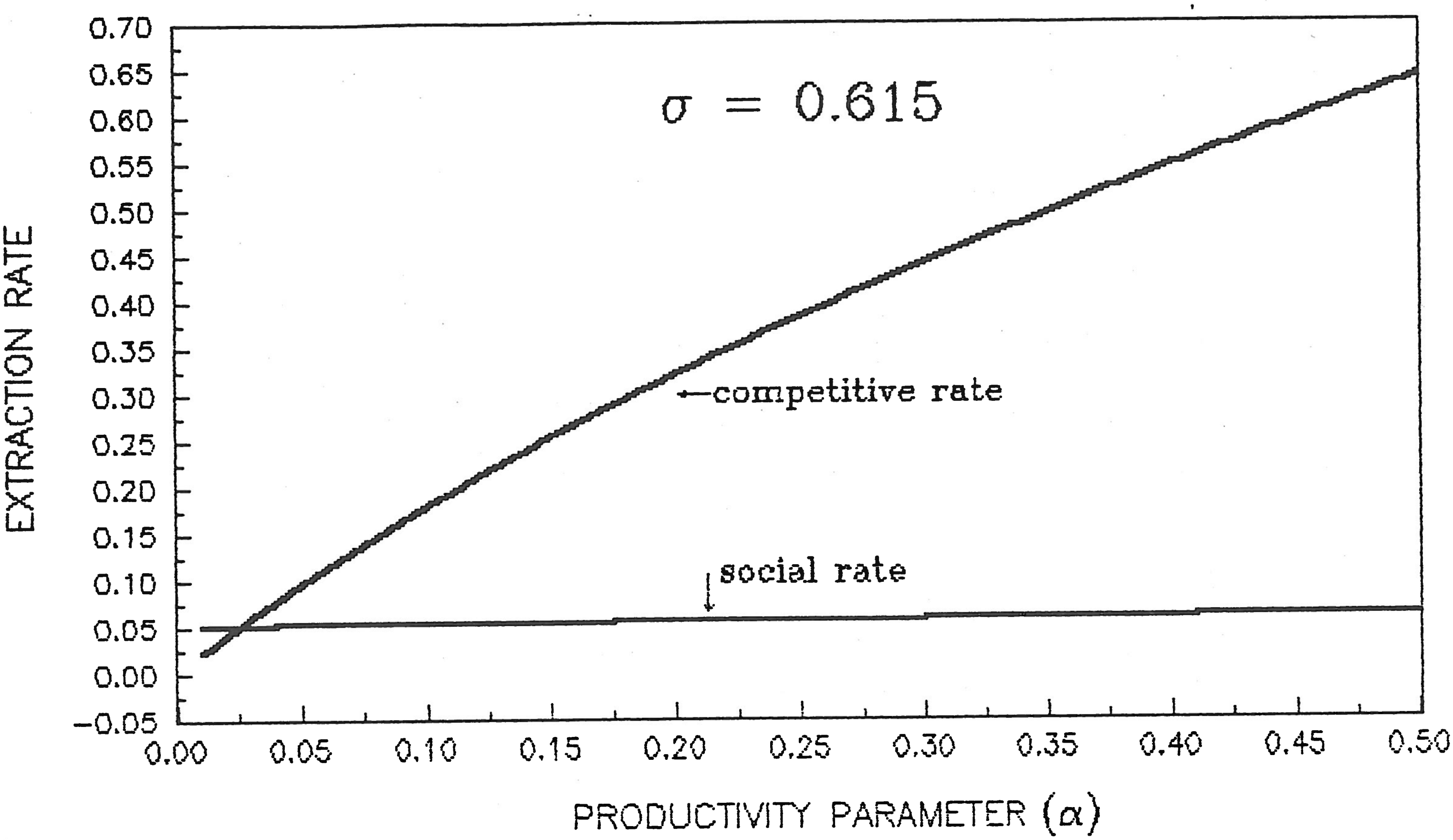
Appendix B: Data generated on extraction rates and derivatives over a spectrum of preference and production parameter values

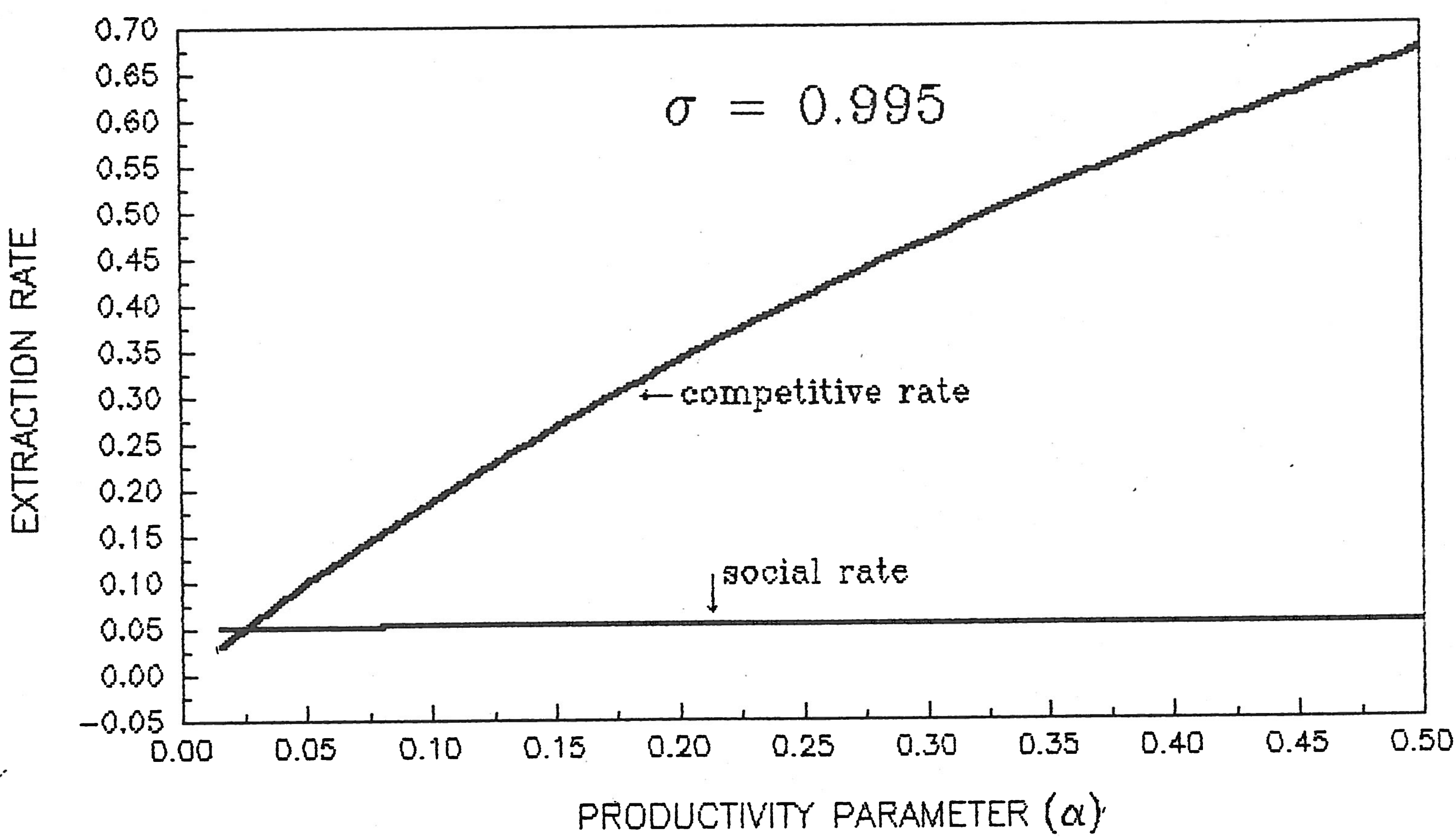
σ	α	ϕ_c	ϕ_s	Converge%	$\partial\mu_{t+1}/\partial\mu_t$
0.255000	0.050000	0.092401	0.051884	0.000009	1.92368
0.255000	0.100000	0.163863	0.053914	0.000010	2.24630
0.255000	0.150000	0.225358	0.056111	0.000010	2.61225
0.255000	0.200000	0.281238	0.058494	0.000009	3.03982
0.255000	0.250000	0.333680	0.061088	0.000009	3.54909
0.255000	0.300000	0.383922	0.063923	0.000010	4.16471
0.255000	0.350000	0.432727	0.067033	0.000010	4.91886
0.255000	0.400000	0.480581	0.070462	0.000010	5.85512
0.255000	0.450000	0.527808	0.074260	0.000009	7.03477
0.255000	0.500000	0.574614	0.078491	0.000009	8.54714
0.610000	0.050000	0.096224	0.050969	0.000008	4.70921
0.610000	0.100000	0.179624	0.051975	0.000009	5.51981
0.610000	0.150000	0.253832	0.053023	0.000009	6.46437
0.610000	0.200000	0.321138	0.054113	0.000009	7.58069
0.610000	0.250000	0.383075	0.055250	0.000010	8.91878
0.610000	0.300000	0.440710	0.056435	0.000009	10.54646
0.610000	0.350000	0.494809	0.057672	0.000010	12.55780
0.610000	0.400000	0.545933	0.058964	0.000009	15.08757
0.610000	0.450000	0.594499	0.060316	0.000010	18.33379
0.610000	0.500000	0.640813	0.061731	0.000010	22.59827
0.965000	0.050000	0.097425	0.050085	0.000009	63.88072
0.965000	0.100000	0.185340	0.050171	0.000009	74.70518
0.965000	0.150000	0.265145	0.050257	0.000009	87.72307
0.965000	0.200000	0.337971	0.050343	0.000010	103.47431
0.965000	0.250000	0.404738	0.050430	0.000009	122.75771
0.965000	0.300000	0.466207	0.050517	0.000010	146.64781
0.965000	0.350000	0.523011	0.050604	0.000010	176.68016
0.965000	0.400000	0.575681	0.050692	0.000009	215.05905
0.965000	0.450000	0.624668	0.050779	0.000010	265.02731
0.965000	0.500000	0.670353	0.050868	0.000010	331.60077
1.005000	0.050000	0.097511	0.049988	0.000006	-455.38403
1.005000	0.100000	0.185765	0.049976	0.000008	-534.70203
1.005000	0.150000	0.266013	0.049963	0.000008	-628.24438
1.005000	0.200000	0.339289	0.049951	0.000010	-742.21899
1.005000	0.250000	0.406457	0.049939	0.000010	-881.87903
1.005000	0.300000	0.468246	0.049927	0.000009	-1055.26868
1.005000	0.350000	0.525273	0.049915	0.000009	-1274.42200
1.005000	0.400000	0.578066	0.049903	0.000009	-1555.41370
1.005000	0.450000	0.627076	0.049891	0.000010	-1924.30530
1.005000	0.500000	0.672696	0.049878	0.000009	-2416.90747

Appendix B: Continued.

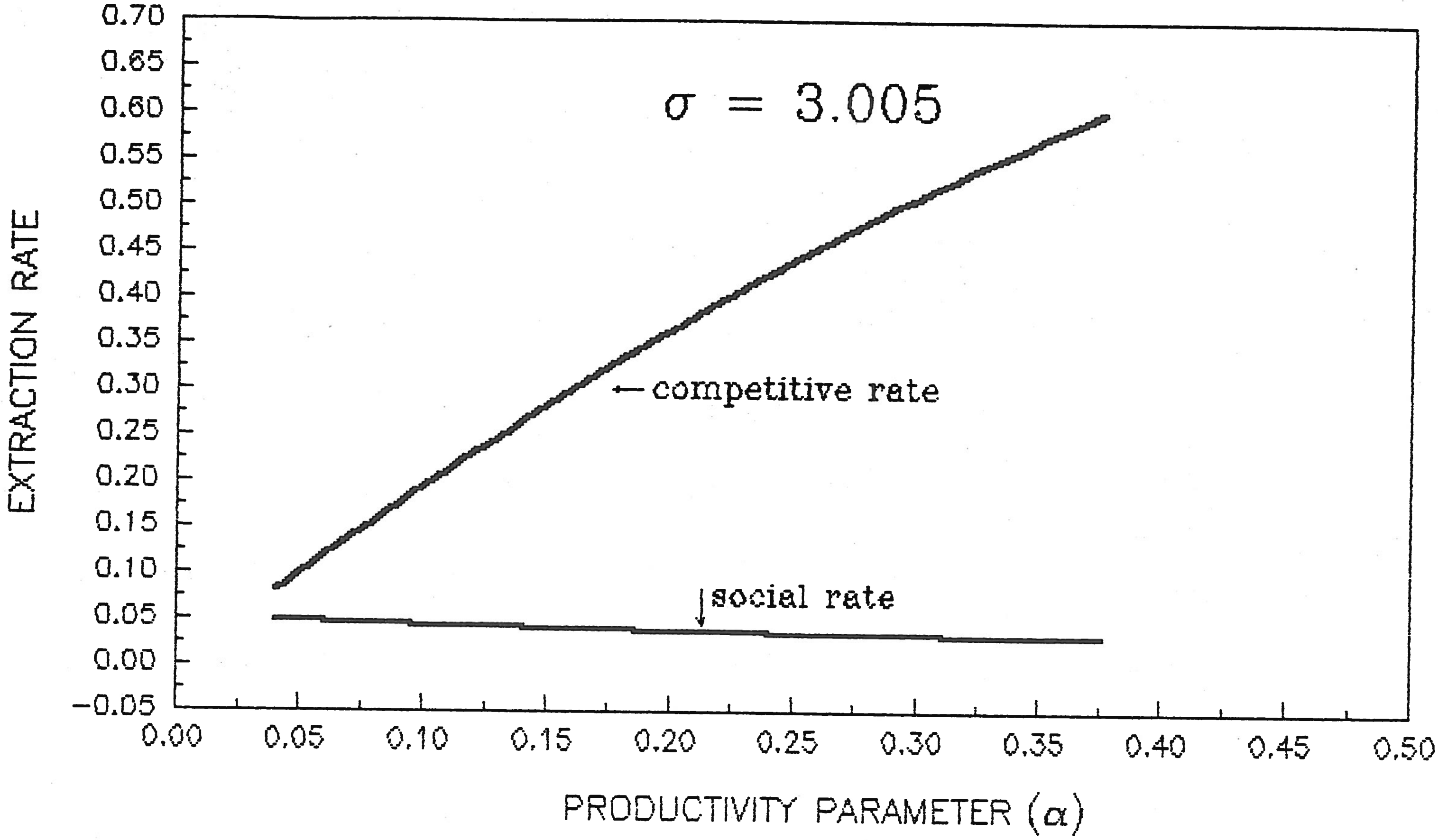
σ	α	ϕ_c	ϕ_s	Converge%	$\partial\mu_{t+1}/\partial\mu_t$
2.005000	0.050000	0.098582	0.047666	0.000003	-3.38166
2.005000	0.100000	0.191305	0.045540	0.000008	-3.93236
2.005000	0.150000	0.277679	0.043595	0.000010	-4.62244
2.005000	0.200000	0.357428	0.041810	0.000009	-5.49296
2.005000	0.250000	0.430494	0.040165	0.000009	-6.59942
2.005000	0.300000	0.497014	0.038644	0.000010	-8.01823
2.005000	0.350000	0.557294	0.037235	0.000009	-9.85680
2.005000	0.400000	0.611759	0.035925	0.000009	-12.27015
2.005000	0.450000	0.660916	0.034703	0.000010	-15.48854
2.005000	0.500000	0.705315	0.033562	0.000009	-19.86538
3.005000	0.050000	0.098957	0.045550	0.000008	-2.25234
3.005000	0.100000	0.193354	0.041827	0.000008	-2.60861
3.005000	0.150000	0.282183	0.038666	0.000009	-3.06421
3.005000	0.200000	0.364663	0.035950	0.000010	-3.65030
3.005000	0.250000	0.440298	0.033590	0.000009	-4.40904
3.005000	0.300000	0.508901	0.031521	0.000010	-5.39841
3.005000	0.350000	0.570583	0.029692	0.000009	-6.69957
3.005000	0.400000	0.625703	0.028063	0.000010	-8.42921
3.005000	0.450000	0.674800	0.026604	0.000009	-10.75984
3.005000	0.500000	0.718520	0.025289	0.000009	-13.95602
4.005000	0.050000	0.099148	0.043614	0.000006	-1.87452
4.005000	0.100000	0.194422	0.038674	0.000005	-2.16528
4.005000	0.150000	0.284577	0.034739	0.000009	-2.54159
4.005000	0.200000	0.368566	0.031531	0.000009	-3.03137
4.005000	0.250000	0.445644	0.028865	0.000010	-3.67250
4.005000	0.300000	0.515424	0.026615	0.000009	-4.51703
4.005000	0.350000	0.577893	0.024690	0.000009	-5.63774
4.005000	0.400000	0.633361	0.023025	0.000009	-7.13877
4.005000	0.450000	0.682389	0.021570	0.000010	-9.17383
4.005000	0.500000	0.725690	0.020288	0.000010	-11.97786
5.005000	0.050000	0.099264	0.041835	0.000010	-1.68534
5.005000	0.100000	0.195078	0.035962	0.000004	-1.94314
5.005000	0.150000	0.286062	0.031535	0.000006	-2.27947
5.005000	0.200000	0.371010	0.028079	0.000009	-2.72061
5.005000	0.250000	0.449014	0.025305	0.000008	-3.30236
5.005000	0.300000	0.519552	0.023030	0.000010	-4.07394
5.005000	0.350000	0.582523	0.021130	0.000010	-5.10399
5.005000	0.400000	0.638208	0.019520	0.000009	-6.49056
5.005000	0.450000	0.687179	0.018138	0.000009	-8.37803
5.005000	0.500000	0.730196	0.016938	0.000010	-10.98668

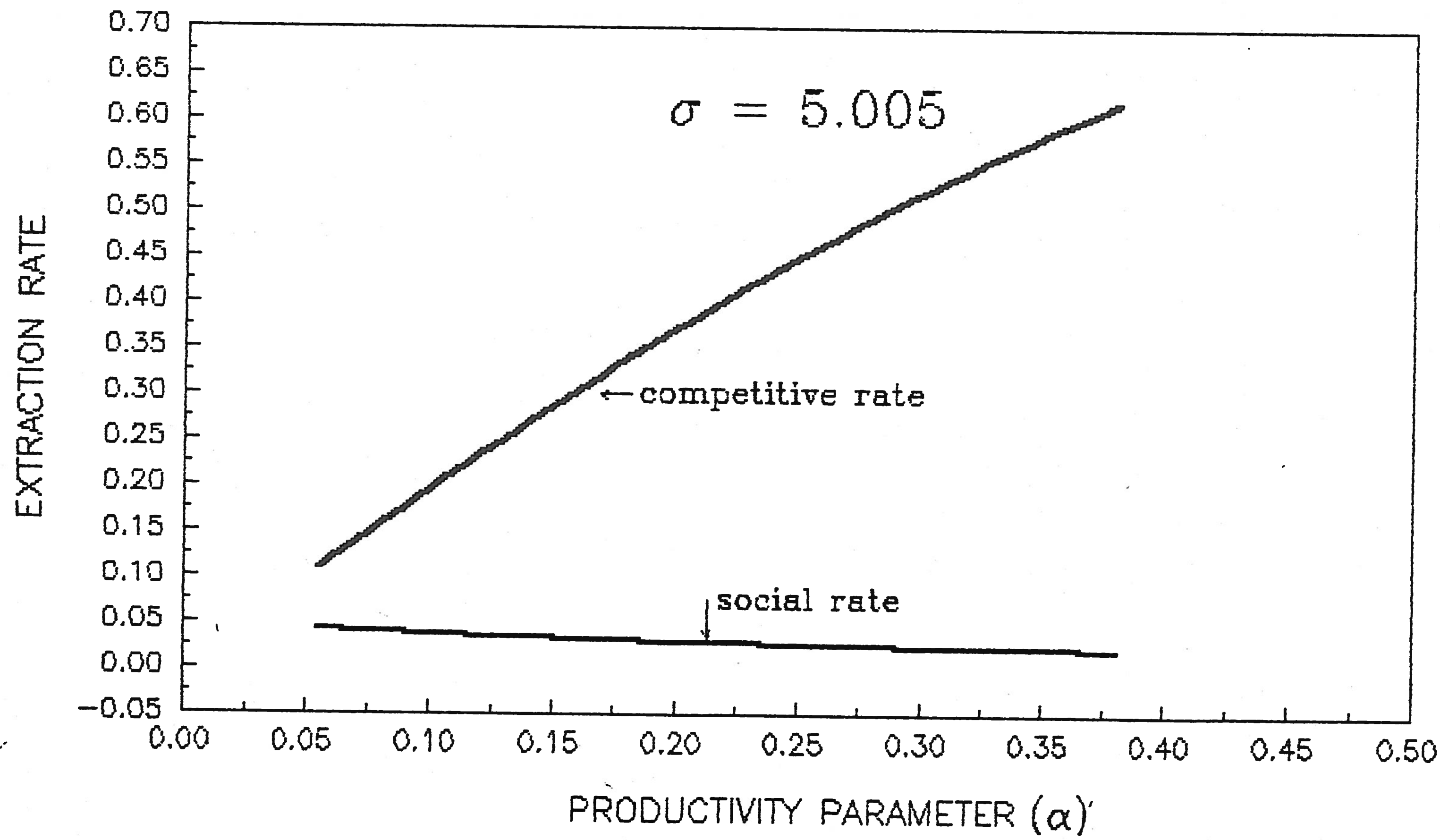
Appendix C: COMPARATIVE EXTRACTION RATES





Appendix C: COMPARATIVE EXTRACTION RATES





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