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Have Middle-Class Earnings Risen in Canada? A Statistical Inference Approach

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Abstract

This paper extends the statistical inference approach developed in Beach (2016) to look at income changes over different regions of an income distribution. Specifically, it looks at relative-mean earnings (RME) ratios and mean earnings levels for lower earners, middle-class (MC) workers and higher earners in Canada since 1970. Formulas are developed for (asymptotic) standard errors of these distributional statistics. The most consistent pattern since 1980 has been the marked decline in RME for MC workers, which has been highly statistically significant. Since 2005, however, real earnings levels have increased significantly and have been broadly shared across these earnings groups.

1. **Introduction**

Rising income inequality is one of the major concerns of the day (for recent overviews, see Beach, 2016; Fortin et al. (2012); Green, 2016; Green and Sand, 2015; Green et al., 2016, and Heisz, 2016). It can lead to political inequality and tilt the rules in favour of those with more resources. If rules of the game get too out of line for the public benefit, overall efficiency costs can occur. A growing economics literature has examined how much greater income inequality and a reduced Middle Class can lead to reduced macroeconomic growth and performance. And substantially greater inequality may lead to a loss of economic opportunity and living standards and a general sense of unfairness.

One aspect of this debate that has received relatively little attention, though, is what has happened to actual income levels over different regions of the income distribution – lower incomes, middle incomes and upper incomes. Most of the discussion has been in terms of income shares or various summary measures of inequality such as the Gini coefficient. Yet it is quite possible for most or even all income groups' (real) incomes to go up over time while income inequality rises as well. If economic well-being and living standards are related to (real) incomes, then what has happened to the incomes themselves is of social interest. Such an analysis can also serve to more directly link empirical distributional analyses to theoretical results on general social-welfare inferences that can reasonably be drawn from observed distributional changes (Jenkins, 1991).

Almost all of the discussion of income inequality change, however, has been carried on in terms of descriptive statistics – such as income shares or various inequality measures. Formal statistical inference principles are hardly ever used. It would thus be a useful analytical advance if one could examine changes in the (real) incomes of various income groups across the distribution in terms of formal statistical inference.

This paper, correspondingly, has several objectives. First, it offers an alternative perspective to income shares and summary inequality measures for studying distributional change by forwarding an examination of the (real) incomes of various income groups. These incomes – both in terms of relative incomes and income-group conditional income levels – can be usefully viewed as a complementary set of statistics to be used along side the more conventional income share and summary inequality statistics. Indeed, this will doubly make sense where it is shown below that the underlying statistical inference formulas for the former statistics can be straightforwardly derived from those of the latter statistics. Second, this paper extends the formal statistical inference methodology introduced in Beach (2016) for income shares and population shares to apply also to relative-mean incomes and conditional income levels of different income groups. It thus complements Beach (2016) and completes the statistical inference formulas for a proposed foursome of distributional statistics: population shares, income shares, relative-mean income ratios, and conditional mean income levels. The paper also illustrates a useful general methodology of statistical inference applied to an analysis of distributional change. While the income groups examined are defined in terms of median income levels – similar to Beach (2016) – the general methodology applies to income groups defined in terms of any quantile cut-offs (such as, say, income quintiles).

The paper proceeds as follows. The next section sets out the problem of how to undertake formal statistical inference for incomes of groups defined in terms of the median (or quantile) incomes, and outlines the basic approach used in this study. Section 3 presents the background data on shares of workers and their corresponding earnings shares. Section 4 applies the approach of this paper to examine relative mean earnings ratios for lower-earnings workers, middle-class earners, and higher-earnings workers (separately for men and women) in Canada using Census data over 1970-2005 and Labour Force Survey (LFS) semi-annual data over 1997-2015. Since the use of LFS data is relatively less common for distributional analysis in Canada, more extensive examination of distributional patterns is provided with the latter data. The statistical approach is then extended in Section 5 to the actual conditional mean earnings levels for these three earnings groups, and the empirical results are presented in Section 6. The final section concludes.

2. Setting Out the Problem and Approach Used

2.1 Set Out the Problem

In the rest of this paper, the form of income that will be examined is workers' earnings. Earnings are the great majority of total income for most non-retired adults in the population, and it has been well established that labour market earnings are the principal source of the widening of income inequality in Canada, the United States and many other developed economies over recent decades. Accordingly, the proportion of workers who receive earnings within some range will be referred to as that group's *workers' share*, and proportion of total earnings going to this earnings group will be referred to as the group's *earnings share*.

This study will examine three broad earnings groups – referred to as lower earners (LE), middle-class workers (MC), and higher earners (HE) – on the basis of their (sex-specific) medians:

Lower earners - those with earnings below 50% of the median

Middle-class earners - those with earnings between 50% and 150% of the median

Higher earners - those with earnings above 200% of the median.

Note that, while these groups are mutually exclusive, they are not exhaustive. Calculations were also done where the upper group was those with earnings above 150 percent of the median, and the results are essentially the same as for the above narrower definition, but not quite as crisp.

The problem this paper addresses is to develop an approach to allow formal statistical inference of standard estimates of the average earnings of these three earnings groups. More specifically, we look at both the relative-mean earnings ratios (RME) and the conditional mean earnings levels (CME) for each of these three earnings groups. The reason why this approach is not straightforward (and that only descriptive statistics estimates are used in the literature) is that both these RME and CME measures for each earnings group are dependent upon an estimate of the median and it is well known in the statistics literature that the asymptotic variance of the sample median is not distribution-free. Hence neither will be the asymptotic variances of the RME and CME estimates. This paper proposes an approach to deal with this problem.

2.2 Review the Lognormal Inference Approach for Shares of Workers and Earnings Shares

Since the approach being forwarded in this paper extends that developed in Beach (2016), it is useful to briefly outline the latter, which applies to income shares and shares of workers. To do so, consider for illustrative purposes the case of the middle-class income group. And to keep the notation the same as in Beach (2016), we will use the term income group and recipients' incomes rather than earnings and population shares rather than share of workers.

If $f(\cdot)$ is a specified income distribution density function, then the population share and the income share of the middle-class group, for example, are:

$PS_{MC} = \int_a^b f(x)dx$ and

$$IS_{MC} = \int_a^b \frac{1}{\propto} xf(x)dx \quad \text{where } \propto \equiv E(x)$$

where [a,b] is the defined range of middle-class income recipients within the distribution f(·). Let ξ be the (population) median and $\hat{\xi}$ the sample estimate as the middle-most observation in the ordered sample of incomes. Then $a = 0.5\,\hat{\xi}$ and $b = 1.5\hat{\xi}$. So $\widehat{PS}_{\mathrm{MC}}$ is the estimate obtained as the proportion of sample observations between 0.5 $\hat{\xi}$ and 1.5$\hat{\xi}$ in the ordered sample of incomes, and $\widehat{IS}_{\mathrm{MC}}$ is the estimate obtained by cumulating all incomes within this range divided by total income in the sample.

The approach taken to establish the (asymptotic) distributions of the random variables $\widehat{PS}_{\mathrm{MC}}$ and $\widehat{IS}_{\mathrm{MC}}$ is based on recognizing that these are both functions of the sample median whose (asymptotic) distribution is well known. More specifically, under fairly broad conditions, $\sqrt{N}(\hat{\xi} - \xi)$ has a limiting normal distribution with mean zero and variance

$$\text{Asy var }(\hat{\xi}) \equiv \theta(\xi)^2 = (.5)(.5)/[f(\xi)]^2 \tag{1}$$

where N is the sample size (Rao, 1965, p. 423). Hence, the (asymptotic) standard error of $\hat{\xi}$ is

$$S.E.\left(\hat{\xi}\right) = \hat{\theta}\left(\hat{\xi}\right)/\sqrt{N} = (.5)/f\left(\hat{\xi}\right) \cdot \sqrt{N}. \tag{2}$$

To link the share formulas to the median, recall from Rao (1965, p. 385) that, if $\hat{\xi}$ has a limiting normal distributions with variance given by (1) and if $g(\hat{\xi})$is a continuous function of $\hat{\xi}$ with a first derivative $g'(\hat{\xi}) \equiv dg(\hat{\xi})/\, d\hat{\xi}$, then the statistic $g(\hat{\xi})$ also has a limiting normal distribution with mean $g(\xi)$ and (asymptotic) variance:

$$\text{Asy var }\left(g\left(\hat{\xi}\right)\right) \equiv [g'(\xi)]^2 \cdot \theta(\xi)^2 \ . \tag{3}$$

The two examples of $g(\hat{\xi})$ we make use of are $PS_{MC}(\hat{\xi})$ and $IS_{MC}(\hat{\xi})$.

To obtain the gradients $g'(\xi)$, we make use of Leibnitz's Rule (Bergin, 2015, p. 467). In the case of the population share,

$$g'(\xi) = f(1.5\xi)(1.5) - f(0.5\xi)(0.5)$$

so that

$$\text{Asy var}\left(\widehat{PS}_{MC}\right) = [f(1.5\xi)(1.5) - f(0.5\xi)(0.5)]^2 \cdot [0.25\ /\ f(\xi)^2]. \tag{4}$$

Hence the estimated asymptotic variance of $\widehat{PS}_{MC}$ is gotten by putting sample estimates into equation (4).

Now to implement equation (4), one needs an expression for $f(\cdot)$. Since we are working with earnings distributions in our applications, it seems quite reasonable to assume that $f(\cdot)$ follows a lognormal distribution:

$$f(x) = [1\ /\ \sigma\sqrt{2\pi}\ \cdot x]\ \cdot exp[-(ln\, x - \mu)^2\ /\ 2\sigma^2] \quad \text{for } x\ \epsilon\ (0, +\infty) \tag{5}$$

where μ is the mean of *ln x* and σ^2 is the variance of *ln x*. Hence,

$$\text{Asy }\widehat{\text{var}}\left(\widehat{PS}_{MC}\right) = [(1.5)\ \cdot\ \hat{f}\left(1.5\hat{\xi}\right) - (0.5)\ \cdot\ \hat{f}\left(0.5\hat{\xi}\right)]^2 \cdot [(0.25)/\ \hat{f}(\hat{\xi})^2] \tag{6a}$$

and thus

$$S.E.\left(\widehat{PS}_{MC}\right) = [Asy\ \widehat{var}\left(\widehat{PS}_{MC}\right)\ /\ N]^{1/2} \tag{6b}$$

$$= \left[\hat{f}\left(1.5\hat{\xi}\right)(1.5) - \hat{f}\left(0.5\hat{\xi}\right)(0.5)\right]\ \cdot [0.5\ /\ \hat{f}(\hat{\xi})]\quad /\sqrt{N}$$

where $\hat{f}(\cdot)$ is obtained by plugging consistent (indeed maximum likelihood) sample estimates of μ and σ^2 into (5), and 1.5 $\hat{\xi}$ and 0.5 $\hat{\xi}$ are the upper and lower bounds on the middle-class earnings range.

Similarly, in the case of the income share, the gradient

$$g'(\xi) = [(1.5\xi) \cdot f(1.5\xi) \cdot 1.5\ /\propto] - [(0.5\xi) \cdot f(0.5\xi) \cdot 0.5\ /\propto]\ , \tag{7}$$

so the estimated (asymptotic) variance is

$$\text{Asy }\widehat{\text{var}}\left(\widehat{IS}_{MC}\right) = ([\left(1.5\hat{\xi}\right)\ \cdot\ \hat{f}\left(1.5\hat{\xi}\right) \cdot\ 1.5/\ \hat{\propto}]$$

$$-\ [\left(0.5\hat{\xi}\right)\ \cdot\ \hat{f}\left(0.5\hat{\xi}\right) \cdot\ 0.5/\ \hat{\propto}])^2 \cdot\ [(0.25)/\ \hat{f}(\hat{\xi})^2],$$

and the standard error is

$$S.E.\left(\widehat{IS}_{MC}\right) = ([(1.5)^2 \cdot (\hat{\xi} \,/\, \hat{\propto}) \cdot \hat{f}(1.5\hat{\xi})] - [(0.5)^2 \,(\hat{\xi}/\, \hat{\propto}) \cdot \hat{f}(0.5\hat{\xi})])$$

$$\cdot\, [0.5 \,/\, \hat{f}(\hat{\xi}\,)]/\sqrt{N} \qquad (8)$$

The standard error formulas for $\widehat{PS}$ and $\widehat{IS}$ over the lower and higher earnings ranges are obtained in similar fashion, and turn out to be simpler in form:

$$S.E.\left(\widehat{PS}_{LE}\right) = [(0.5) \cdot \hat{f}(0.5\hat{\xi})] \cdot [0.5/\, \hat{f}(\hat{\xi})] \;/\sqrt{N} \qquad (9)$$

$$S.E.\left(\widehat{IS}_{LE}\right) = [(0.5) \cdot (0.5\hat{\xi}) \cdot f(0.5\hat{\xi})/\, \hat{\propto}] \;\cdot\; [0.5/\, \hat{f}(\hat{\xi})] \;/\sqrt{N} \qquad (10)$$

$$S.E.\left(\widehat{PS}_{HE}\right) = [2 \;\cdot \hat{f}(2\hat{\xi})] \;\cdot\; [0.5/\, \hat{f}(\hat{\xi})] \;/\sqrt{N} \qquad (11)$$

$$S.E.\left(\widehat{IS}_{HE}\right) = [2 \cdot (2\hat{\xi}) \cdot \hat{f}(2\hat{\xi})/\, \hat{\propto}] \;\cdot\; [0.5/\, \hat{f}(\hat{\xi})] \;/\sqrt{N} \qquad (12)$$

To perform the calculations for the standard errors, first compute the median earnings level for the distribution ($\hat{\xi}$) and the mean and standard deviation of log earnings ($\hat{\mu}, \hat{\sigma}$). Plug the latter two values into the formula for the lognormal density to get $\hat{f}(\cdot)$. Then compute the various cut-off bounds as functions of $\hat{\xi}$ and evaluate $\hat{f}(\cdot)$ at its sample median $\hat{\xi}$ and at the required bound values, and calculate the standard error estimates from equations (6), (8), and (9) - (12). In case of weighted samples, estimates of ξ, μ, σ and $\propto$ should all be calculated in weighted fashion.

2.3 Applying the Lognormal Inference Approach to Relative-Mean Incomes

Now consider applying the lognormal inference approach to relative-mean incomes. If x represents individual incomes, then, from first principles,

$$\frac{E(x \,/\, x \,\in i)}{E(x)} = \left[\int_a^b x\, f(x)dx \,/\, \int_a^b f(x)dx\right] / \int_0^\infty x\, f(x)dx \qquad (13)$$

where f(·) is the underlying density function of the distribution of income, a is the lower bound of the incomes of individuals in group i and b is the upper bound of the incomes for group i =

LE, MC, HE. In the case of middle-class income recipients, the bounds are defined in terms of a = 0.5ξ and b = 1.5ξ where ξ is the (population) median income level. So

$$\frac{E(x\ /\ MC)}{E(x)} = \left[\int_{0.5\xi}^{1.5\xi} \frac{1}{\propto}\ x\ f(x)dx\right] / \int_{0.5\xi}^{1.5\xi} x\ f(x)dx \qquad (14)$$

$$= IS_{MC}\ /\ PS_{MC}$$

where $\propto = E(x)$. But we already know the (asymptotic) distributions of sample estimates of IS_{MC} and PS_{MC} as established in the previous section. This suggests that a simple way to establish the (asymptotic) distribution of the sample relative-mean income ratio is in terms of the distributions of $\widehat{IS}_{MC}$ and $\widehat{PS}_{MC}$ since

$$\widehat{RMY}_{MC} = (\widehat{PS}_{MC})^{-1} \cdot \widehat{IS}_{MC}\ . \qquad (15)$$

Equation (15) has an interesting interpretation. Since $\widehat{PS}$ is measured along the horizontal axis of a Lorenz curve diagram and $\widehat{IS}$ is measured along the vertical axis, $\widehat{RMY}$ is the average slope along a segment of the Lorenz curve. So if 0.5$\hat{\xi}$ and 1.5$\hat{\xi}$ map out a region along the horizontal axis corresponding to the middle-class set of workers, then $\widehat{RMY}_{MC}$ is the slope of the straight line segment of the sample Lorenz curve subtended by these two middle-class bounds. It is thus not surprising that $\widehat{RMY}$ is a useful supplementary bit of information beyond simply looking at $\widehat{PS}$ and $\widehat{IS}$.

In order to establish the (asymptotic) distribution of $\widehat{RMY}_{MC}$, however, one needs to extend the argument in the last section to establish the joint distribution of $\widehat{PS}_{MC}$ and $\widehat{IS}_{MC}$ together. But this can be done by making use of a multivariate form of the Rao Linkage Theorem (Rao, 1965, p. 388). Let $g(\xi) = [g_1(\xi), g_2(\xi)]^1$ be a 2x1 vector where $g_1(\xi) \equiv PS_{MC}(\xi)$ and $g_2(\xi) \equiv IS_{MC}(\xi)$ are both continuous differentiable functions of ξ. Then the above theorem establishes that the joint (asymptotic) distribution of $g(\xi) = [\widehat{PS}_{MC},\ \widehat{IS}_{MC}]^1$ is joint bivariate normal with an (asymptotic) variance-covariance matrix given by

$$\text{Asy. Var}\big(\widehat{PS}_{MC}, \widehat{IS}_{MC}\big) = G\ \Lambda\ G^1 \qquad \text{(16a)}$$

where $G = [\partial g_1\ /\ \partial\xi,\ \partial g_2\ /\ \partial\xi]^1$ is a 2x1 vector of first derivatives and Λ is a scalar of $v\ \equiv$ $\text{asy. var}\ \big(\hat{\xi}\big) = (0.5)(0.5)\ /\ [f(\xi)]^2$. Then

$$\text{Asy. Var}\big(\widehat{PS}_{MC}, \widehat{IS}_{MC}\big) = \begin{bmatrix} (g_1^1)^2 v & (g_1^1 g_2^1) v \\ (g_1^1 g_2^1) v & (g_2^1)^2 v \end{bmatrix} \qquad \text{(16b)}$$

where the superscript primes indicate partial derivatives with respect to ξ. So

$$g_1^1\ \equiv \frac{\partial g_1}{\partial \xi} = \frac{\partial PS_{MC}}{\partial \xi}$$

$$= f(1.5\xi)(1.5) - f(0.5\xi)(0.5), \text{ and} \qquad \text{(17)}$$

$$g_2^1\ \equiv \frac{\partial g_2}{\partial \xi} = \frac{\partial IS_{MC}}{\partial \xi}$$

$$= \left[\frac{(1.5\xi)}{\propto} \cdot f(1.5\xi)\right](1.5) - \left[\frac{(0.5\xi)}{\propto} \cdot f(0.5\xi)\right](0.5)\ . \qquad \text{(18)}$$

This result provides both the (asymptotic) variances of $\widehat{PS}_{MC}$ and $\widehat{IS}_{MC}$, but also the (asymptotic) covariance between the two.

To establish the asymptotic distribution of $\widehat{RMY}_{MC}$ from (15), we can again make use of the Rao Linkage Theorem. Since $\hat{q} \equiv q\big(\widehat{PS}_{MC},\ \widehat{IS}_{MC}\big)\ \equiv\ \big(\widehat{PS}_{MC}\big)^{-1} \cdot\ \widehat{IS}_{MC}$ is again a continuous differentiable function of $\widehat{PS}_{MC}$ and $\widehat{IS}_{MC}$ where the latter are (asymptotically) joint normally distributed, then $\sqrt{N}(\hat{q} - q)$ also has a limiting normal distribution with mean zero and variance

$$\text{Asy. Var}(\hat{q}) = Q\ W\ Q^1 \qquad \text{(19)}$$

where now the (1x2) gradient vector $Q = [q_1, q_2]$ where

$$q_1\ \equiv \frac{\partial q}{\partial PS_{MC}}\ (PS_{MC}, IS_{MC}) = \frac{-\ IS_{MC}}{(PS_{MC})^2}$$

$$q_2 \equiv \frac{\partial q}{\partial IS_{MC}} (PS_{MC}, IS_{MC}) = \frac{1}{PS_{MC}},$$

$$q \equiv q(PS_{MC}, IS_{MC}) = (PS_{MC})^{-1} \cdot IS_{MC}, \text{ and}$$

$$W \equiv \begin{bmatrix} \text{Asy. var.}(\widehat{PS}_{MC}) & \text{Asy. cov.}(\widehat{PS}_{MC}, \widehat{IS}_{MC}) \\ \text{Asy. cov.}(\widehat{PS}_{MC}, \widehat{IS}_{MC}) & \text{Asy. var.}(\widehat{IS}_{MC}) \end{bmatrix}$$

$$= \begin{bmatrix} w_{11} & w_{21} \\ w_{12} & w_{22} \end{bmatrix}, \text{ where}$$

$w_{11} = (g_1^1)^2 v$ with g_1^1 from equation (17)

$w_{22} = (g_2^1)^2 v$ with g_2^1 from equation (18)

$w_{12} = (g_1^1 \cdot g_2^1)\, v$ from both equations (17) and (18).

Therefore,

$$\text{Asy. Var}(\hat{q}) = (q_1)^2\, w_{11} + (q_1 q_2 w_{12} + q_1 q_2 w_{21}) + (q_2)^2\, w_{22}\,. \tag{20}$$

Consequently, the (asymptotic) standard error of $\hat{q} \equiv \widehat{RMY}_{MC}$ is given by

$$(\text{Asy.})\, S.E.(\hat{q}) \equiv S.E.(\widehat{RMY}_{MC})$$

$$= \left[\frac{\text{Asy.}\hat{\ }\text{Var}\ (\hat{q})}{N}\right]^{1/2} \tag{21}$$

where N is again the sample size and the “hat” indicates that all of the unknowns have been estimated as done in the previous section.

The corresponding formulas for the relative-mean income ratios for the lower-earnings (LE) and higher-earnings (HE) groups are derived in the same way, but turn out to be simpler (because only one integral bound of PS and IS is a function of the median). In the case of the lower-earnings group,

$$RMY_{LE} = \frac{E(x\ /\ LE)}{E(x)} = \left[\int_0^{0.5\xi} \frac{1}{\propto} x\, f(x)dx \,/\, \int_0^{0.5\xi} f(x)dx\right]$$

$$= IS_{LE}\ /\ PS_{LE}\,, \tag{22}$$

where again $\propto \equiv E(x)$, and RMY_{LE} is estimated as

$$\widehat{RMY}_{LE} = (\widehat{PS}_{LE})^{-1} \cdot \widehat{IS}_{LE} . \tag{23}$$

As before,

$$\text{Asy. var}\left(\widehat{PS}_{LE}\right) = [(0.5) \cdot f(0.5\xi)]^2 \cdot (0.25) / f(\xi)^2 \text{ , and}$$

$$\text{Asy. var}\left(\widehat{IS}_{LE}\right) = [(0.5) \cdot (0.5\xi) \cdot f(0.5\xi) / \propto]^2 \cdot (0.25) / f(\xi)^2 .$$

But also now $\widehat{PS}_{LE}$ and $\widehat{IS}_{LE}$ are asymptotically joint normal with

$$\text{Asy. cov}\left(\widehat{PS}_{LE}, \widehat{IS}_{LE}\right) = [(0.5) \cdot f(0.5\xi)] [(0.5) \cdot (0.5\xi) \cdot f(0.5\xi) / \propto] \cdot$$

$$(0.25) / f(\xi)^2 . \tag{24}$$

Thus the (asymptotic) variance now of $\hat{q}_{LE} \equiv (\widehat{PS}_{LE})^{-1} \cdot \widehat{IS}_{LE}$ is gotten from the same formula as in equations (19) and (20). So

$$(\text{Asy.})\ S.E.(\hat{q}_{LE}) \equiv S.E.(\widehat{RMY}_{LE})$$

$$= \left[\frac{\text{Asy.}\widehat{\ }\text{Var}\ (\hat{q}_{LE})}{N}\right]^{1/2} . \tag{25}$$

In the case of the higher-earnings group,

$$RMY_{HE} = \frac{E(x / HE)}{E(x)} = \left[\int_{2.0\xi}^{\infty} \frac{1}{\propto} x\, f(x)dx \,/\, \int_{2.0\xi}^{\infty} f(x)dx\right] \tag{26}$$

$$= IS_{HE} / PS_{HE} ,$$

and RMY_{HE} is estimated by

$$\widehat{RMY}_{HE} = (\widehat{PS}_{HE})^{-1} \cdot \widehat{IS}_{HE} . \tag{27}$$

As before,

$$\text{Asy. var}\left(\widehat{PS}_{HE}\right) = [2 \cdot f(2\xi)]^2 \cdot (0.25) / f(\xi)^2 \text{ , and}$$

$$\text{Asy. var}\left(\widehat{IS}_{HE}\right) = [2 \cdot (2\xi) \cdot f(2\xi) / \propto]^2 \cdot (0.25) / f(\xi)^2 .$$

But also now $\widehat{PS}_{HE}$ and $\widehat{IS}_{HE}$ are asymptotically joint normal as well with

$$\text{Asy. cov}\left(\widehat{PS}_{HE}, \widehat{IS}_{HE}\right) = [2 \cdot f(2\xi)] \cdot [2 \cdot (2\xi) \cdot f(2\xi) / \propto] \cdot (0.25) / f(\xi) \quad . \tag{27}$$

Thus the (asymptotic) variance of $\hat{q}_{HE} \equiv (\widehat{PS}_{HE})^{-1} \cdot \widehat{IS}_{HE}$ is gotten as well from the same formula as in equations (19) and (20). So

$$(\text{Asy.})\ S.E.(\hat{q}_{HE}) \equiv S.E.(\widehat{RMY}_{HE}) = \left[\frac{\text{Asy.} \widehat{\ } \text{Var}\ (\hat{q}_{HE})}{N}\right]^{1/2} . \tag{28}$$

A STATA file of coding to perform all the calculations in this section is available upon request from the author.

Formulas for asymptotic variances and standard errors for percentile-based statistics are presented in Appendix B of this paper.

3. Basic Data Used for Shares of Workers and Earnings Shares

The data used for the study come from Canadian Census Public Use Microdata Files for Individuals for 1971, 1981, 1991, 2001, and 2006, and from monthly Labor Force Survey (LFS) microdata files (for May) for each year over 1997-2015. In the Census files, earnings refers to total annual wage and salary income plus net self-employment income in the previous year. In the LFS files, earnings refers to usual weekly wage and salary income of paid employees who are not currently full-time students. The latter thus excludes net self-employment income and the former aggregates earnings over a full year.

The paper considers two types of workers: all workers (henceforth AW) and full-time workers aged 25-59 (henceforth FT). The empirical analysis of this section then examines the relative-mean earnings of these two types of workers, separately for males and females, and how this ratio has changed over the period 1970-2015. Illustrative cut-offs for the three earnings groups for the FT sample (in earnings per week) in May 2015 are:

	LE	MC	HE	Median
Males	$553	$553-$1658	$2211	$1105.4
Females	$441	$441-$1323	$1764	$881.8

Full sets of summary statistics for the analysis samples of this study for the two data sources appear in the on-line appendices of Beach (2016) and are reproduced as appendix Tables A1-A4.

Graphs of the semi-annual LFS calculated shares of workers and earnings shares over 1997-2015 appear in Figures 1-12. The first three time-series figures are for shares of workers for all workers and the next three graphs refer to the shares of workers for full-time workers. The first of each triple of graphs illustrates the proportions of workers in the lower-earnings (LE) group, the second shows the shares for the middle-class (MC) earnings group, and the third presents the proportions of workers in the higher-earnings (HE) group. Figures 7-12 provide similar time-series graphs for the earnings shares of workers, with again the first three graphs referring to all workers and the second three graphs referring to the sample of full-time workers. Each of these twelve graphs contains two lines – for male and for female workers separately. By simple inspection, one can see, for example, that for the LE group, the shares of workers are much greater than their corresponding shares of earnings; for the MC group, their shares of earnings are slightly less than their corresponding proportion of workers; and for the HE group, their earnings shares far exceed their corresponding shares of workers in the workforce. Since the full-time sample constitutes a more homogeneous group of workers than for all workers as a whole, the middle-class shares are larger in the former sample while LE and HE shares appear larger in the latter sample. Similarly among all workers, there is relatively greater heterogeneity among female than male workforces, so again MC shares appear larger for male than for female workers.

Two general findings can be observed from these figures. First, the patterns of change over time in these series are most marked and clear cut among male and among full-time female workers – the relatively higher labour-cost groups in the Canadian labour market – and they show similar patterns. Second, while the patterns for the LE group's shares are relatively ragged, those for the MC and HE groups of male and full-time female workers are very clear cut – the middle-class shares of workers and earnings have markedly declined over this period, while the corresponding higher-earnings shares of both workers and earnings have moved up substantially. This set of patterns is indeed highlighted in Figures 13-16 which show declining middle-class earnings shares measured on the left-hand vertical axis and the rising HE earnings shares measured on the right-hand axis of each diagram. These figures also show that the above patterns of change were stronger until about 2007-08, and since then have apparently considerably attenuated.

4. Relative-Mean Earnings Results

Tables 1 and 2 report relative-mean earnings ratios for the various groups of workers. The first table refers to Census data over the period 1970-2005. The second table refers to LFS data for five-year intervals over 2000-2015. The layout of both tables is the same. The four columns refer to the different analysis samples – males, females, AW and FT. The three row panels refer to results for lower-earning workers (LE), middle-class workers (MC), and higher-earnings workers (HE). The reported figures show the relative-mean earnings ratios expressed as proportions for each earnings group and analysis sample for given years. Since earnings distributions are skewed to the right, the median is always less than the sample mean, so it shouldn't be surprising that the average earnings of middle-class workers are below the overall mean earnings levels, resulting in the middle-class RME ratio being always less than one. At the bottom of each panel is the change in RME ratio for that earnings group over the time period covered by that table. Figures in parentheses are conventional "t-ratios" for changes in the RME ratios over the covered periods. These allow us to determine if corresponding reported changes in RME ratios are actually statistically significant. A full set of estimated standard errors on the relative-mean earnings ratios discussed in this study (based on the formulas of the previous section) is presented in appendix Tables A5-A6 at the end of the current paper.

Several results are evident from Table 1. First, the relative-mean earnings of lower-earnings and middle-class workers declined significantly over the full 1970-2005 period, and especially so for males over 1980-2005, while the RME for higher-earnings workers rose significantly, again especially so over the 1980-2005 period. For example, over the latter period, the RME for lower-earnings male workers fell by 17.1% for all workers (AW) and 17.4% for full-time (FT) workers, that for middle-class workers fell by 22.0% and 16.2% respectively, while the RME of higher-earnings male workers went up by 5.8% and 23.7%, respectively. These changes are all highly statistically significant (because of their large magnitude and the large sample sizes of microdata on which the estimates are based).

Second, the declines in RME for LE and MC workers were much larger among males than females, while the rise in RME for HE workers was much more marked for full-time workers, and especially for male full-time workers. For example, among full-time workers over the 1980-2005 period, lower-earnings workers' RME went down by 17.4% for males and by 4.6% for females, middle-class workers' RME declined by 16.2% for males and by 10.3% for females, while higher-earnings workers' RME went up by 23.7% for males vs 13.8% for females. Also, the declines among lower-earnings and middle-class workers occurred across both full-time and all worker samples, while increases among higher-earnings workers were much more marked for just full-time workers.

Table 2 is set up in similar fashion to Table 1, but here the results cover the period 2000-2015, the data come from the Labour Force Survey, and the earnings definition refers to usual weekly wage and salary income. Again, though, the reported figures are relative-mean earnings ratios.

It can be seen from the results in Table 2 that the previous pattern of change in relative-mean earnings ratios – declines in lower-earnings and middle-class earnings ratios combined with rises in higher-earnings ratios – has largely continued since 2000 and was still frequently statistically significant, especially so for the declines in the earnings ratio for middle-class workers (both males and females and both FT and AW samples). But this general pattern of change has apparently been reduced or attenuated. For example, over the full 2000-2015 period, the declines for the lower-earnings workers were less than 2% and half the time not statistically significant, the increases for higher-earnings workers were between 0.4% and 3.0% and again often not significant, and among middle-class workers, the declines lie between 1.9% and 6.0% but continue to be quite significant.

The changes in Tables 1 and 2 are not directly comparable, however, in that they generally refer to different-length time intervals. One can make them more comparable by expressing all changes in terms of average change per year over the various intervals covered. These results appear in Table 3. It can be seen from these results that major distributional changes of the pattern already highlighted didn't start occurring until the 1980s. The pattern of the 1970s was much more one of increased equalization continuing on from the 1960s. Indeed, when one breaks down patterns of distributional change into various sub-periods, the most consistent pattern of change is the quite marked decline in relative-mean earnings of middle-class workers – both for men and women and for all workers and full-time workers samples. The 1990s were also the decade of most rapid middle-class RME decline, though for some groups (such as full-time workers) the pace of middle-class RME decline is continuing apace.

For the period since 1997, the above patterns are highlighted in more detail with the semi-annual LFS data in Figures 17-22. Again, the first three diagrams refer to all workers as a whole, while Figures 20-22 refer to full-time workers in the labour market.

As can be seen, the relative-mean earnings of the lower-earnings (LE) workers saw a slight decline between the two end years, but closer inspection shows that the decline pretty well ceased by 2005-06 and has since stabilized. The relative-mean earnings of middle-class (MC) workers, in contrast, has continued to decline since 2006 or so, though at a generally slower rate. The relative-mean earnings of the higher-earnings (HE) group, interestingly, has not substantially risen over this period. Since these ordinates are all relative to the mean, the only way this can occur is that the relative-mean earnings of the one group of workers not covered by the three LE, MC and HE groups – is, of workers with earnings between 1.5 median and 2.0 median levels – rose significantly. That is, the big winners in terms rising relative-mean earnings over this recent period has been not the very top earners, but those near the lower end of the set of higher earners more broadly defined.

5. Applying the Lognormal Inference Approach to Conditional Mean Incomes

While relative-mean income ratios are a useful supplementary set of statistics for analyzing patterns of distributional change, intuitively readers may relate more readily to actual income *levels* across a distribution and how these levels may have changed for different income groups. Workers' attitudes and sense of insecurity and economic uncertainty may depend on what has happened to their actual wages and earnings levels and not so much on relative wages/earnings compared to some economy-wide mean. Furthermore, the social welfare or economic well-being of individuals may be viewed as a function of actual income levels rather than relative-income figures. It is quite possible, for example, that actual income levels have risen for some groups while their relative-mean income figures have indeed declined over some period. Both income levels and relative-mean income ratios would seem to be useful supplementary distributional statistics that should be viewed together for a more informative picture of what is going on.

The problem we want to look at then is to work out the formulas that will allow statistical inference for estimates of conditional mean income levels

$$E(x \,/\, x \in i) = \int_{LB}^{UB} x\, f(x)dx \,/\, \int_{LB}^{UB} f(x)dx \qquad (29)$$

for income group i = LE, MC, HE, whose lower and upper bounds are given by LB and UB. In the case of middle-class income recipients, the bounds are given by LB = 0.5ξ and UB = 1.5ξ for population median income level ξ. The sample estimate of conditional mean income, $\widehat{CMY}$, is obtained, as before, by ordering the observations by their income, calculating the sample median $\hat{\xi}$, then computing the total income received by those individuals whose incomes lie within the designated income range – $0.5\hat{\xi}$ to $1.5\hat{\xi}$ in the case of the MC group – divided by the total number of individual falling within this group.

The derivation of the (asymptotic) standard error of $\widehat{CMY}$, once again assuming the underlying income density function, $f(\cdot)$, is lognormal, proceeds along similar lines as followed earlier in Section 2. Again we outline the development in terms of the middle-class income group, so

$$\widehat{CMY}_{MC} = \hat{E}(x \,/\, x \in MC) = \int_{0.5\hat{\xi}}^{1.5\hat{\xi}} x\, f(x) dx \,/\, \int_{0.5\hat{\xi}}^{1.5\hat{\xi}} f(x) dx \,. \qquad (30)$$

To derive the (asymptotic) standard error of $\widehat{CMY}_{MC}$, one needs to work out the variance of the limiting distribution of $\widehat{CMY}_{MC}$. This is done by considering the joint distribution of the numerator and denominator in (30) and then looking at their ratio.

So define the numerator of (30) as

$$\hat{N}_{MC} \equiv N_{MC}(\hat{\xi}) \equiv \int_{0.5\hat{\xi}}^{1.5\hat{\xi}} x\, f(x) dx$$

and the denominator as

$$\hat{D}_{MC} \equiv D_{MC}(\hat{\xi}) \equiv \int_{0.5\hat{\xi}}^{1.5\hat{\xi}} f(x) dx \,.$$

Note, incidentally, that $\hat{D}_{MC} \equiv \widehat{PS}_{MC}$ is the population share of the middle-class income group that we have already worked with in Section 2. So under the assumption that $f(\cdot)$ is lognormal, we can view $\hat{D}_{MC} \equiv \hat{N}_{MC}$ as (continuous differentiable) functions of $\hat{\xi}$, the sample estimate of the median income level. In Beach (2016) it was established that $\hat{D}_{MC} \equiv \widehat{PS}_{MC}$ was asymptotically normally distributed with mean D_{MC} and asymptotic variance:

$$\text{Asy. var}\left(\hat{D}_{MC}\right) = [f(1.5\xi)(1.5) - f(0.5\xi)(0.5)]^2 \cdot (0.25) \,/\, f(\xi)^2 \,.$$

But, if we go back to the multivariate Rao Linkage Theorem (Rao, 1965, p. 388), if we can also establish the asymptotic joint distribution of $\hat{N}_{MC}$ and $\hat{D}_{MC}$. If

$$\widehat{N}_{MC} \equiv h_1(\hat{\xi}) = \int_{0.5\hat{\xi}}^{1.5\hat{\xi}} x\, f(x) dx \text{ and}$$

$$\widehat{D}_{MC} \equiv h_2(\hat{\xi}) = \int_{0.5\hat{\xi}}^{1.5\hat{\xi}} f(x) dx ,$$

and $\mathrm{h}(\hat{\xi}) = [h_1(\hat{\xi}), h_2(\hat{\xi})]^1$ is a 2x1 vector of continuous differentiable functions of $\hat{\xi}$, then the above theorem establishes that the (joint) asymptotic distribution of $\mathrm{h}(\hat{\xi})$ is bivariate normal with an (asymptotic) variance-covariance matrix given by

$$\text{Asy. Var}(\widehat{N}_{MC}, \widehat{D}_{MC}) = H \, \Lambda \, H^1 \equiv W \tag{31a}$$

$$= \begin{bmatrix} \partial h_1 \,/\, \partial \xi \\ \partial h_2 \,/\, \partial \xi \end{bmatrix} \cdot v \cdot \left[\frac{\partial h_1}{\partial \xi}, \frac{\partial h_2}{\partial \xi}\right]$$

$$= \begin{bmatrix} (h_1^1)^2 \, v & (h_1^1 h_2) \, v \\ (h_1^1 h_2^1) \, v & (h_2^1)^2 \, v \end{bmatrix} \tag{31b}$$

where

$$h_1^1 \equiv \frac{\partial h_1}{\partial \xi} = \frac{\partial N_{MC}}{\partial \xi} = [(1.5\xi) \cdot f(1.5\xi)](1.5) - [(0.5\xi) \cdot f(0.5\xi)](0.5) \tag{32}$$

$$h_2^1 \equiv \frac{\partial h_2}{\partial \xi} = \frac{\partial D_{MC}}{\partial \xi} = f(1.5\xi)(1.5) - f(0.5\xi)(0.5) \tag{33}$$

and the scalar $v \equiv asy.var(\hat{\xi}) = (0.5)(0.5) \,/\, [f(\xi)]^2$. Again, equations (32) and (33) are gotten by Leibnitz's Rule. Note also, once again, that this result establishes the (asymptotic) covariance as well between $\widehat{N}_{MC}$ and $\widehat{D}_{MC}$.

But what we want is the limiting distribution of $\widehat{CMY}_{MC}$ which is the *ratio* of these two terms. Reasoning as in Section 2, let $q \equiv E(x \,/\, x \in MC)$ and

$$\hat{q} \equiv q(\widehat{N}_{MC}, \widehat{D}_{MC}) \equiv (\widehat{D}_{MC})^{-1} \cdot \widehat{N}_{MC} , \tag{34}$$

so that one can again use Rao's Linkage Theorem to establish now that $\sqrt{NOB}(\hat{q} - q)$ also has a limiting normal distribution with mean zero and variance

$$\text{Asy. Var}(\hat{q}) = Q \; W \; Q^1 \tag{35}$$

where now the (1x2) gradient vector $Q = [q_1, q_2]$ where

$$q_1 = \frac{\partial q(N_{MC}, D_{MC})}{\partial N_{MC}} = \frac{1}{D_{MC}}$$

$$q_2 = \frac{\partial q(N_{MC}, D_{MC})}{\partial D_{MC}} = \frac{-N_{MC}}{D_{MC}^2}, \text{ and}$$

$$W \equiv \begin{bmatrix} \text{Asy. var.}(\widehat{N}_{MC}) & \text{Asy. cov.}(\widehat{N}_{MC}, \widehat{D}_{MC}) \\ \text{Asy. cov.}(\widehat{N}_{MC}, \widehat{D}_{MC}) & \text{Asy. var.}(\widehat{D}_{MC}) \end{bmatrix}$$

$$= \begin{bmatrix} (h_1^1)^2\, v & (h_1^1 h_2)\, v \\ (h_1^1 h_2^1)\, v & (h_2^1)^2\, v \end{bmatrix} \equiv \begin{bmatrix} w_{11} & w_{21} \\ w_{12} & w_{22} \end{bmatrix}$$

$w_{11} = (h_1^1)^2 v$ with h_1^1 from equation (32)

$w_{22} = (h_2^1)^2 v$ with h_2^1 from equation (33)

$w_{12} = (h_1^1 \cdot h_2^1)\, v$ from both equations (32) and (33).

Therefore,

$$\text{Asy. Var}(\hat{q}) = (q_1)^2\, w_{11} + 2(q_1 q_2) w_{12} + (q_2)^2\, w_{22} \tag{35}$$

Consequently, the (asymptotic) standard error of $\hat{q}$ is given by

$$(\text{Asy.})\ S.E.(\hat{q}) \equiv S.E.(\hat{E}(x \,/\, x \in MC))$$

$$= \left[\frac{\widehat{\text{Asy. Var}}\,(\hat{q})}{NOB}\right]^{1/2} \tag{36}$$

where NOB is again the sample size and the "hat" indicates that all of the unknowns have been estimated as done before.

The corresponding formulas for the conditional-mean income levels for the lower-earnings (LE) and higher-earnings (HE) groups are derived in the same way, and again turn out to be simpler. In the case of the lower-earnings group,

$$CMY_{LE} = E(x \,/\, x \in LE) = \int_0^{0.5\xi} x\, f(x)dx \,/\, \int_0^{0.5\xi} f(x)dx\,,$$

so that

$$h_1(\xi) \equiv N_{LE} = \int_0^{0.5\xi} x\, f(x) dx \;,$$

and $$h_2(\xi) \equiv D_{LE} = \int_0^{0.5\xi} f(x) dx \,.$$

Then the vector $h(\hat{\xi}) = [h_1(\hat{\xi}), h_2(\hat{\xi})]^1$ is

(asymptotically) joint normally distributed with (asymptotic) variance-covariance matrix

$$\text{Asy. Var}(\widehat{N}_{LE}, \widehat{D}_{LE}) \equiv W = H \Lambda H^1 \tag{37}$$

$$= \begin{bmatrix} (h_1^1)^2\, v & (h_1^1 h_2^1)\, v \\ (h_1^1 h_2^1)\, v & (h_2^1)^2\, v \end{bmatrix} \qquad \text{where}$$

$$H \equiv [h_1^1, h_2^1]^1$$

$$h_1^1 \equiv \frac{\partial h_1}{\partial \xi} = \frac{\partial N_{LE}}{\partial \xi} \tag{38}$$

$$= (0.5\xi) \cdot f(0.5\xi) \cdot (0.5)$$

$$h_2^1 \equiv \frac{\partial h_2}{\partial \xi} = \frac{\partial D_{LE}}{\partial \xi} \tag{39}$$

$$= f(0.5\xi) \cdot (0.5),$$

and again $v \equiv asy.var(\hat{\xi}) = (0.5)(0.5) \,/\, f(\xi)^2$.

Now again using the Rao Linkage Theorem, if

$$\hat{q}_{LE} = \widehat{N}_{LE} \,/\, \widehat{D}_{LE} \,,$$

then $$\text{Asy. Var}(\hat{q}_{LE}) = Q\, W\, Q^1 \tag{40}$$

where $Q = [q_1, q_2]$

and $$q_1 \equiv \frac{\partial q_{LE}}{\partial N_{LE}} = \frac{1}{D_{LE}} \tag{41}$$

$$q_2 \equiv \frac{\partial q_{LE}}{\partial D_{LE}} = \frac{-N_{LE}}{D_{LE}^2} \,. \tag{42}$$

Therefore, $\text{Asy. Var}\left(\hat{E}(x \,/\, x \in LE)\right) \equiv \text{asy. var}(\hat{q}_{LE})$

$$= [q_1, q_2] \cdot W \cdot \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}$$

$$= (q_1)^2 w_{11} + 2(q_1 q_2) w_{12} + (q_2)^2 w_{22}$$

$$= (q_1)^2 (h_1^1)^2 v + 2(q_1 q_2)(h_1^1 h_2^2) v + (q_2)^2 (h_2^1)^2 v \qquad (43)$$

where the terms in equation (43) come from expressions (38), (39), (41), (42), and

$v \equiv \text{asy.var}(\hat{\xi}) = (0.5)(0.5) / f(\xi)^2$. Hence,

$$(\text{Asy.})\ S.E.(\hat{q}_{LE}) \equiv S.E.(\widehat{CMY}_{LE})$$

$$= \left[\frac{\text{Asy.}\widehat{\ }\text{Var}(\hat{q}_{LE})}{NOB}\right]^{1/2}. \qquad (44)$$

Similarly, in the case of the higher-earnings group,

$$CMY_{HE} = E(x / x \in HE) = \int_{2.0\xi}^{\infty} x f(x) dx / \int_{2.0\xi}^{\infty} f(x) dx$$

so that

$$h_1(\xi) \equiv N_{HE} = \int_{2.0\xi}^{\infty} x f(x) dx ,$$

and $$h_2(\xi) \equiv D_{HE} = \int_{2.0\xi}^{\infty} f(x) dx .$$

Therefore, $\text{Asy.Var}(\hat{N}_{HE}, \hat{D}_{HE}) \equiv W$

$$= \begin{bmatrix} (h_1^1)^2 v & (h_1^1 h_2^1) v \\ (h_1^1 h_2^1) v & (h_2^1)^2 v \end{bmatrix} \quad \text{where}$$

$$h_1^1 \equiv \frac{\partial h_1}{\partial \xi} = \frac{\partial N_{HE}}{\partial \xi} \qquad (45)$$

$$= (2\xi) \cdot f(2\xi) \cdot (-2)$$

$$h_2^1 \equiv \frac{\partial h_2}{\partial \xi} = \frac{\partial D_{HE}}{\partial \xi} \qquad (46)$$

$$= f(2\xi)(-2).$$

Consequently,

$$\text{Asy.Var}\left(\hat{E}(x / x \in HE)\right) \equiv \text{asy.var}(\hat{q}_{HE})$$

$$= (q_1)^2 (h_1^1)^2 v + 2(q_1 q_2)(h_1^1 h_2^1) v + (q_2)^2 (h_2^1)^2 v \qquad (47)$$

where $q_1 = \frac{1}{D_{HE}}$ (48)

and $q_2 = \frac{-N_{HE}}{D_{HE}^2}$. (49)

Hence, (Asy.) $S.E.(\hat{q}_{HE}) \equiv S.E.(\widehat{CMY}_{HE})$

$$= \left[\frac{\widehat{\text{Asy.Var}}\ (\hat{q}_{HE})}{NOB}\right]^{1/2}. \quad (50)$$

The standard errors formulas for CMY_i and RMY_i ($i = LE, MC, HE$), however, are closely linked. A more intuitive approach may be to reason as follows. In the case of the MC income group,

$$CMY_{MC} \equiv E(x \mid x \in MC)$$

$$= \int_{0.5\xi}^{1.5\xi} x\, f(x)dx \Big/ \int_{0.5\xi}^{1.5\xi} f(x)dx$$

$$= \propto \cdot \int_{0.5\xi}^{1.5\xi} \frac{1}{\propto}\, x\, f(x)dx \Big/ \int_{0.5\xi}^{1.5\xi} f(x)dx$$

$$= \propto \cdot IS_{MC} / PS_{MC} \quad (51)$$

where $\propto \equiv E(x)$. But we have already established the asymptotic distribution of $(\widehat{PS}_{MC})^{-1} \cdot \widehat{IS}_{MC}$ in section 2.3 above. So the limiting distribution of $\propto \cdot (\widehat{PS}_{MC})^{-1} \cdot \widehat{IS}_{MC}$ is also normal with an asymptotic variance that is simply $\propto^2$ times the asymptotic variance of $\hat{q}$ given by eq. (20). And the asymptotic standard error of $\widehat{CMY}_{MC}$ is then given by $\hat{\propto} \cdot S.E.(\widehat{RMY}_{MC})$. This can be verified by comparing the derivatives for g_1^1 and g_2^1 term-by-term in eqs. (17)-(18) versus those for h_1^1 and h_2^1 in eqs. (32)-(33). Corresponding results hold for CMY_{LE} and CMY_{HE}.

Again, a STATA file of coding to perform all these calculations is available upon request from the author.

And again, formulas for percentile-based statistics are provided in Appendix B.

6. Conditional-Mean Earnings Results for Canada 1970-2015

A full set of estimated standard errors on the conditional-mean earnings levels (based on the above formulas) is presented in appendix Tables A7-A8 at the end of the paper.

Table 4 presents conditional-mean earnings levels (all in real 2015 dollars) for the various groups of workers based on Census data over the 1970-2005 period. The layout of the table is the same as for the earlier tables in Section 4.

The first result that comes out of the figures in Table 4 – in contrast with those in Table 1 above – is that, while relative-mean earnings of lower-earnings and middle-class groups declined over the full period 1970-2005, actual conditional-mean earnings *levels* for these two earnings groups *increased*. However, when viewed over the shorter period of 1980-2005, actual *decreases* in earnings levels occurred among male lower-earnings and middle-class workers that are both substantial and highly statistically significant. For example, over the latter period, male LE workers' earnings fell by 4-6% and male MC workers' earnings went down by 4-10% (with the larger declines occurring among non-FTFY workers). The mean earnings levels of higher-earnings workers, by contrast, increases substantially (by 22-54% for male earners) over both time periods.

Second, very marked differences in the pattern of earnings changes occurred between female and male workers, with the earnings levels of female workers increasing much faster then for male workers in the labour market. Over the 1980-2005 period, the earnings of female lower-earners (instead of falling as for male LE workers) rose by 18-33%, and the earnings of the female middle-class workers also went up by 11-25% as well. Among FTFY higher-earners, however, male and female average earnings went up by essentially the same amount (41%). All these results are highly statistically significant.

From Table 5 based on LFS data, it can be seen that, since 2000 or 2005 earnings, increases have been much more broadly shared across all three earnings groups (LE, MC, and HE) than in the previous two decades, with female earnings increases continuing to exceed those of males. For example, over 2000-2015 among full-time male workers, LE earnings went up by 8.3%, MC earnings rose by 6.1% and HE earnings increased by 10.3%. But earnings increases for middle-class workers over this period are still on average less than those for LE and HE workers across all four samples. It still remains the case, though, that even by 2015, the average female earnings levels fall considerably short of those for males. For example, among full-time LE workers, females earn on average 84.8% that of males, among FT middle-class workers 78.8%, and among FT higher-earnings 77.8% (the ratios are lower among the all-worker samples which lack controls for hours worked).

Again, Table 6 expresses percentage changes in comparable per-year terms. The figures are larger than in Table 3 because the percent figures are for earnings levels rather than relative-earnings ratios. One can see that the negative changes in males' earnings were concentrated in the 1980-2005 period. The 1970s experienced large increases in earnings for all groups. Since 2005, all groups have again experienced (real) earnings increases, though generally at a much lower rate than in the 1970s. The exception is high-earnings males whose earnings have, on average, increased at a faster rate than in the 1970s.

Figures 23-28 further illustrate the above patterns with LFS data since 1997. The sequencing and lay-out of the diagrams follow that of the relative-mean earnings graphs in Figures 17-22. Once again, they show the generally broad earnings gains across the distribution since around 2005-06, the more moderate gains for middle-class male earners, and the generally larger earnings increases for higher-earning workers in the Canadian labour market.

7. Conclusions

This paper forwards a complementary set of distributional statistics – relative-mean income ratios and conditional-mean income levels – that can usefully be employed to supplement standard empirical distributional analyses of income shares and the proportion of individuals within various income groups. It also develops a statistical methodology—called the lognormal inference approach – for allowing for the standard principles of statistical inference to be applied to this new set of distributional statistics. This is based upon an assumption of a lognormal income or earnings distribution, and thus extends the approach made use of in Beach (2016).

This new set of distributional tools or statistics can be used to identify if some income groups have experienced real income losses (or market gains) and if these losses (or gains) are statistically significant. They can be used to identify which income/earnings groups have lost out (or markedly gained) relative to other income/earnings groups across the distribution and the actual size of their gains or losses. And they serve to more directly link empirical distributional analyses to theoretical results on general social-welfare inferences that can reasonably be drawn from observed distributional changes. They also provide more sensitive indicators of income changes and turning points than typically done simply by income shares or workers' shares.

As to the question in the title of this paper, since 2005 (real) earnings increases have indeed occurred and been fairly broadly shared across all earnings groups in the analysis, though at a much reduced rate than earnings increases in the 1970s, the previous period of broad earnings gains. Over the 1980-2005 period, however, actual decreases in real earnings levels were experienced by male lower-earnings and middle-class workers that were both substantial and highly statistically significant.

The second major empirical finding is that *relative*-mean earnings ratios over the 1980-2005 period declined for lower-earnings and middle-class workers while increasing for higher-earners (both male and female). Since 2005, however, changes in relative-mean earnings for lower-earnings and higher-earnings workers have been attenuated, mixed in sign, and often not significant. The most consistent pattern of change since 1980 has been the quite marked decline in relative-mean earnings for middle-class workers, which has continued since 2005 and remains statistically significant.

Third, very marked differences occurred between male and female earnings changes throughout the period covered, with female relative earnings increases substantially exceeding those of male workers. Over 1980-2005, while the earnings levels of lower-earnings and middle-class male workers declined, corresponding female earnings levels increased. Since 2005, this pattern has largely continued, though the earnings levels of male full-time higher-earnings workers have been moving up as fast as or faster than that for females.

Table 1
Relative Mean Earnings of Male and Female Workers by Earnings Level, Canada, 1970-2005: Census Data on Annual Earnings

	Males		Females	
	All Workers	**FTFY Workers**	**All Workers**	**FTFY Workers**
Lower Earnings **<u>(below 50% of median)</u>**				
1970	.1980	.2757	.2007	.2875
1980	.2050	.2898	.1925	.2820
1990	.1996	.2669	.1924	.2863
2000	.1850	.2626	.1773	.2711
2005	.1699	.2394	.1843	.2690
Change 1970-2005	-.0281 [-14.2%]	-.0363 [-13.2%]	-.0164 [-8.2%]	-.0185 [-6.4%]
	(12.52)	(10.43)	(6.91)	(2.63)
Change 1980-2005	-.0351 [-17.1%]	-.0504 [-17.4%]	-.0082 [-4.3%]	-.0130 [-4.6%]
	(22.82)	(21.14)	(5.40)	(3.38)
Middle-Class Earnings **<u>(with 50% of median)</u>**				
1970	.9043	.8635	.9049	.9060
1980	.9275	.8999	.8589	.9100
1990	.9021	.8847	.8183	.9037
2000	.8339	.8482	.8047	.8768
2005	.7232	.7539	.7718	.8167
Change 1970-2005	-.1811 [-20.0%]	-.1096 [-12.7%]	-.1331 [-14.7%]	-.0893 [-9.9%]
	(51.68)	(38.43)	(21.60)	(18.95)
Change 1980-2005	-.2043 [-22.0%]	-.1460 [-16.2%]	-.0871 [-10.1%]	-0.0933 [-10.3%]
	(74.43)	(66.92)	(22.87)	(31.44)
Higher Earnings **<u>(above 200% of median)</u>**				
1970	2.8000	2.7504	2.4708	2.4591
1980	2.6236	2.5522	2.4211	2.3022
1990	2.6014	2.6130	2.3764	2.3922
2000	2.5252	2.5321	2.3783	2.3167
2005	2.7760	3.1558	2.5510	2.6201
Change 1970-2005	-.0240 [-0.9%]	.4054 [+14.7%]	.0802 [+3.2%]	.1611 [+6.6%]
	(1.47)	(15.48)	(7.79)	(4.92)
Change 1980-2005	+.1524 [+5.8%]	+.6036 [+23.7%]	+.1299 [5.4%]	+.3179 [+13.8%]
	(15.33)	(30.00)	(19.23)	(18.93)

Source: Statistics Canada, Census of Canada Individual PUMF files for 1971, 1981, 1991, 2001, and 2006.

FTFY = full-time, full-year

Figures in parentheses are absolute (asymptotic) “t-ratios” based on the standard errors in appendix Table A5.

Figures in square brackets are percentage changes in RME figures over indicated years.

Table 2
Relative Mean Earnings of Male and Female Workers by Earnings Level, Canada, 2000-2015: LFS Data on Weekly Earnings

	Males		Females	
	All Workers	**FT Workers**	**All Workers**	**FT Workers**
Lower Earnings (below 50% of median)				
2000	.2796	.3779	.2687	.3884
2005	.2758	.3744	.2707	.3797
2010	.2728	.3759	.2644	.3798
2015	.2831	.3726	.2638	.3847
Change 2000-2015	+.0035 [+1.8%]	-.0053 [-1.4%]	-.0049 [-1.8%]	-.0037 [-1.0%]
	(1.08)	(3.17)	(1.68)	(2.61)
Change 2000-2010	-.0068 [-2.4%]	-.0020 [-0.5%]	-.0043 [-1.6%]	-.0086 [-2.2%]
	(1.87)	(1.15)	(1.31)	(5.45)
Middle-Class Earnings (within 50% of median)				
2000	.9244	.8951	.8708	.8732
2005	.8948	.8889	.8726	.8646
2010	.8776	.8701	.8597	.8487
2015	.8691	.8647	.8542	.8294
Change 2000-2015	-.0553 [-6.0%]	-.0304 [-3.4%]	-.0166 [-1.9%]	-.0438 [-5.0%]
	(13.12)	(10.24)	(3.50)	(12.81)
Change 2000-2010	-.0468 [-5.1%]	-.0250 [-2.8%]	-.0111 [-1.3%]	-.0245 [-2.8%]
	(9.81)	(7.45)	(2.05)	(6.53)
Higher Earnings (above 200% of median)				
2000	2.3571	2.2672	2.2471	2.1185
2005	2.3214	2.2381	2.3235	2.2173
2010	2.2727	2.2284	2.2821	2.1664
2015	2.2856	2.2773	2.2623	2.1573
Change 2000-2015	-.0716 [-3.0%]	.0101 [+0.4%]	.0152 [+0.7%]	.0388 [+1.8%]
	(5.20)	(0.68)	(1.57)	(3.19)
Change 2000-2010	-.0844 [-3.6%]	-.0388 [-1.7%]	+.0350 [+1.6%]	+.0479 [+2.3%]
	(5.59)	(2.60)	(3.08)	(3.91)

Note: Based on May Labour Force Surveys.

Figures in parentheses are absolute (asymptotic) "t-ratios" based on the standard errors in appendix Table A6.

Figures in square brackets are percentage changes in RME figures over indicated years.

Table 3
Average Annual Change in Relative Mean Earnings of Male and Female Workers by Earnings Level, Canada, 1970-2015

	Males		Females	
	All Workers	**FT Workers**	**All Workers**	**FT Workers**
Lower Earnings **(below 50% of median)**				
1970-80	+.0007*	+.0014*	-.0008*	-.0006
1980-90	-.0005*	-.0023*	-.0000	+.0004
1990-00	-.0015*	-.0004	-.0015*	-.0015*
2000-05 (SCF)	-.0008	-.0007	+.0004	-.0017*
2005-10	-.0006	+.0003	-.0013*	+.0000
2010-15	+.0021*	-.0007*	-.0001	+.0010*
Middle-Class Earnings **(within 50% of median)**				
1970-80	+.0023*	+.0036*	-.0046*	+.0004
1980-90	-.0025*	-.0015*	-.0041*	-.0006*
1990-00	-.0068*	-.0037*	-.0014*	-.0027*
2000-05 (SCF)	-.0059*	-.0012*	+.0004	-.0017*
2005-10	-.0034*	-.0038*	-.0026*	-.0032*
2010-15	-.0017*	-.0011	-.0011	-.0039*
Higher Earnings **(above 200% of median)**				
1970-80	-.0176*	-.0198*	-.0050*	-.0157*
1980-90	-.0022*	+.0061*	-.0045*	+.0090*
1990-00	-.0076*	-.0081*	+.0002	-.0076*
2000-05 (SCF)	-.0071*	-.0058*	+.0153*	+.0198*
2005-10	-.0097*	-.0019	-.0083*	-.0102*
2010-15	+.0026	+.0098*	-.0040	-.0018

Note: See data sources in Tables 1 and 2.

Asterisks indicate statistical significance (2-tailed test) at 95% level of confidence. Actual “t-ratios” for each of these changes are found in appendix Table A9.

Table 4
Conditional Mean Earnings of Male and Female Workers by Earnings Level, Canada, 1970-2005: Census Data on Annual Earnings (real 2015 dollars)

	Males		**Females**	
	All Workers	**FTFY Workers**	**All Workers**	**FTFY Workers**
Lower Earnings				
(below 50% of median)				
1970	8310.	15,316.	4063.	9476.
1980	10,074.	18,909.	4952.	11,863.
1990	9768.	17,245.	5655.	12,618.
2000	9248.	17,163.	5795.	12,820.
2005	9617.	17,832.	6597.	14,007.
Change 1970-2005	1307.[15.7%]	2516. [16.4%]	2534. [63.4%]	4540. [48.0%]
	(13.32)	(12.43)	(47.24)	(19.03)
Change 1980-2005	-457. [-4.5%]	-1077. [-5.7%]	1645. [33.2%]	2144. [18.1%]
	(5.83)	(6.67)	(37.38)	(12.80)
Middle-Class Earnings				
(within 50% of median)				
1970	37,946.	47,967.	18,320.	29,838.
1980	45,586.	58,709	22,094.	38,279.
1990	44,145.	57,162.	24,052.	39,833.
2000	41,675.	55,444.	26,304.	41,456.
2005	40,924.	56,146.	27,623.	42,530.
Change 1970-2005	2978. [7.8%]	8179. [17.1%]	9303. [50.8%]	12,962. [43.4%]
	(18.52)	(46.81)	(67.62)	(74.04)
Change 1980-2005	-4662. [-10.2%]	-2563. [-4.4%]	5529. [25.0%]	4251. [11.1%]
	(32.55)	(16.86)	(50.51)	(31.15)
Higher Earnings				
(above 200% of median)				
1970	117,496.	152,789.	50,021.	80,982.
1980	128,950.	166,498.	62,279.	96,841.
1990	127,297.	168,831.	69,853.	105,440.
2000	126,200.	165,511.	77,744.	109,533.
2005	157,087.	235,026.	91,300.	136,447.
Change 1970-2005	39,591. [33.7%]	82,237. [53.8%]	41,279. [82.5%]	55,465. [68.5%]
	(54.42)	(49.75)	(164.5)	(47.68)
Change 1980-2005	28,137. [21.8%]	68,528. [41.2%]	29,021. [46.6%]	39,606. [40.9%]
	(54.20)	(47.90)	(138.2)	(50.73)

Source: Statistics Canada, Census of Canada Individual PUMF files for 1971, 1981, 1991, 2001, and 2006.

FTFY = full-time, full-year

Figures in parentheses are absolute (asymptotic) “t-ratios” based on the standard errors in appendix Table A7.

Figures in square brackets are percentage changes in RME figures over indicated years.

Table 5
Conditional Mean Earnings of Male and Female Workers by Earnings Level, Canada, 2000-2015: LFS Data on Weekly Earnings (real 2015 dollars)

	Males		Females	
	All Workers	**FT Workers**	**All Workers**	**FT Workers**
Lower Earnings				
(below 50% of median)				
2000	270.9	418.1	182.3	328.8
2005	266.3	415.8	191.4	334.3
2010	281.3	445.5	204.2	364.6
2015	298.3	452.9	211.4	384.0
Change 2000-2015	27.4 [10.1%]	34.8 [8.3%]	29.1 [16.0%]	55.2 [16.8%]
	(8.72)	(18.83)	(14.20)	(45.44)
Change 2000-2010	10.4 [3.8%]	27.4 [6.6%]	21.9 [12.0%]	35.8 [10.9%]
	(2.88)	(14.00)	(9.40)	(25.92)
Middle-Class Earnings				
(within 50% of median)				
2000	895.3	990.4	590.7	739.2
2005	864.0	987.1	617.2	761.4
2010	905.0	1031.1	664.0	814.7
2015	915.6	1051.1	684.5	827.8
Change 2000-2015	20.3 [2.3%]	60.7 [6.1%]	93.8 [15.9%]	88.6 [12.0%]
	(4.82)	(17.59)	(27.48)	(28.16)
Change 2000-2010	9.7 [1.1%]	40.7 [4.1%]	73.3 [12.4%]	75.5 [10.2%]
	(2.03)	(10.50)	(18.72)	(22.06)
Higher Earnings				
(above 200% of median)				
2000	2283.1	2508.7	1524.2	1793.3
2005	2241.5	2485.4	1643.4	1952.6
2010	2343.6	2640.9	1762.7	2079.5
2015	2408.1	2768.1	1812.7	2153.0
Change 2000-2015	125.0 [5.5%]	259.4 [10.3%]	288.5 [18.9%]	359.7 [20.1%]
	(9.10)	(14.74)	(40.46)	(30.36)
Change 2000-2010	60.5 [2.6%]	132.2 [5.3%]	238.5 [15.6%]	286.2 [16.0%]
	(4.02)	(7.58)	(28.45)	(24.82)

Note: Based on May Labour Force Surveys.

Figures in parentheses are absolute (asymptotic) “t-ratios” based on the standard errors in appendix Table A8.

Figures in square brackets are percentage changes in RME figures over indicated years.

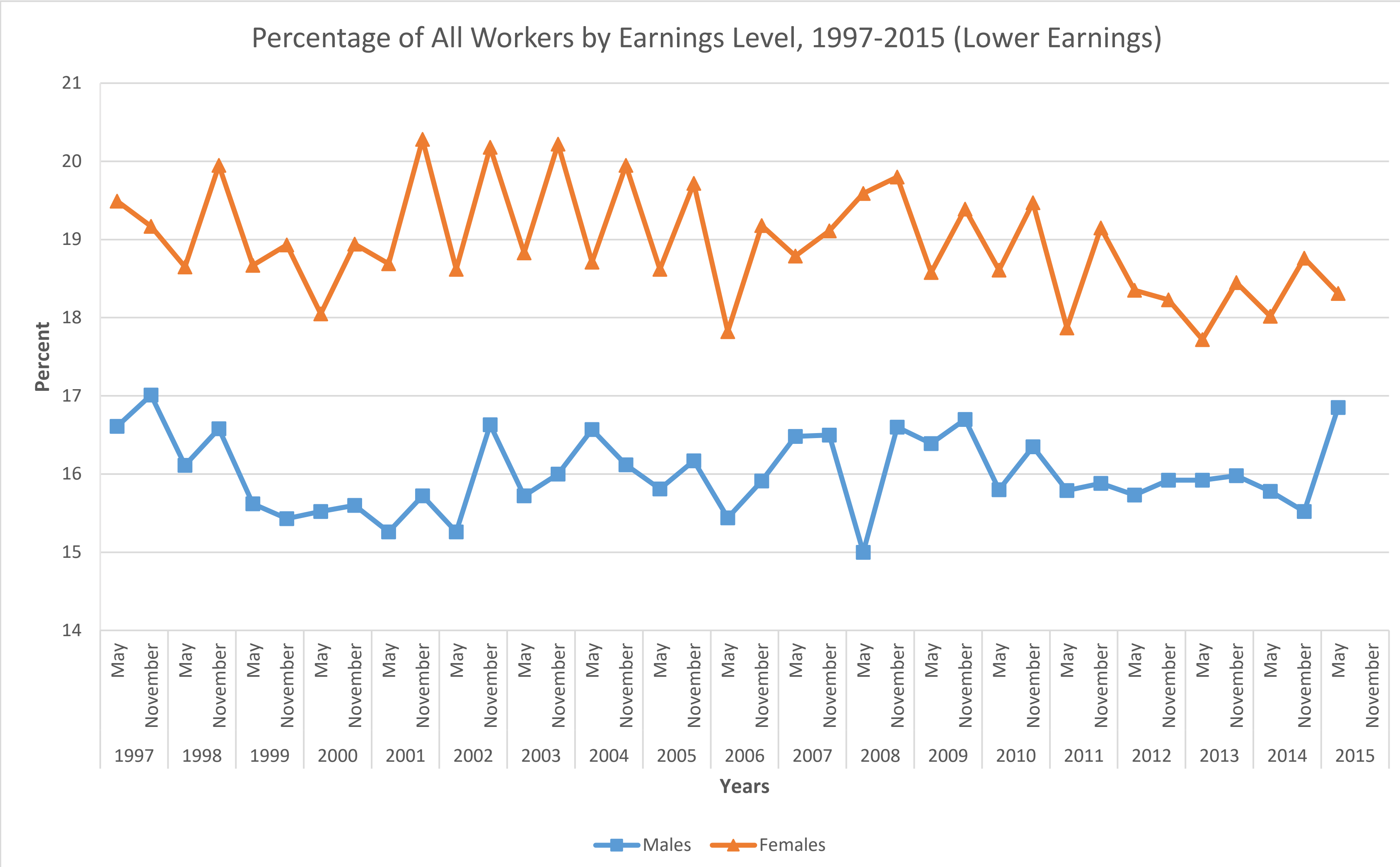
Table 6
Average Annual Percentage Change in Conditional Mean Earnings of Male and Female Workers by Earnings Level, Canada, 1970-2015

	Males		Females	
	All Workers	**FT Workers**	**All Workers**	**FT Workers**
Lower Earnings **(below 50% of median)**				
1970-80	+2.12*	+2.35*	+2.19*	+2.53*
1980-90	-0.30*	-0.88*	+1.42*	+0.64*
1990-00	-0.53*	-0.05	+0.25*	+0.16*
2000-05 (SCF)	-0.34	-0.11	+1.00*	+0.33*
2005-10	+1.13*	+1.43*	+1.34*	+1.81*
2010-15	+1.21*	+0.33*	+0.71*	+1.06*
Middle-Class Earnings **(within 50% of median)**				
1970-80	+2.01*	+2.24*	+2.06*	+2.83*
1980-90	-0.32*	-0.26*	+0.89*	+0.40*
1990-00	-0.56*	-0.30*	+0.94*	+0.41*
2000-05 (SCF)	-0.70*	-0.07	+0.90*	+0.60*
2005-10	+0.95*	+0.89*	+1.52*	+1.40*
2010-15	+0.23*	+0.39*	+0.62*	+0.32*
Higher Earnings **(above 200% of median)**				
1970-80	+0.97*	+0.90*	+2.45*	+1.96*
1980-90	-0.13*	+0.14*	+1.22*	+0.89*
1990-00	-0.09*	-0.20*	+1.13*	+0.39*
2000-05 (SCF)	-0.36*	-0.19	+1.56*	+1.78*
2005-10	+0.91*	+1.25*	+1.45*	+1.30*
2010-15	+0.55*	+0.96*	+0.57*	+0.71*

Note: See data sources in Tables 1 and 2.

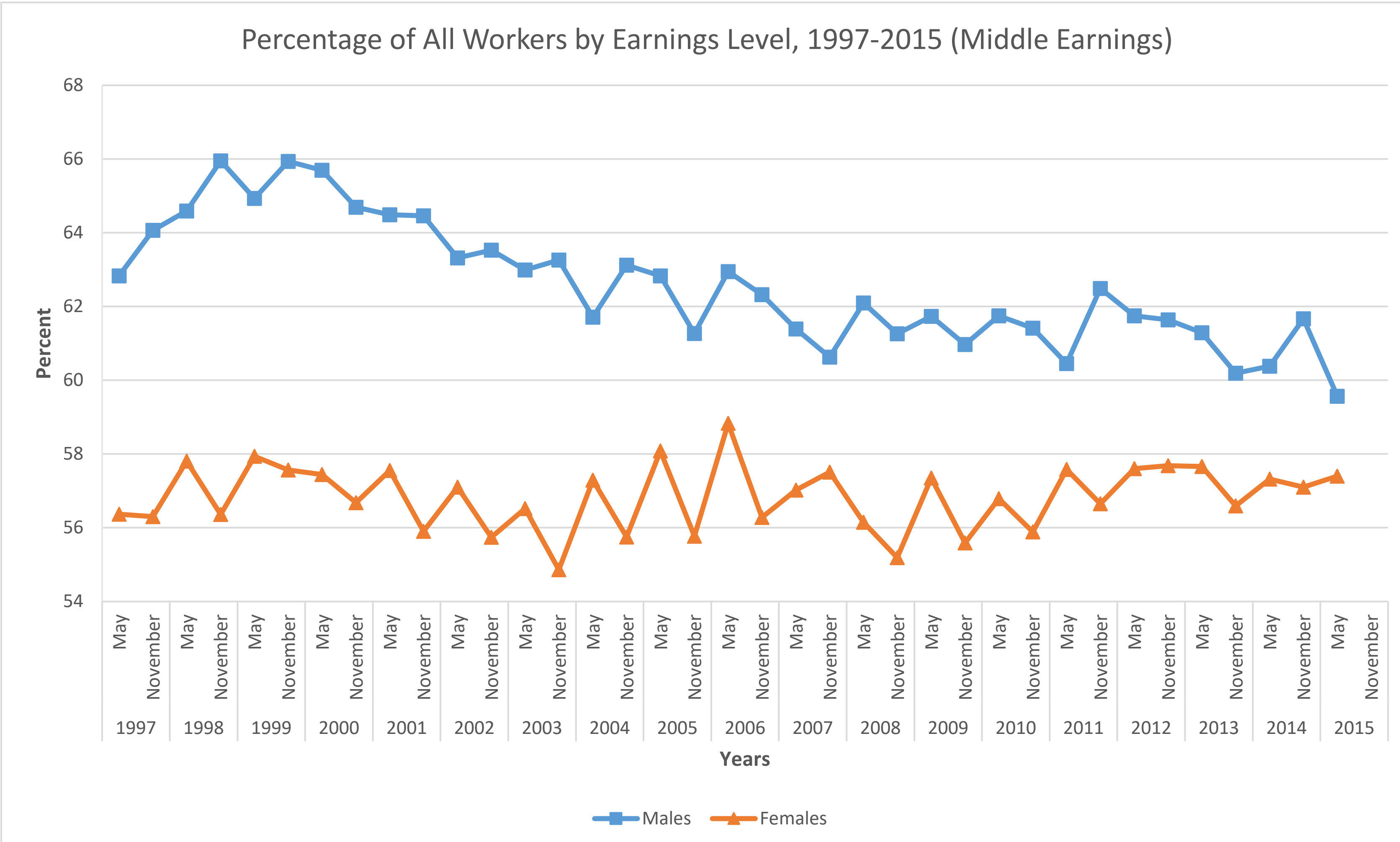
Asterisks indicate statistical significance (2-tailed test) at 95% level of confidence. Actual “t-ratios” for each of these changes are found in appendix Table A10.

Figure 1



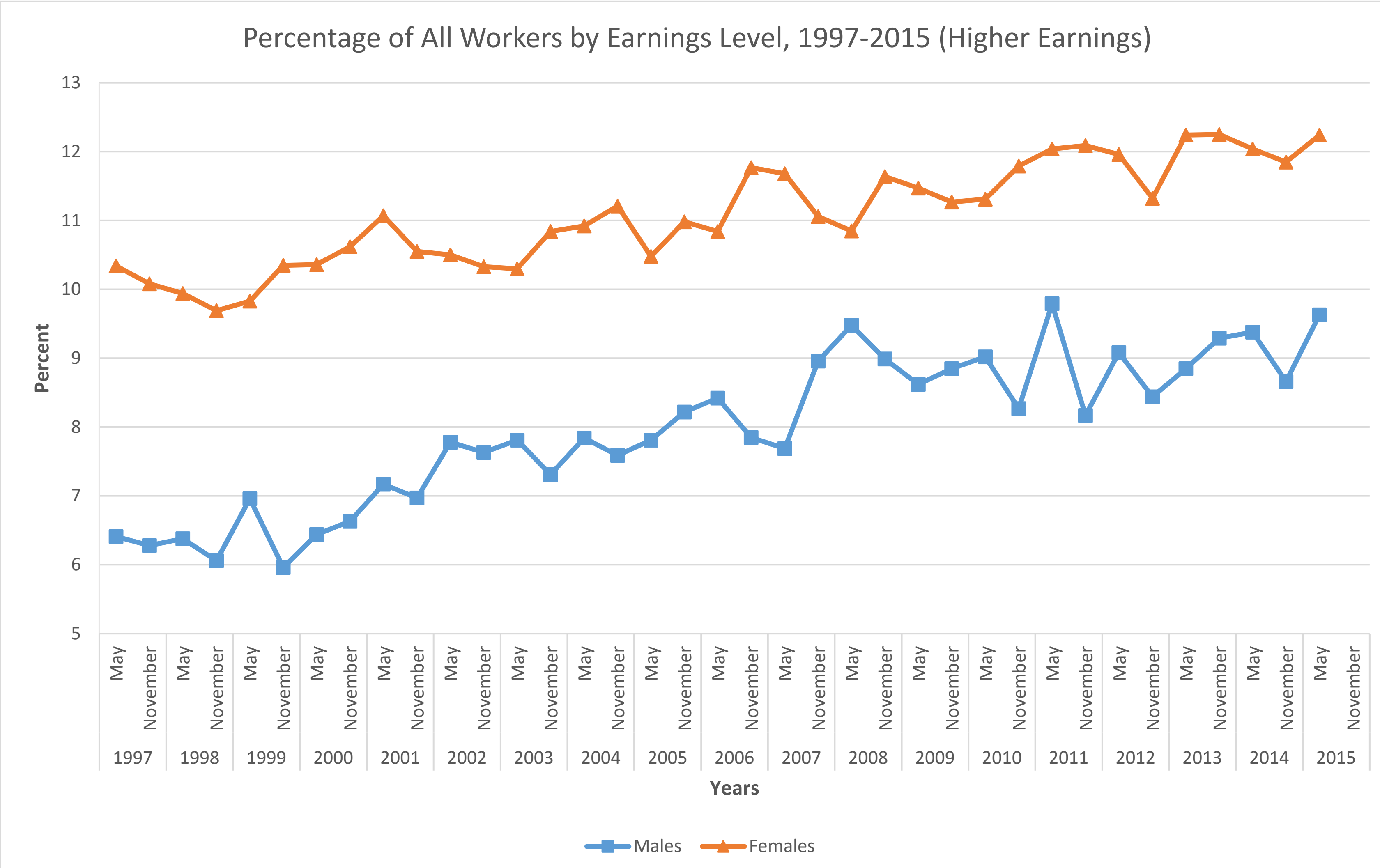
Source: LFS microdata files; calculations by author.

Figure 2



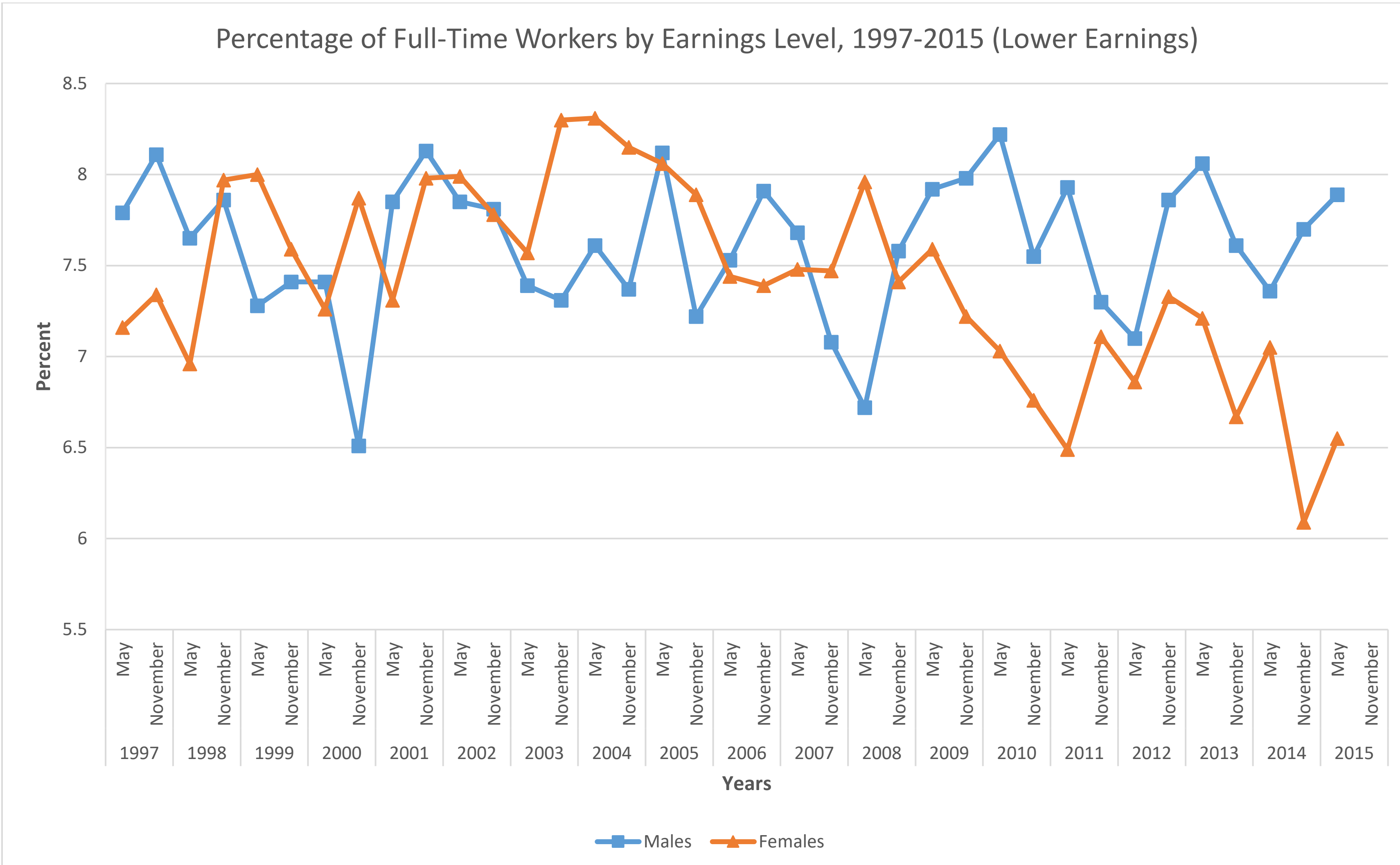
Source: LFS microdata files; calculations by author.

Figure 3



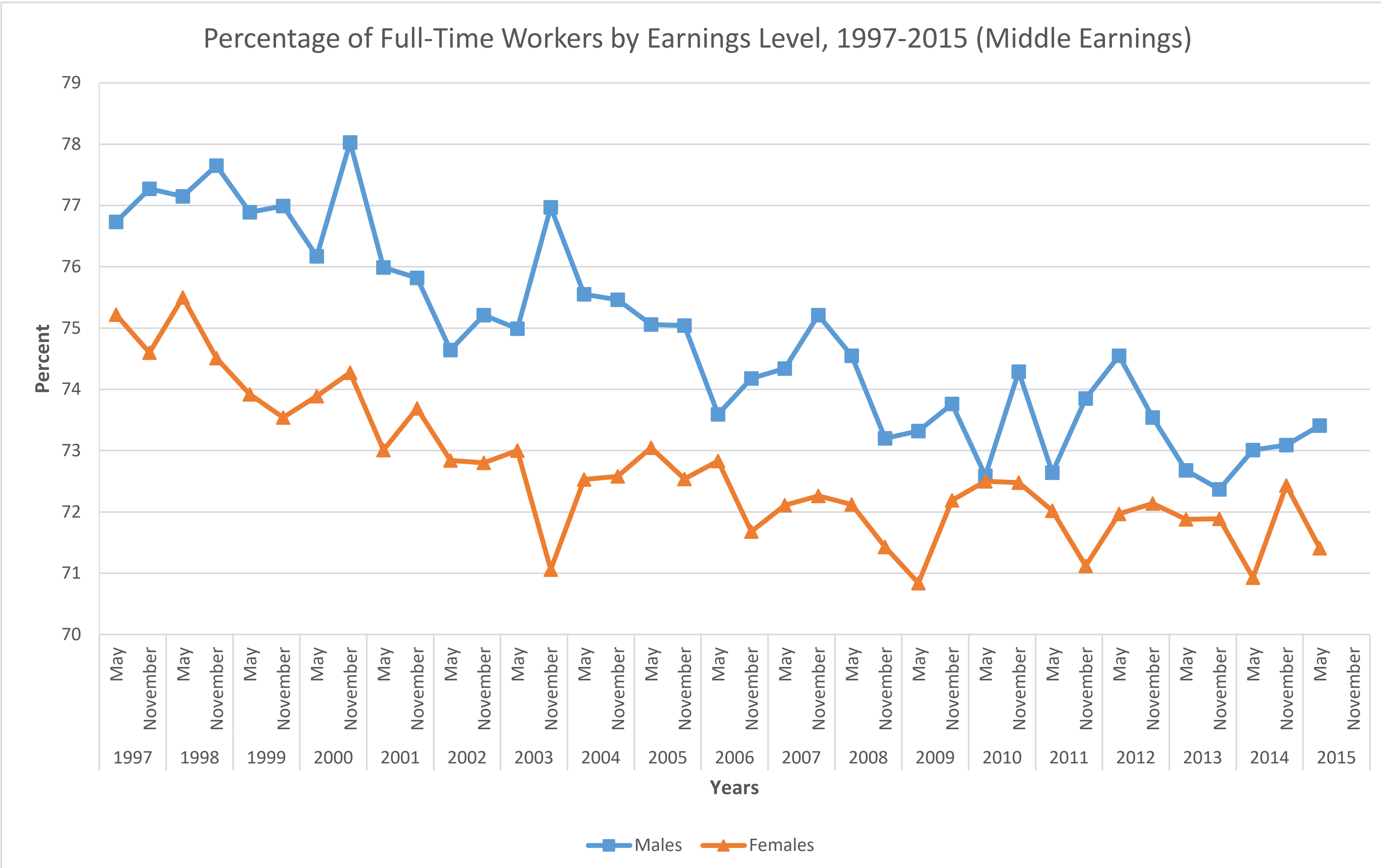
Source: LFS microdata files; calculations by author.

Figure 4



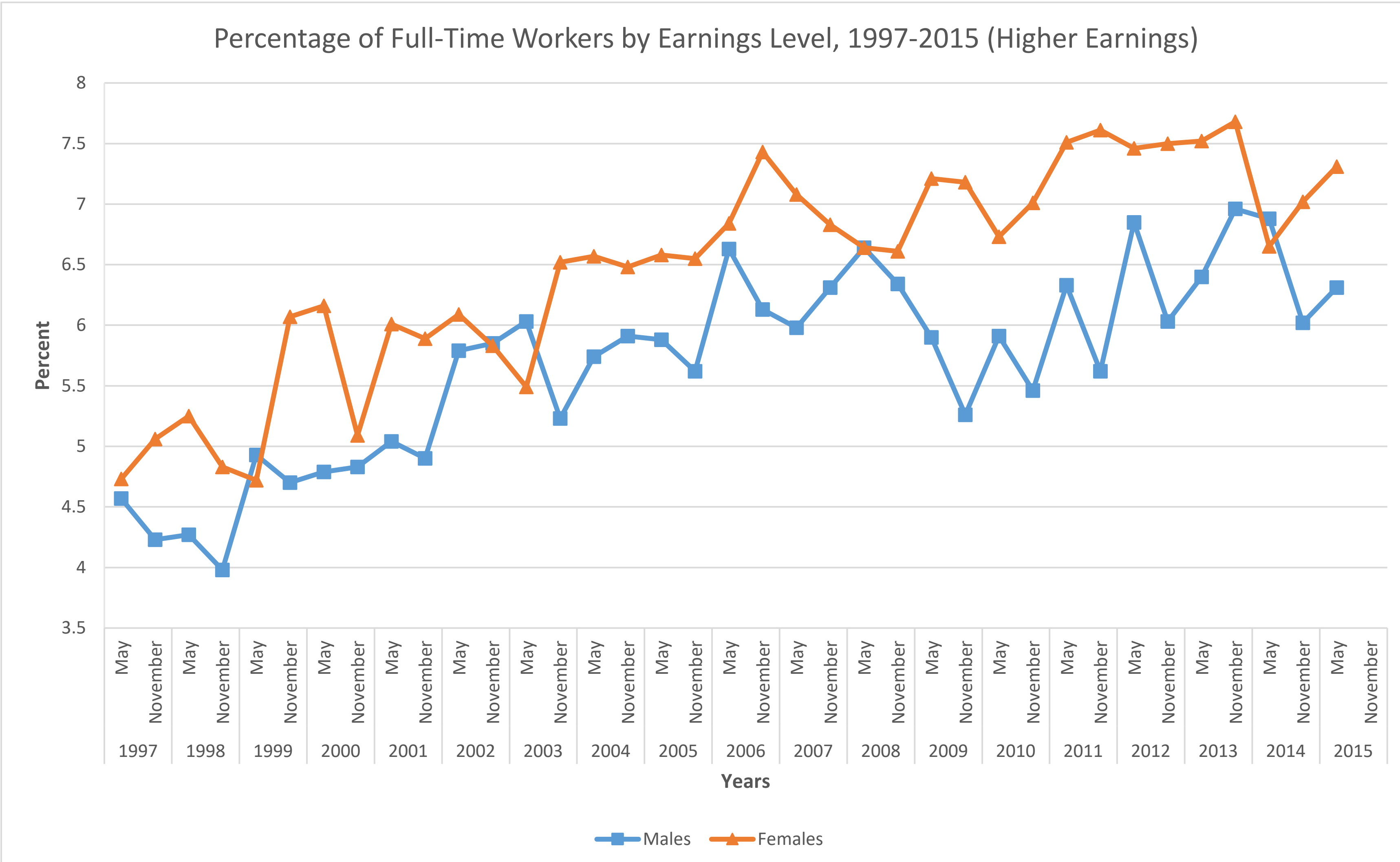
Source: LFS microdata files; calculations by author.

Figure 5



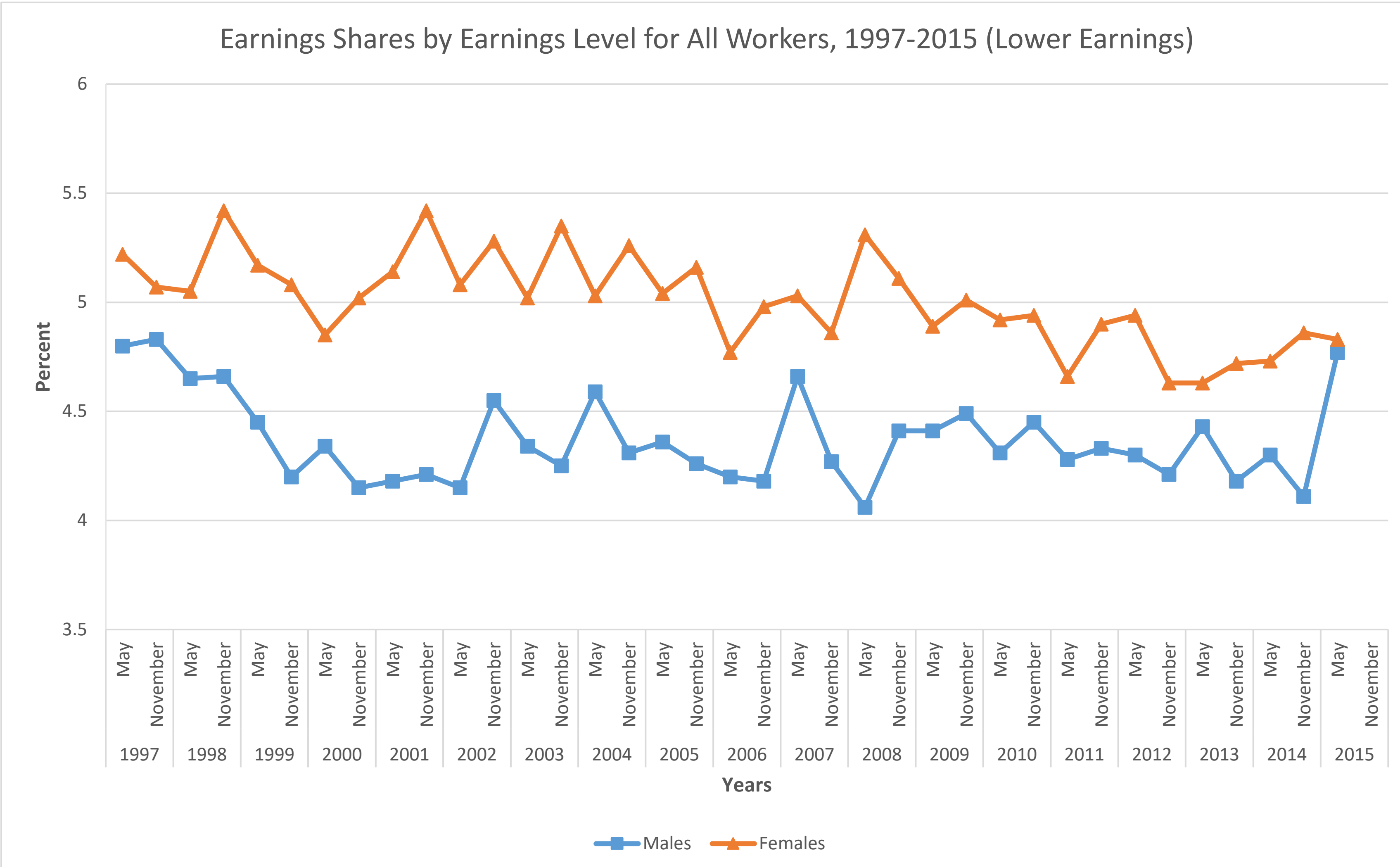
Source: LFS microdata files; calculations by author.

Figure 6



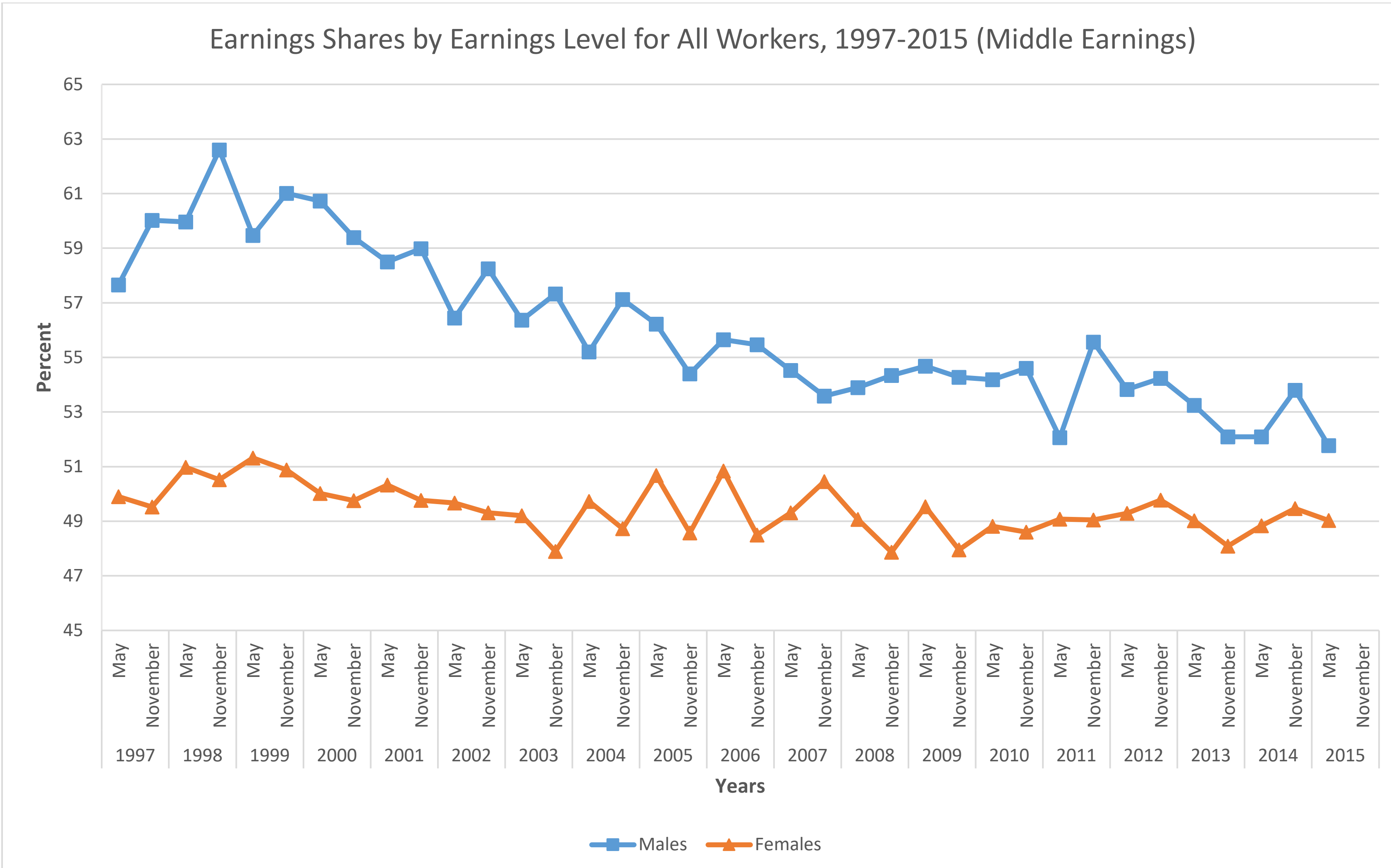
Source: LFS microdata files; calculations by author.

Figure 7



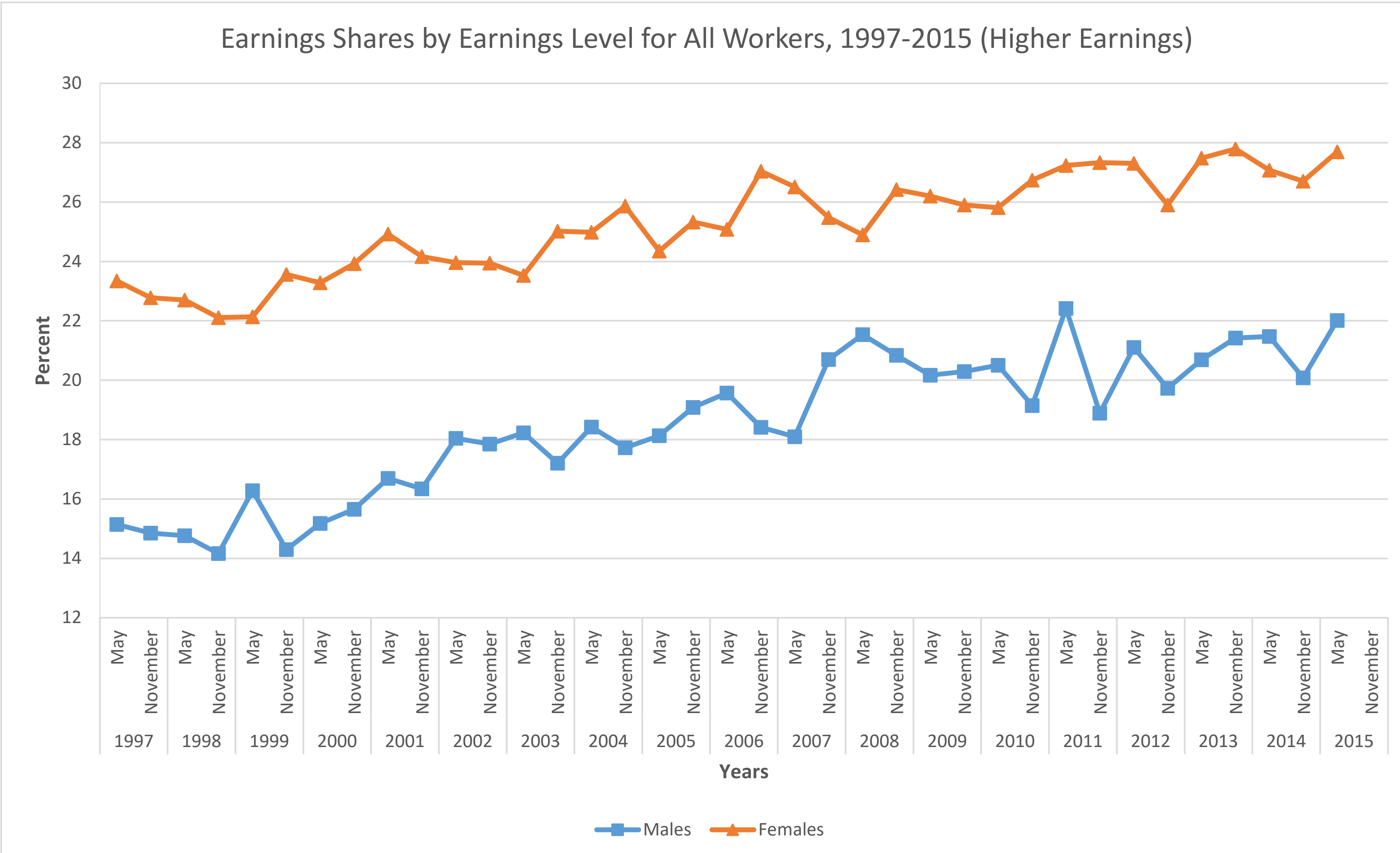
Source: LFS microdata files; calculations by author.

Figure 8



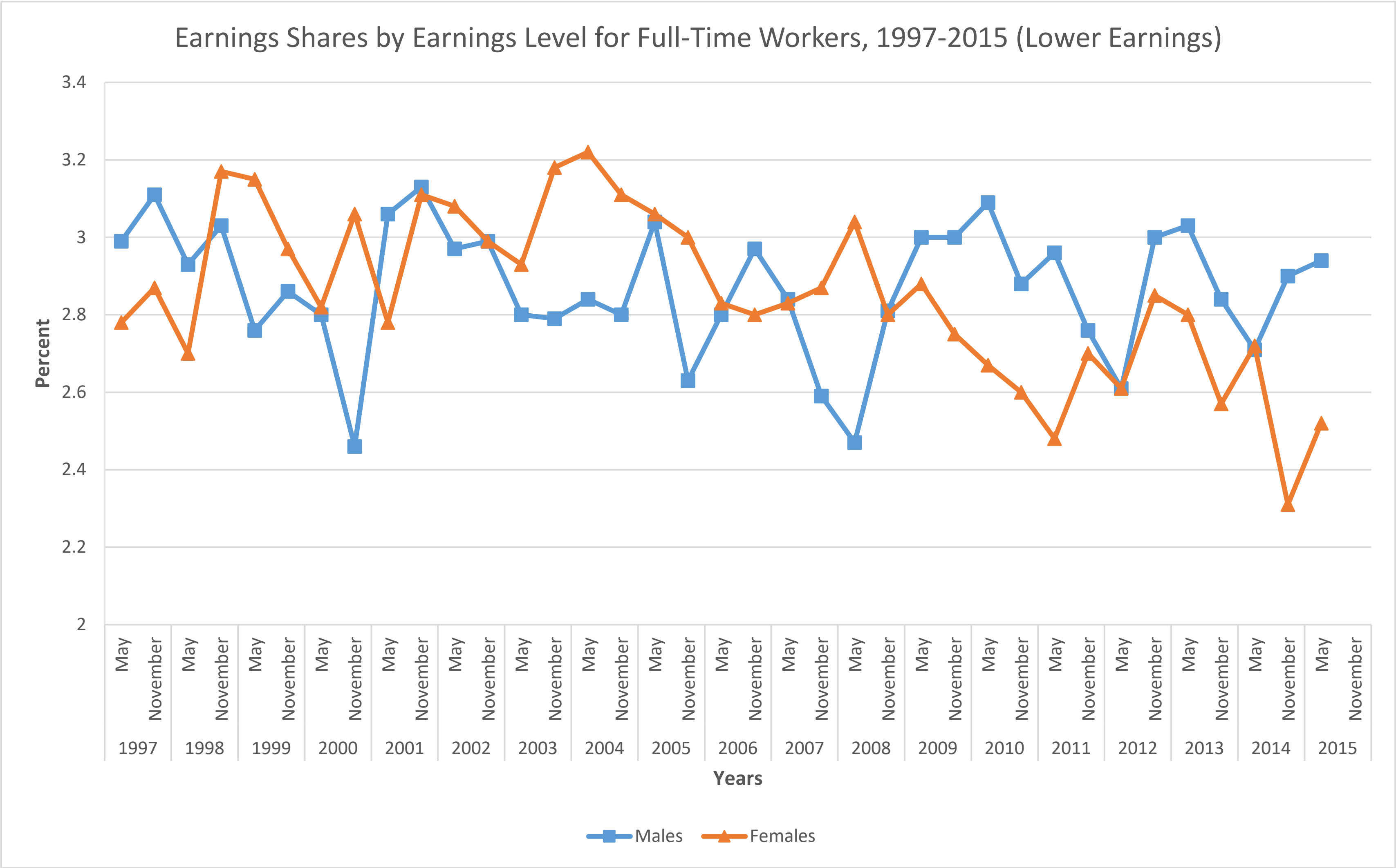
Source: LFS microdata files; calculations by author.

Figure 9



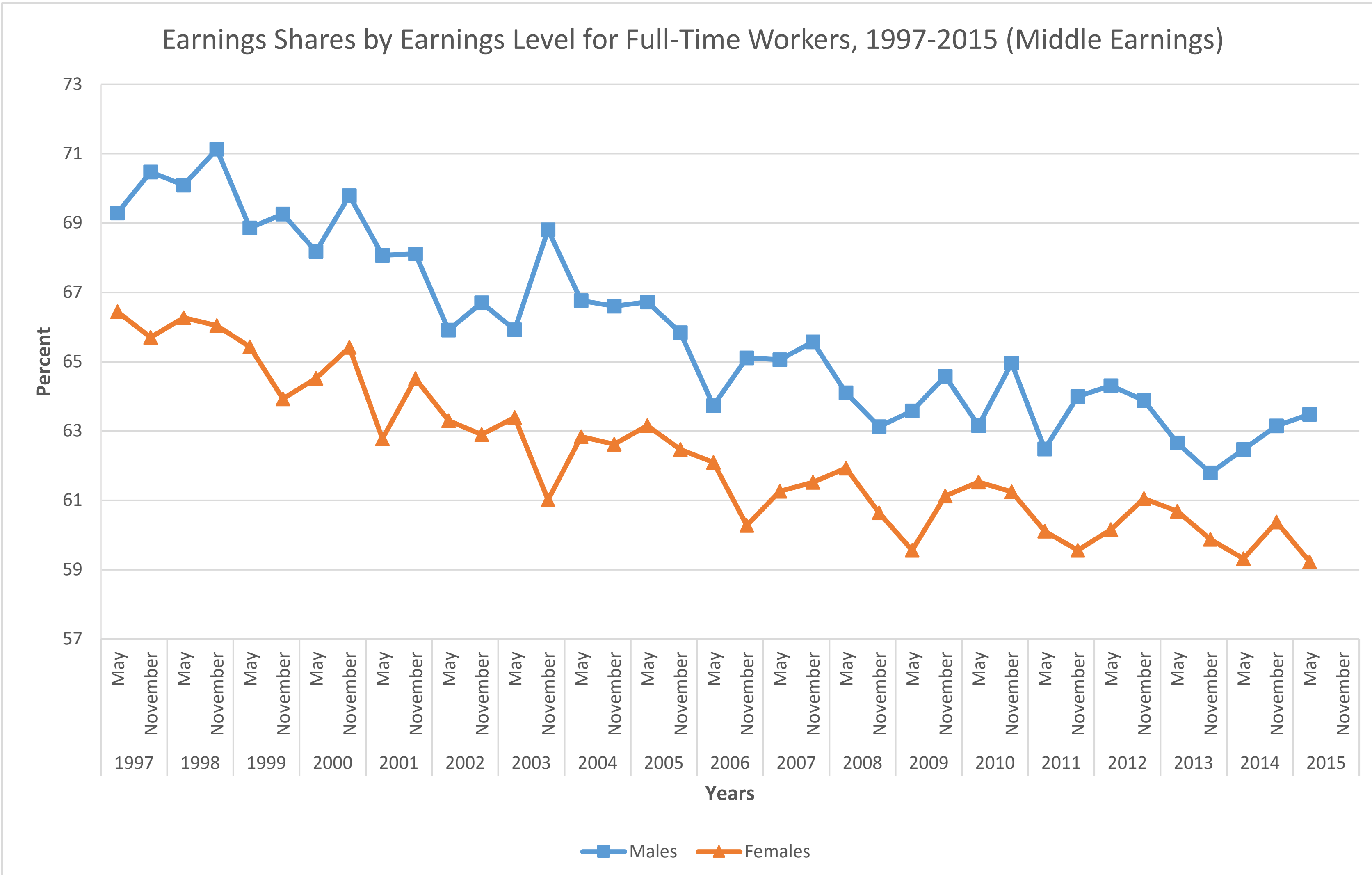
Source: LFS microdata files; calculations by author.

Figure 10



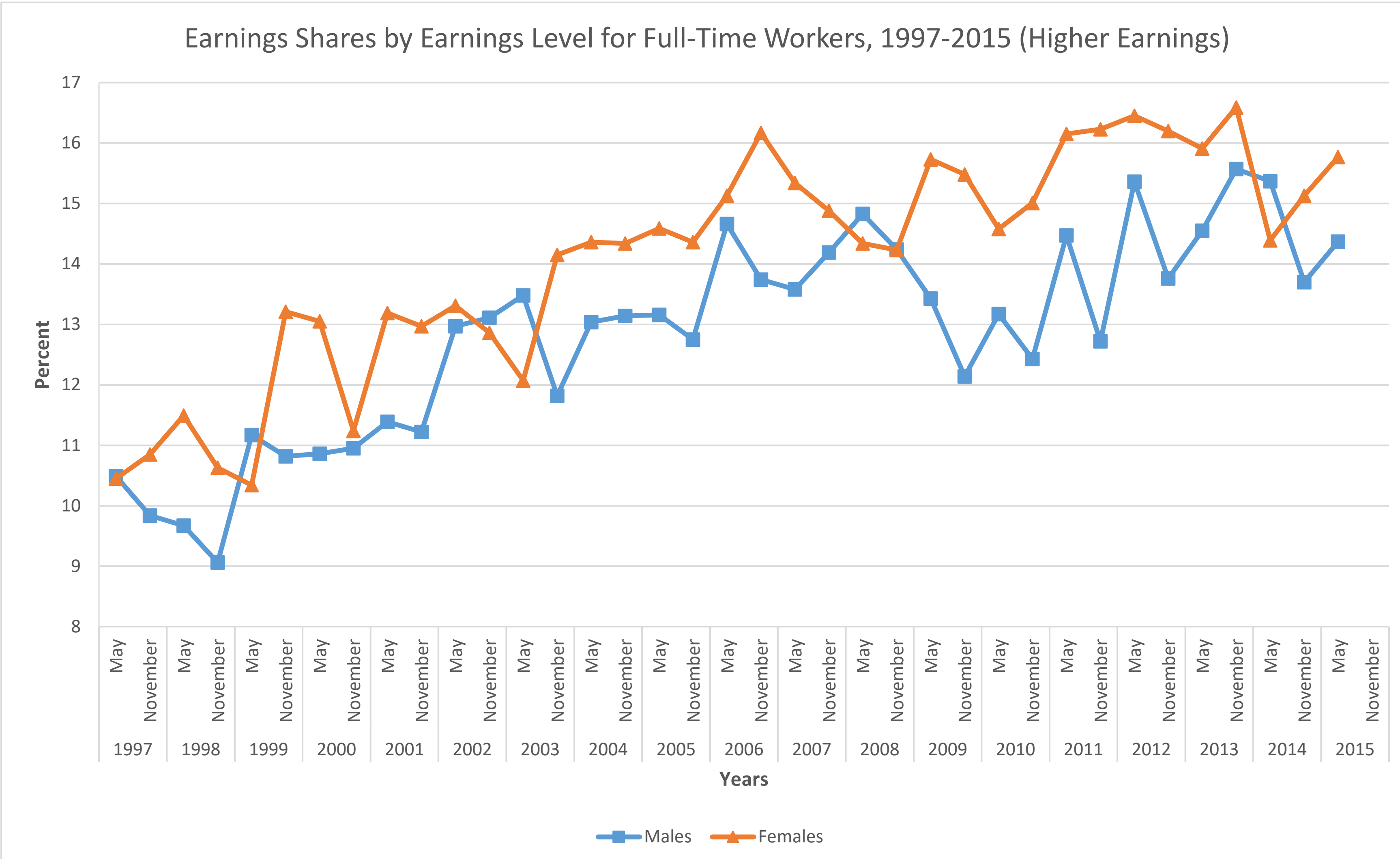
Source: LFS microdata files; calculations by author.

Figure 11



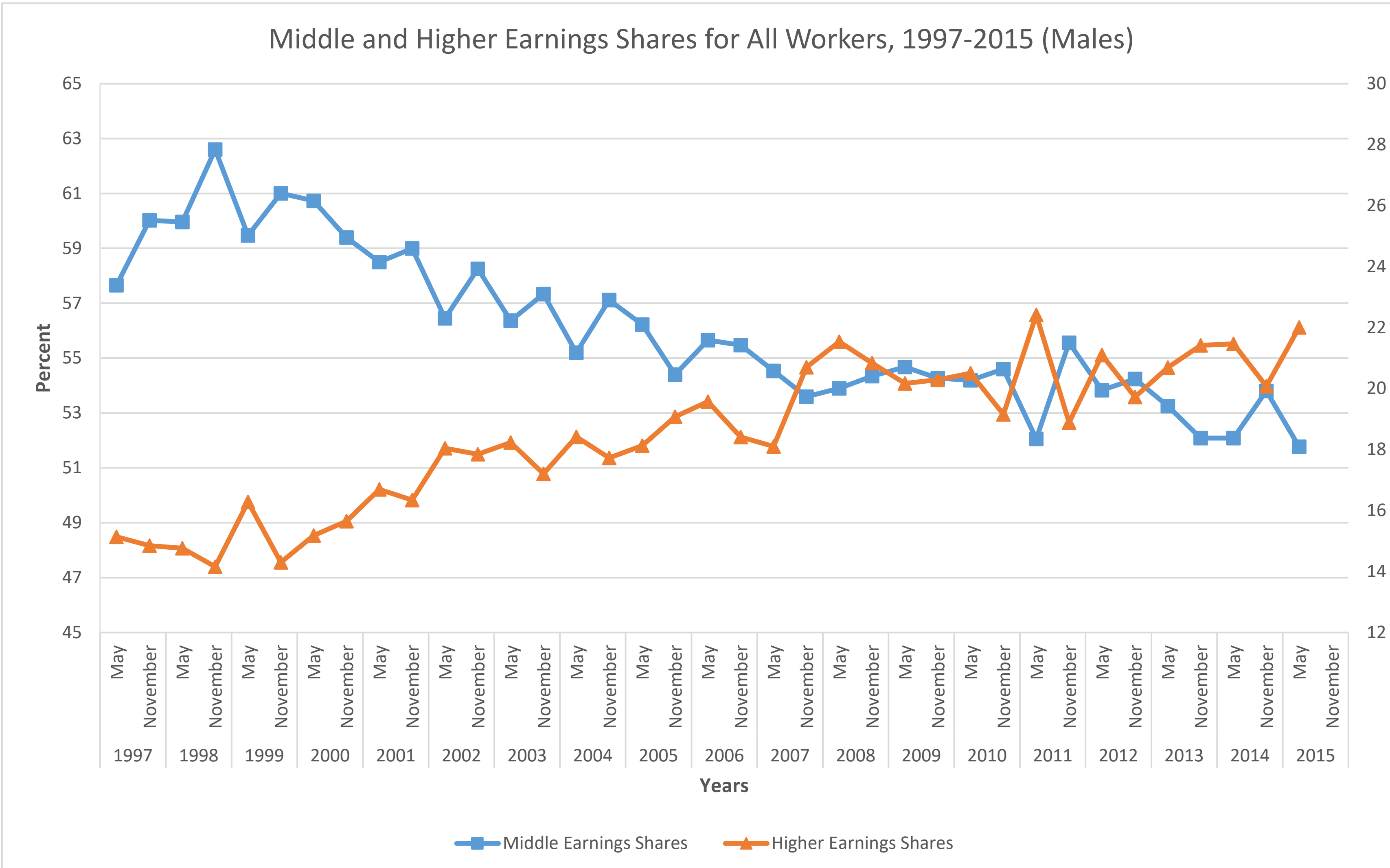
Source: LFS microdata files; calculations by author.

Figure 12



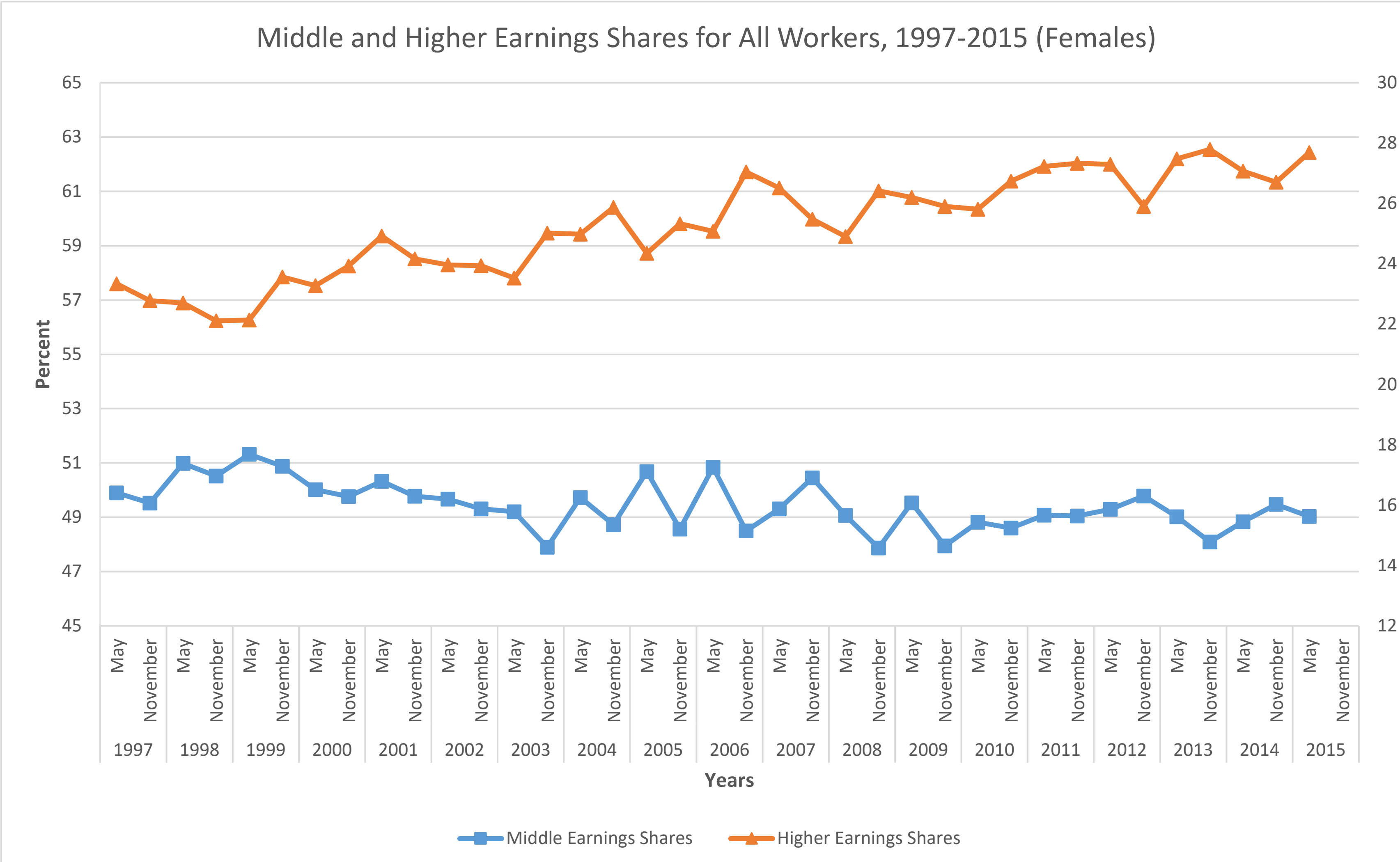
Source: LFS microdata files; calculations by author.

Figure 13



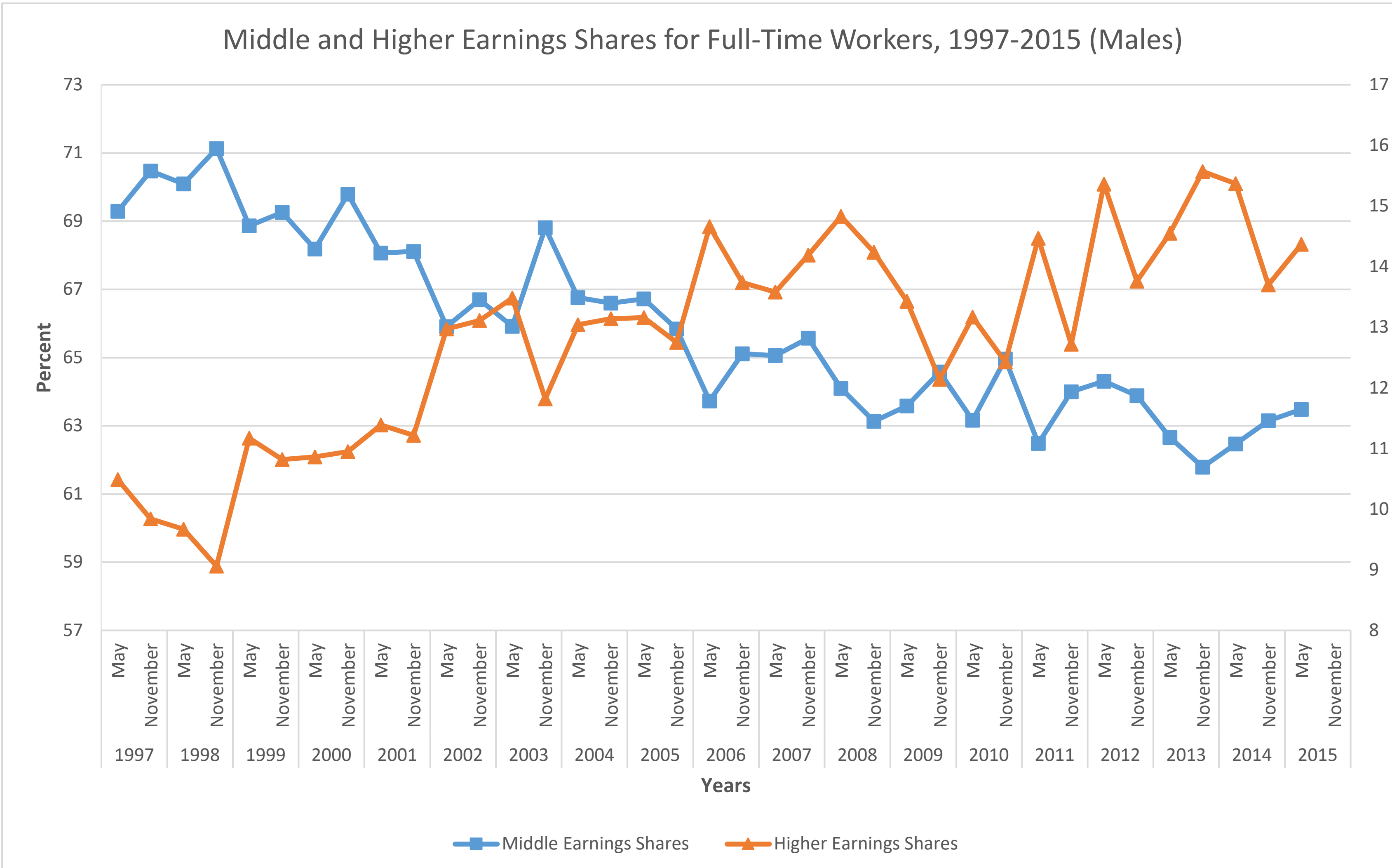
Source: LFS microdata files; calculations by author.

Figure 14



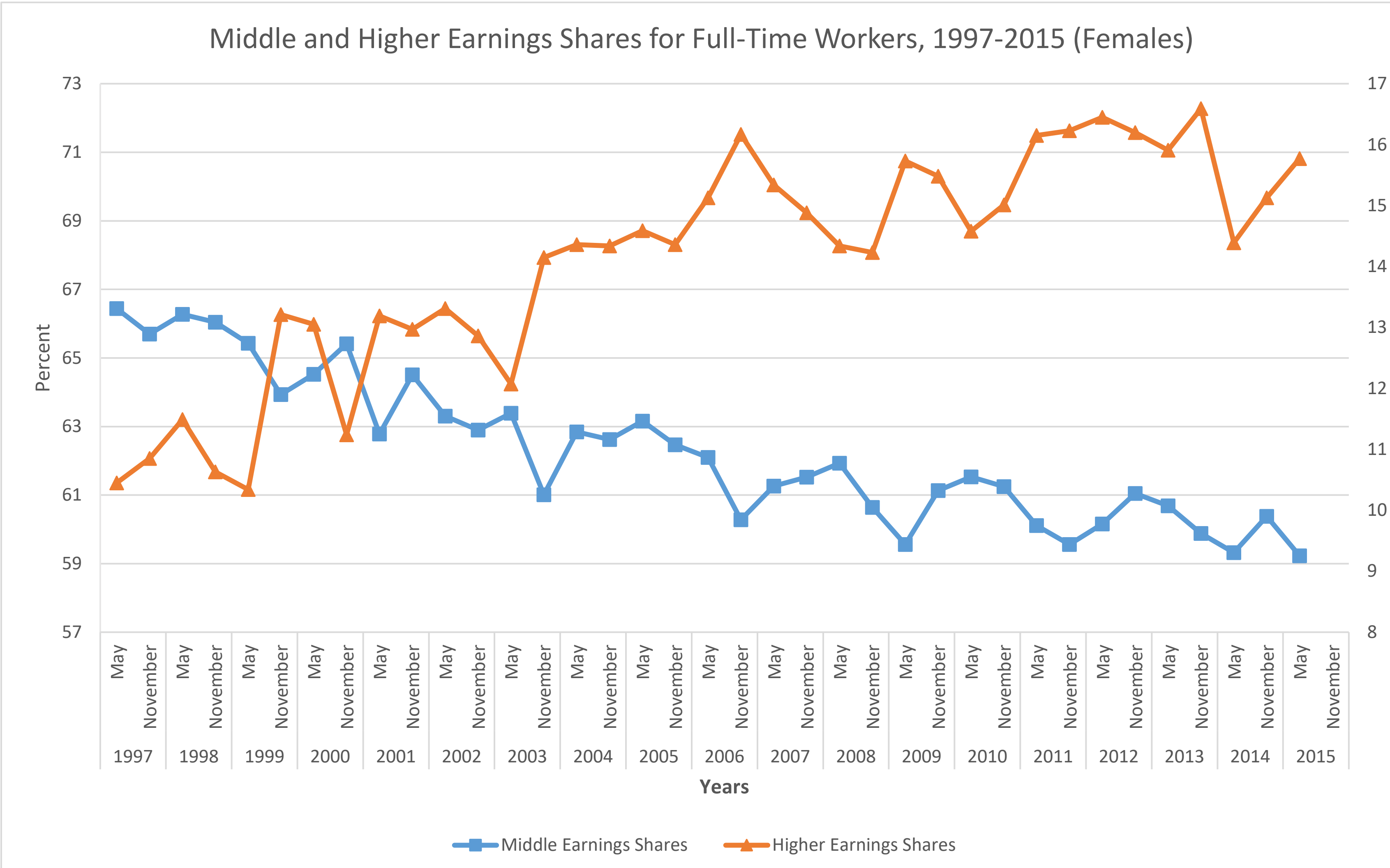
Source: LFS microdata files; calculations by author.

Figure 15



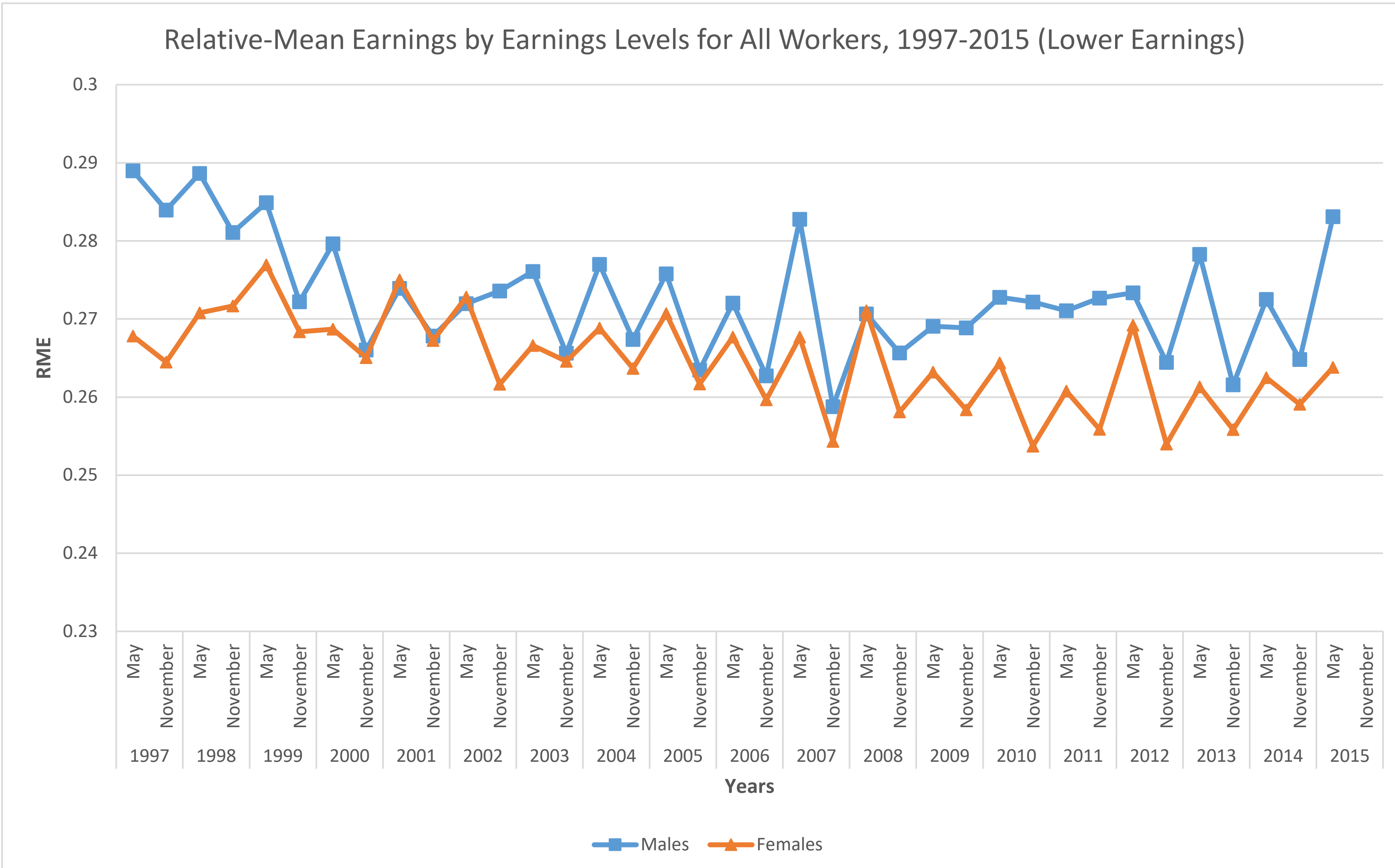
Source: LFS microdata files; calculations by author.

Figure 16



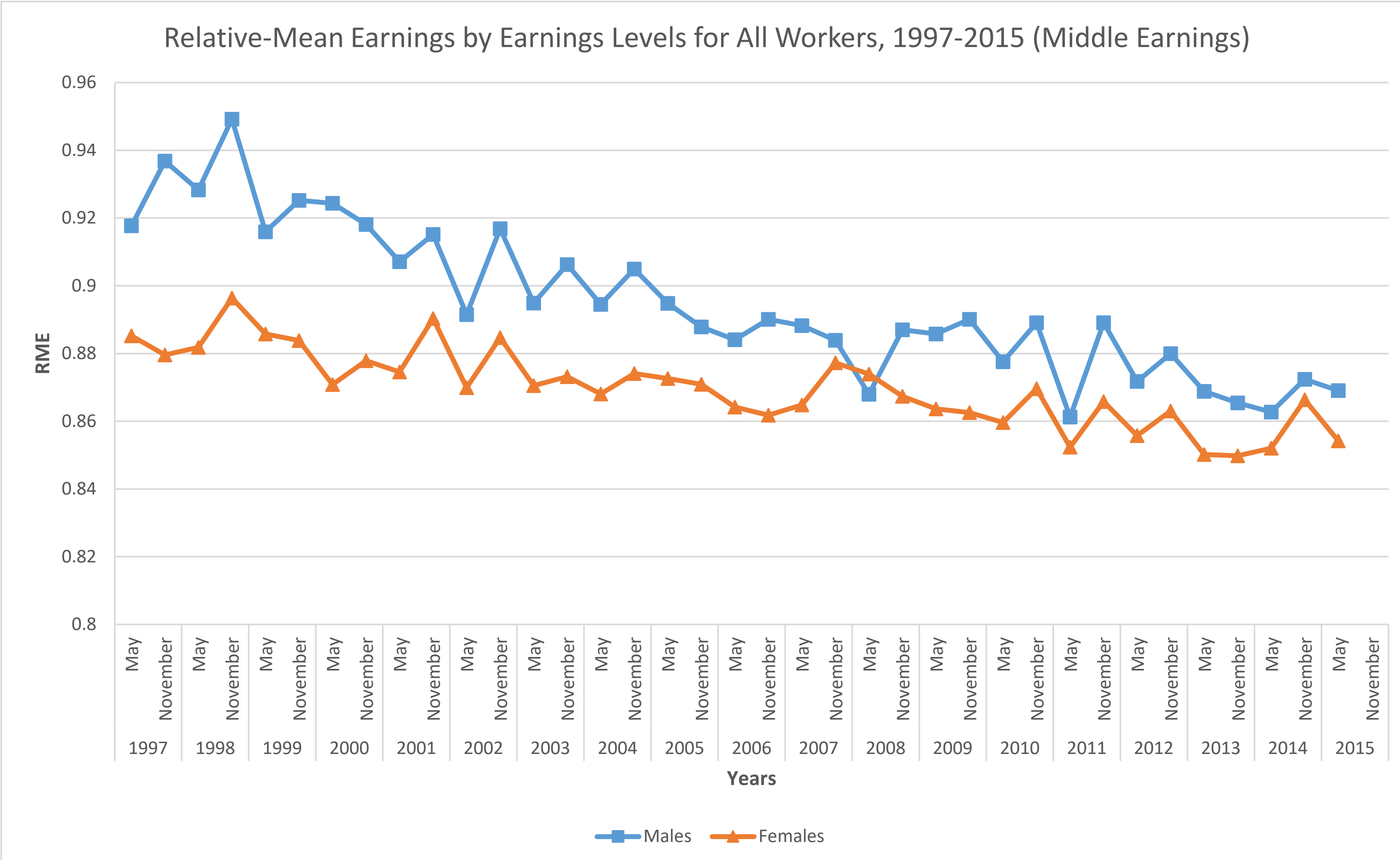
Source: LFS microdata files; calculations by author.

Figure 17



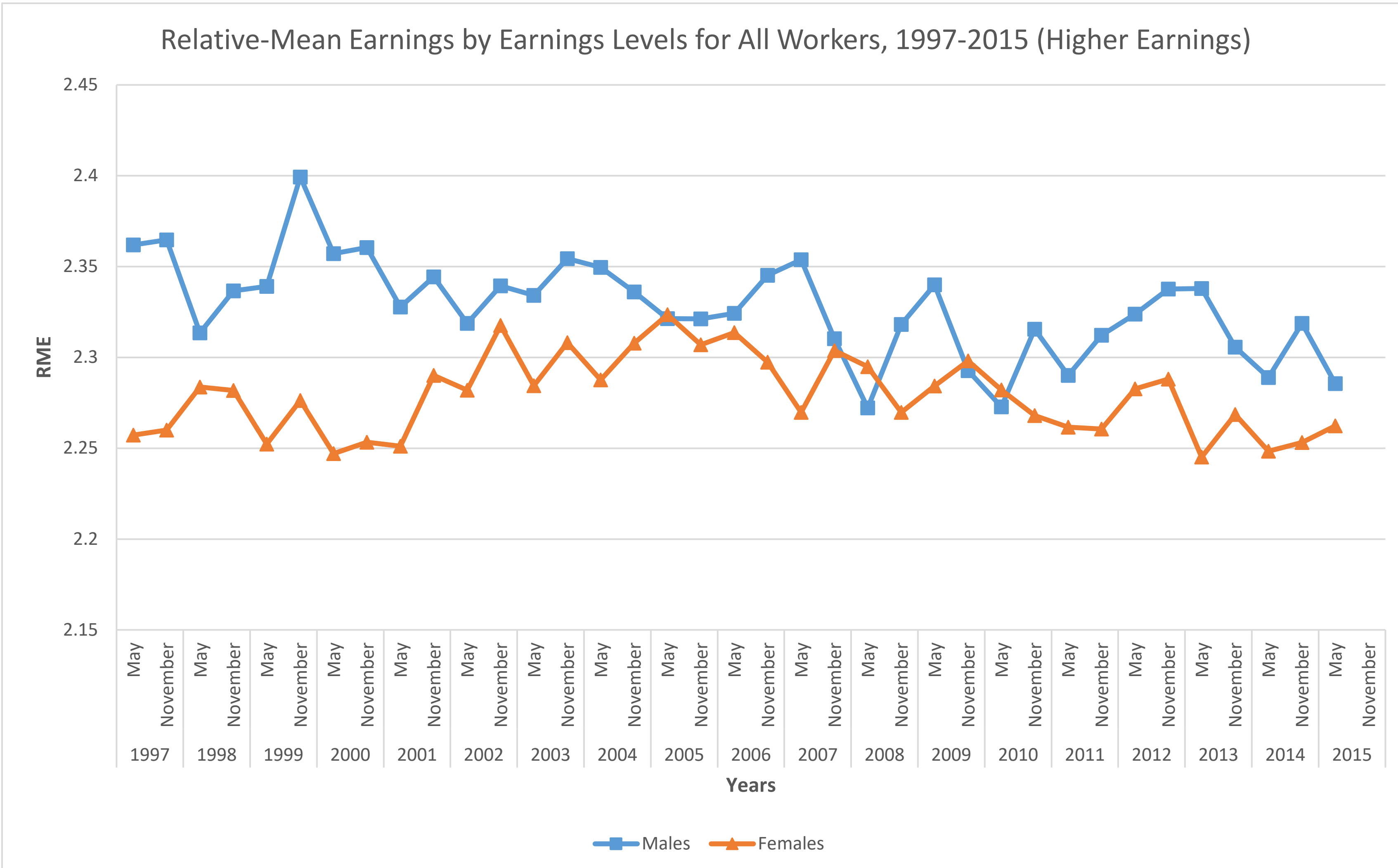
Source: LFS microdata files; calculations by author.

Figure 18



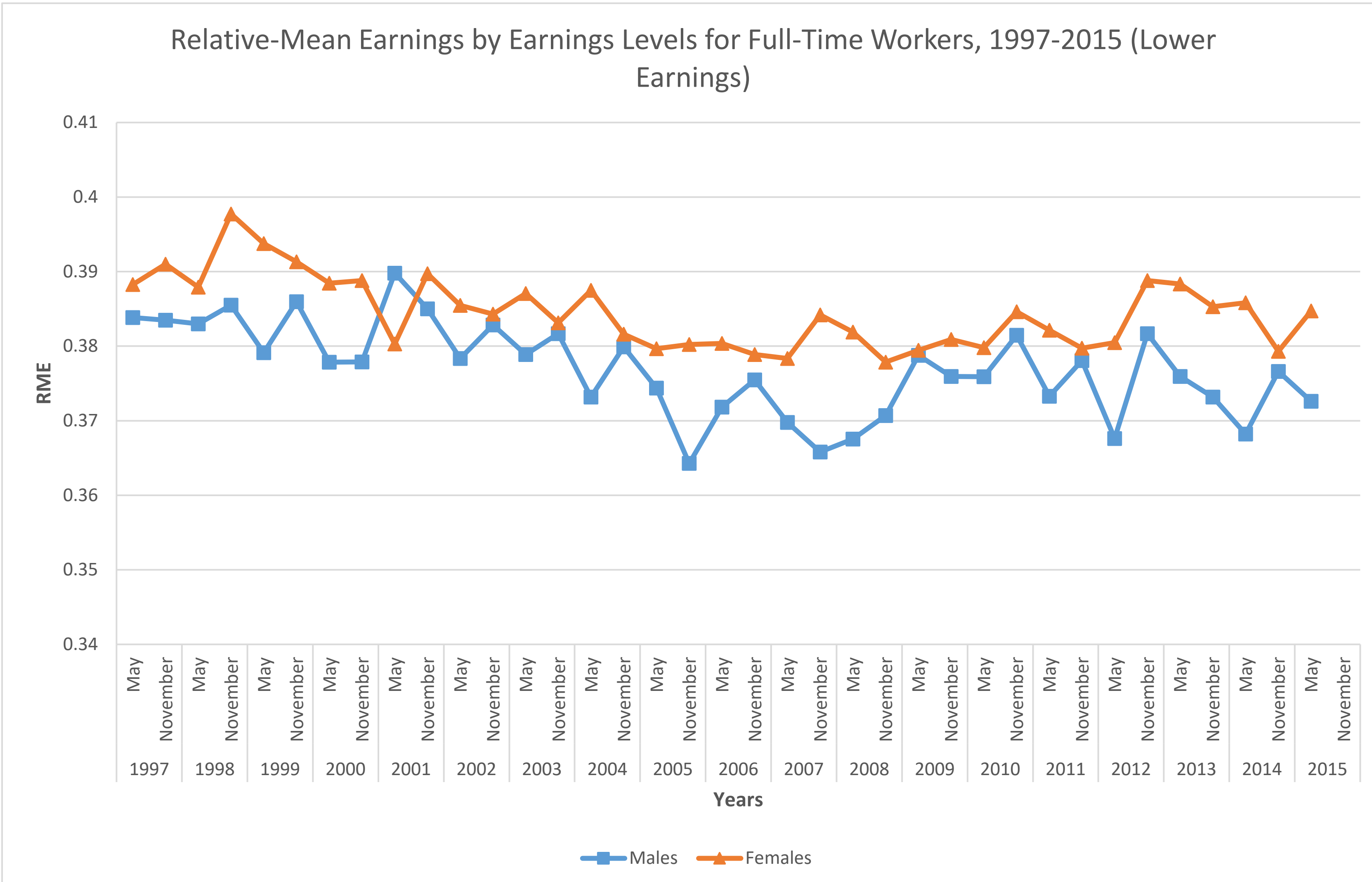
Source: LFS microdata files; calculations by author.

Figure 19



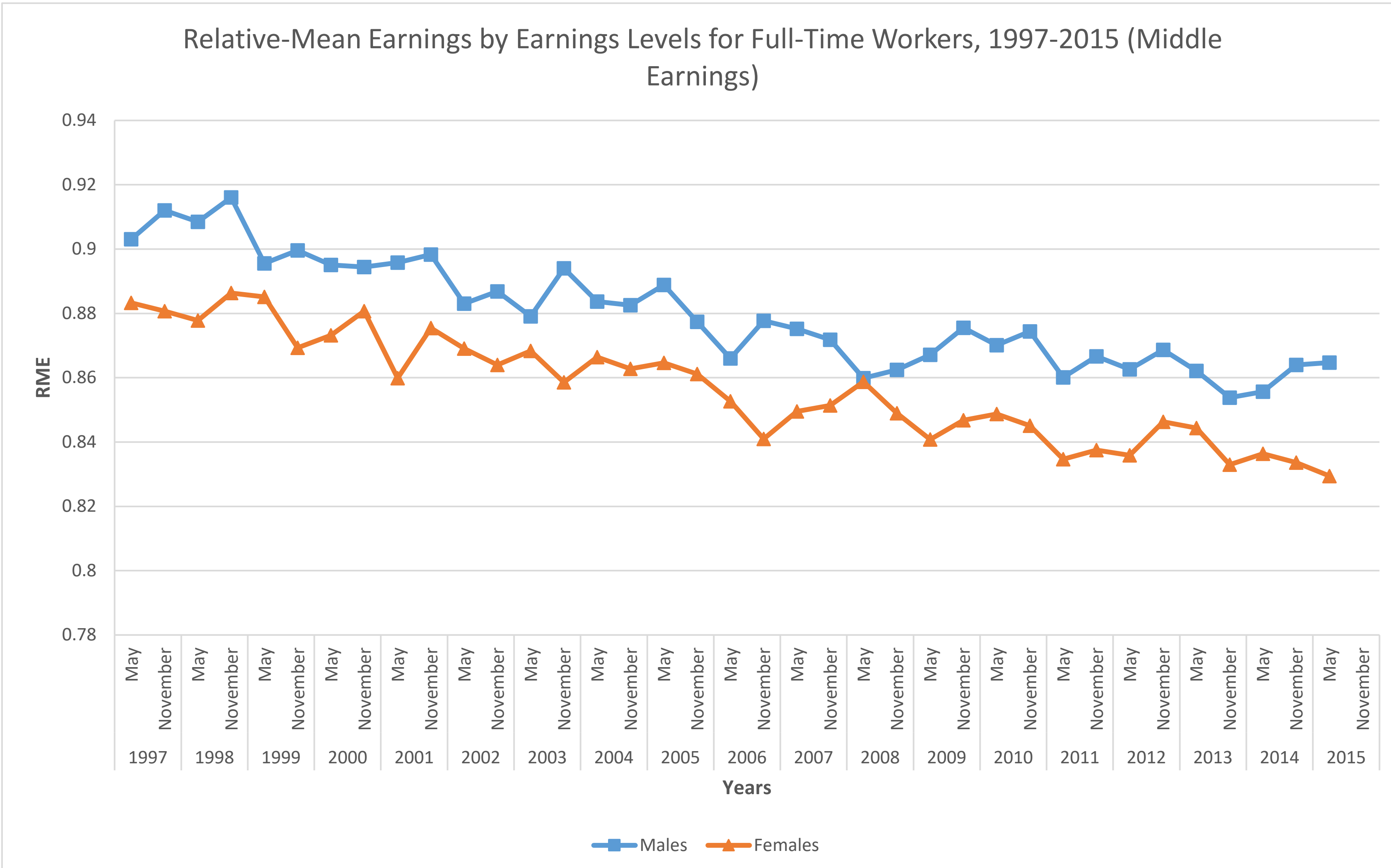
Source: LFS microdata files; calculations by author.

Figure 20



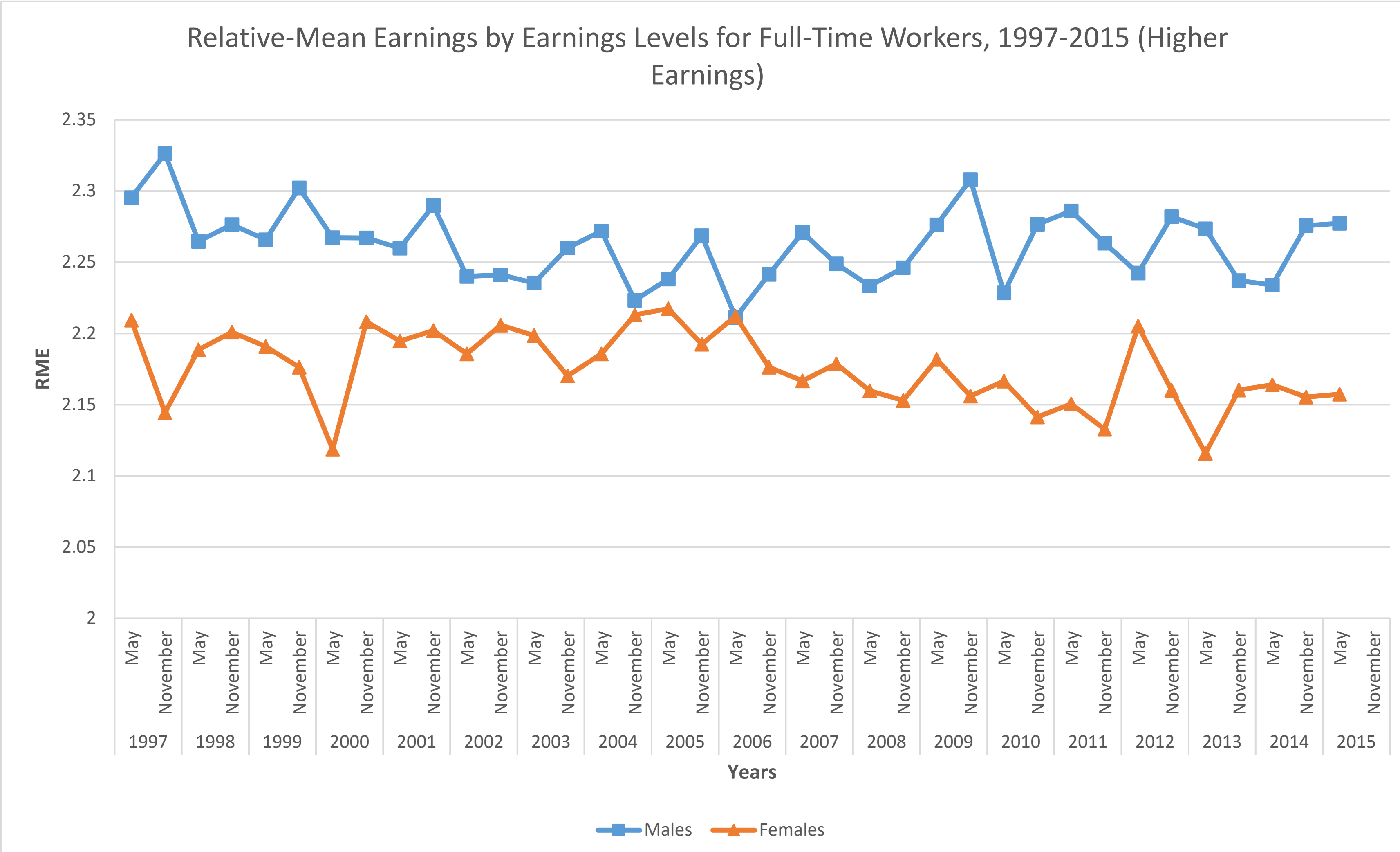
Source: LFS microdata files; calculations by author.

Figure 21



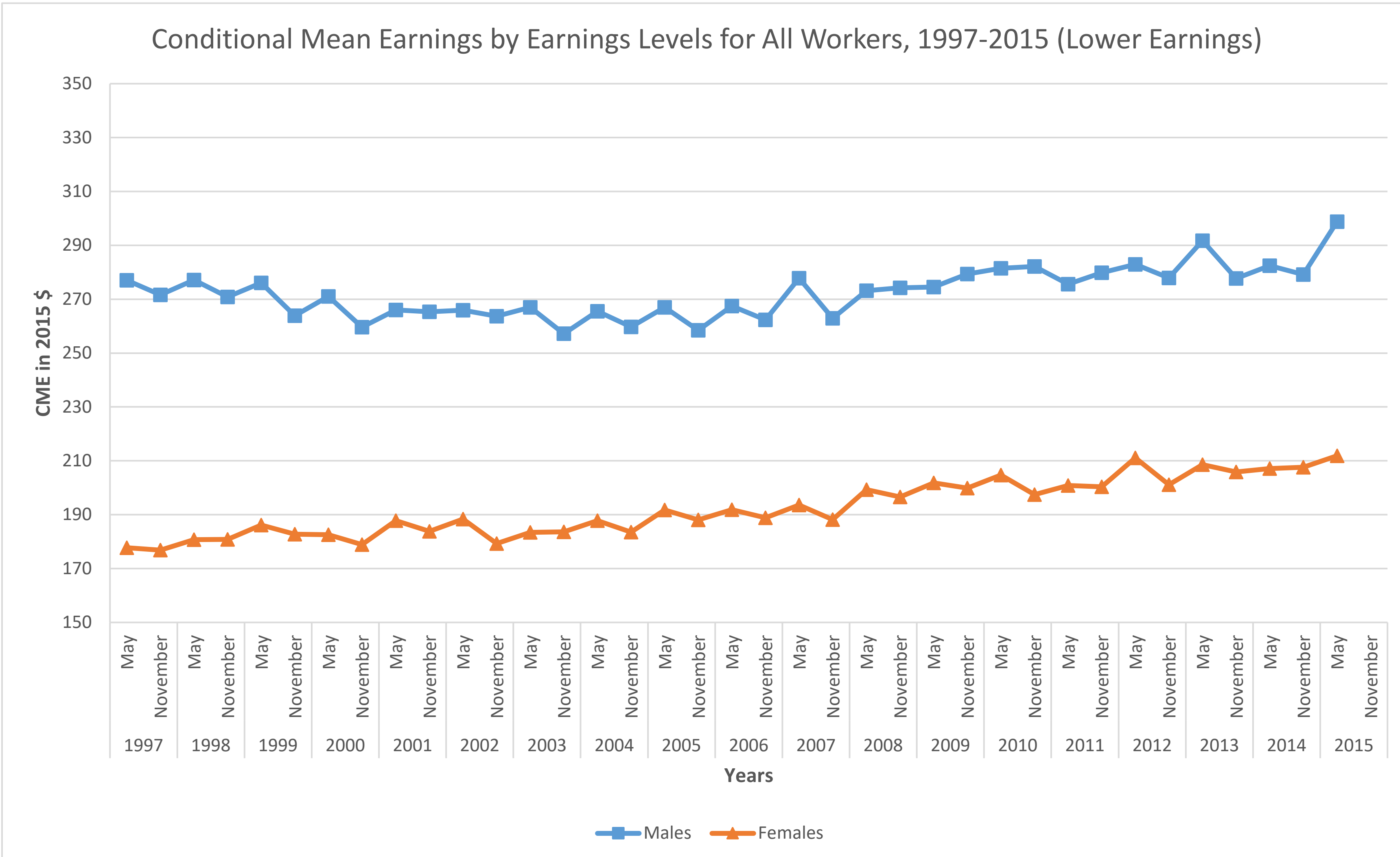
Source: LFS microdata files; calculations by author.

Figure 22



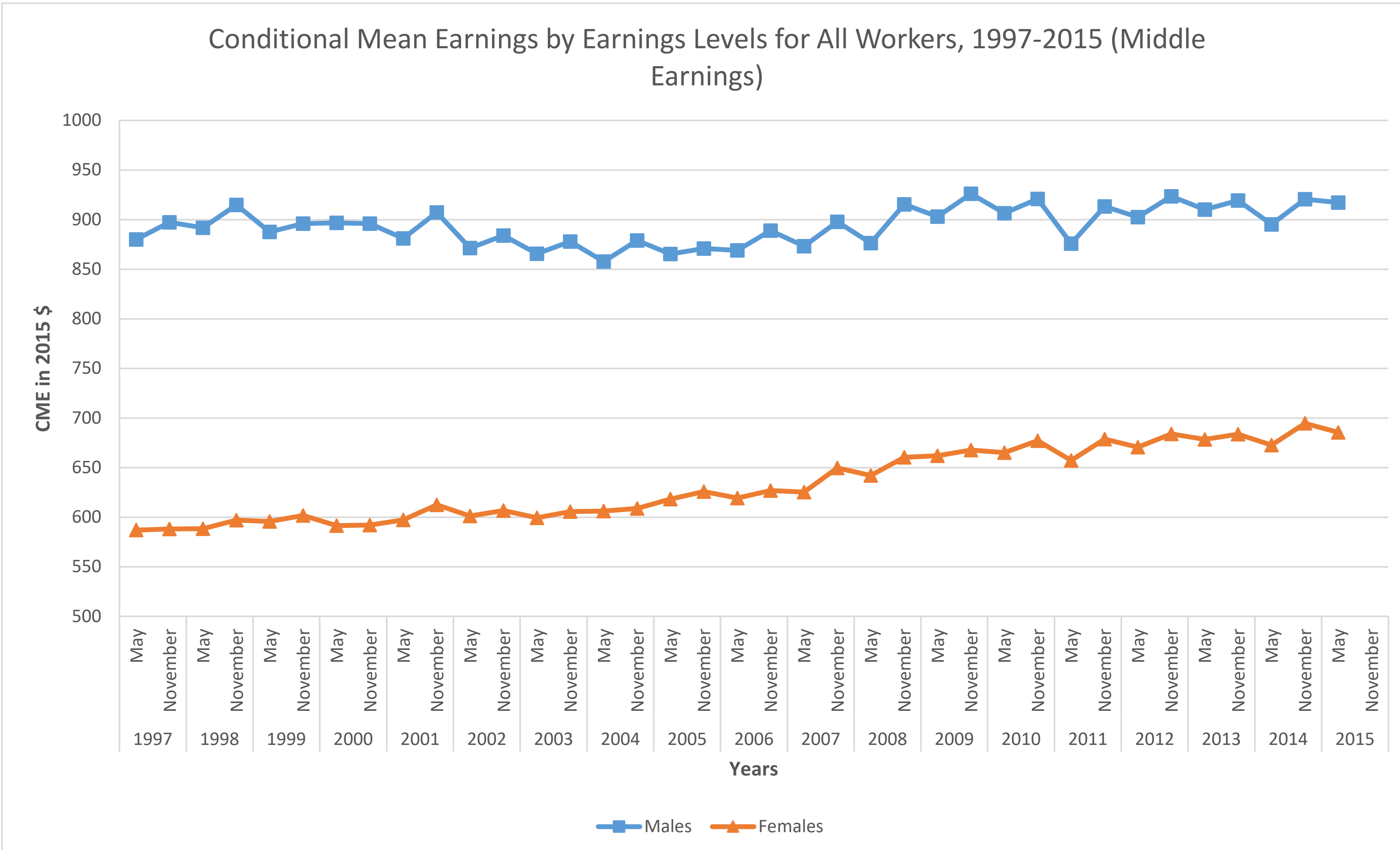
Source: LFS microdata files; calculations by author.

Figure 23



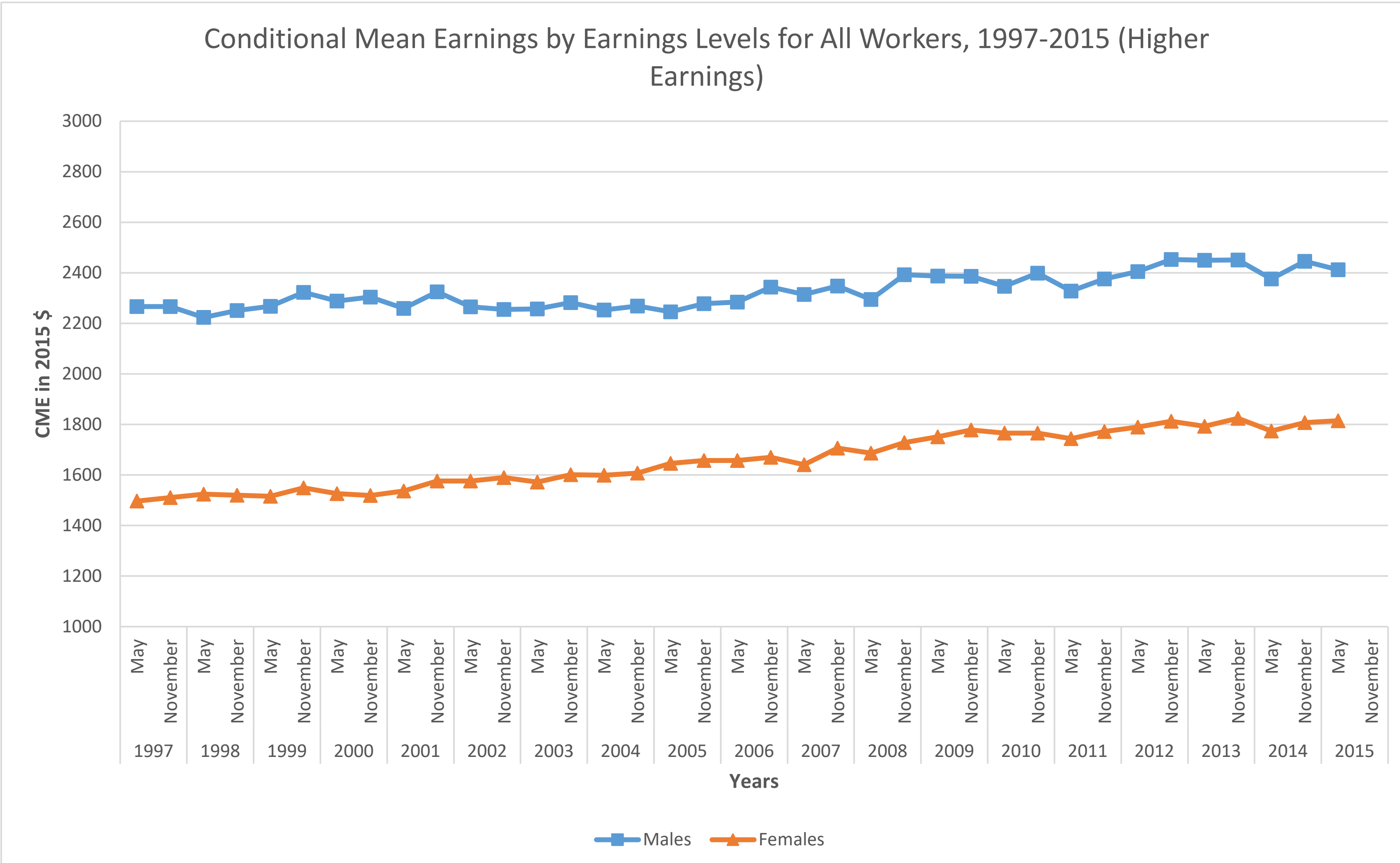
Source: LFS microdata files; calculations by author.

Figure 24



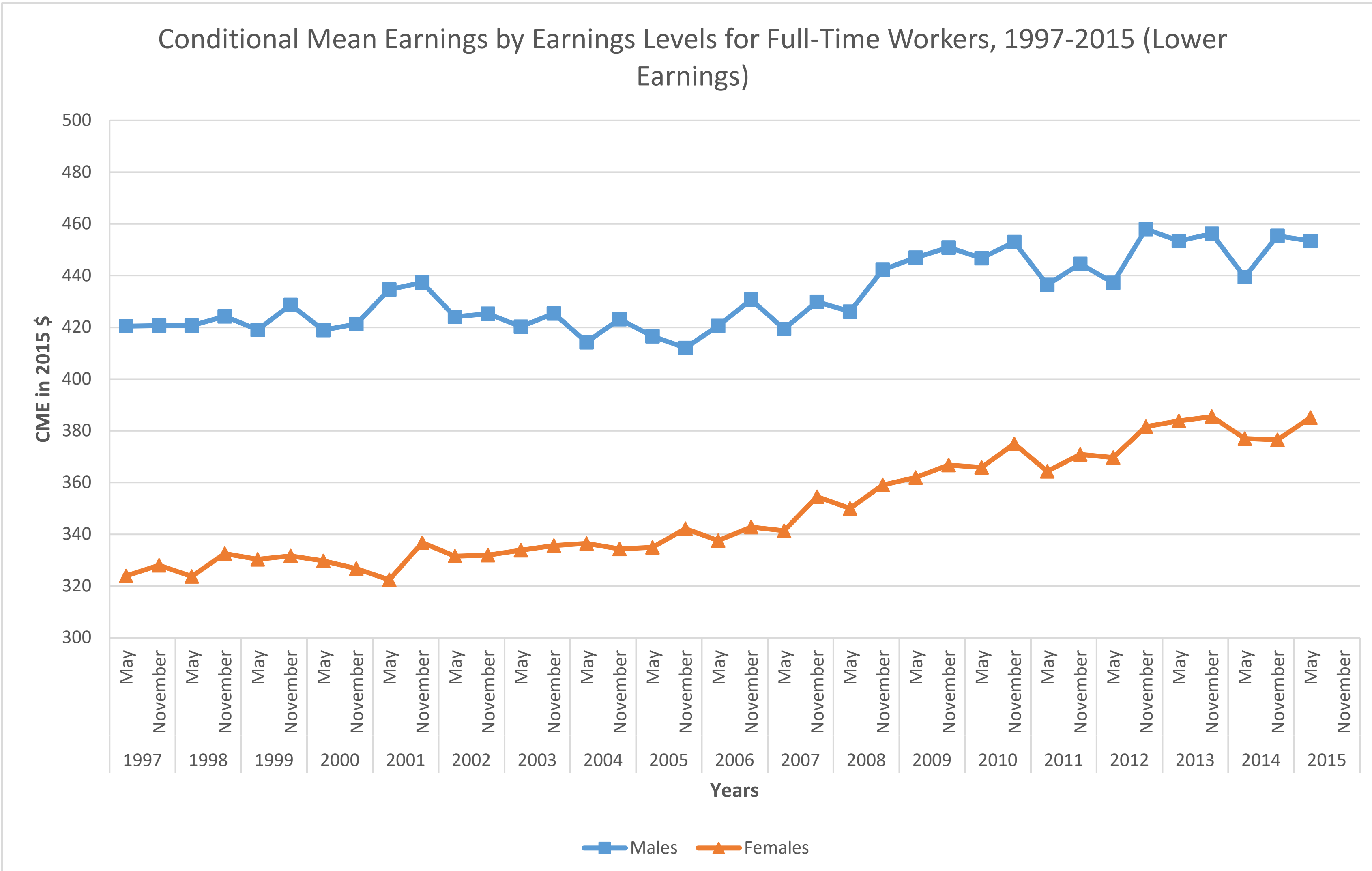
Source: LFS microdata files; calculations by author.

Figure 25



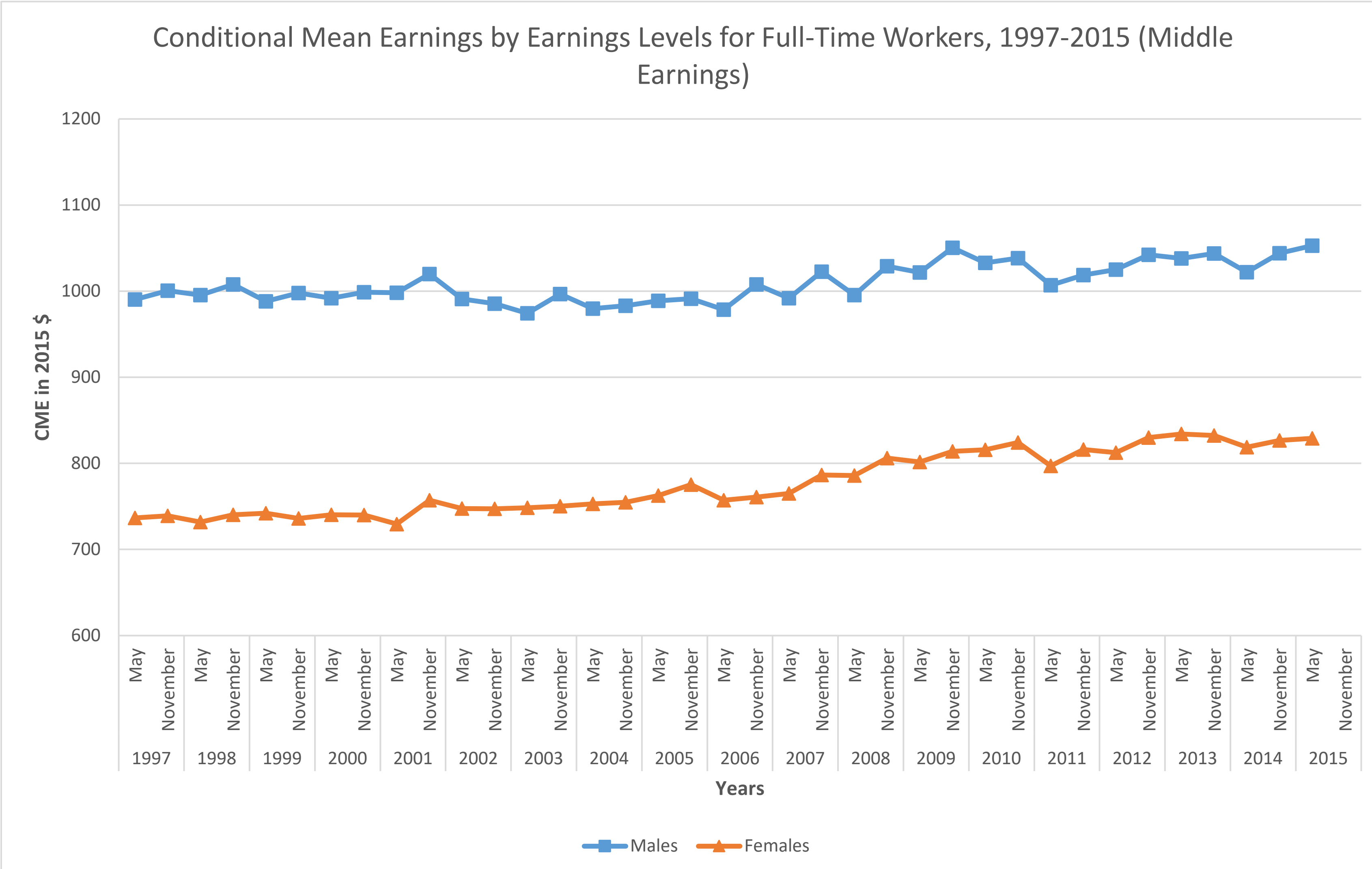
Source: LFS microdata files; calculations by author.

Figure 26



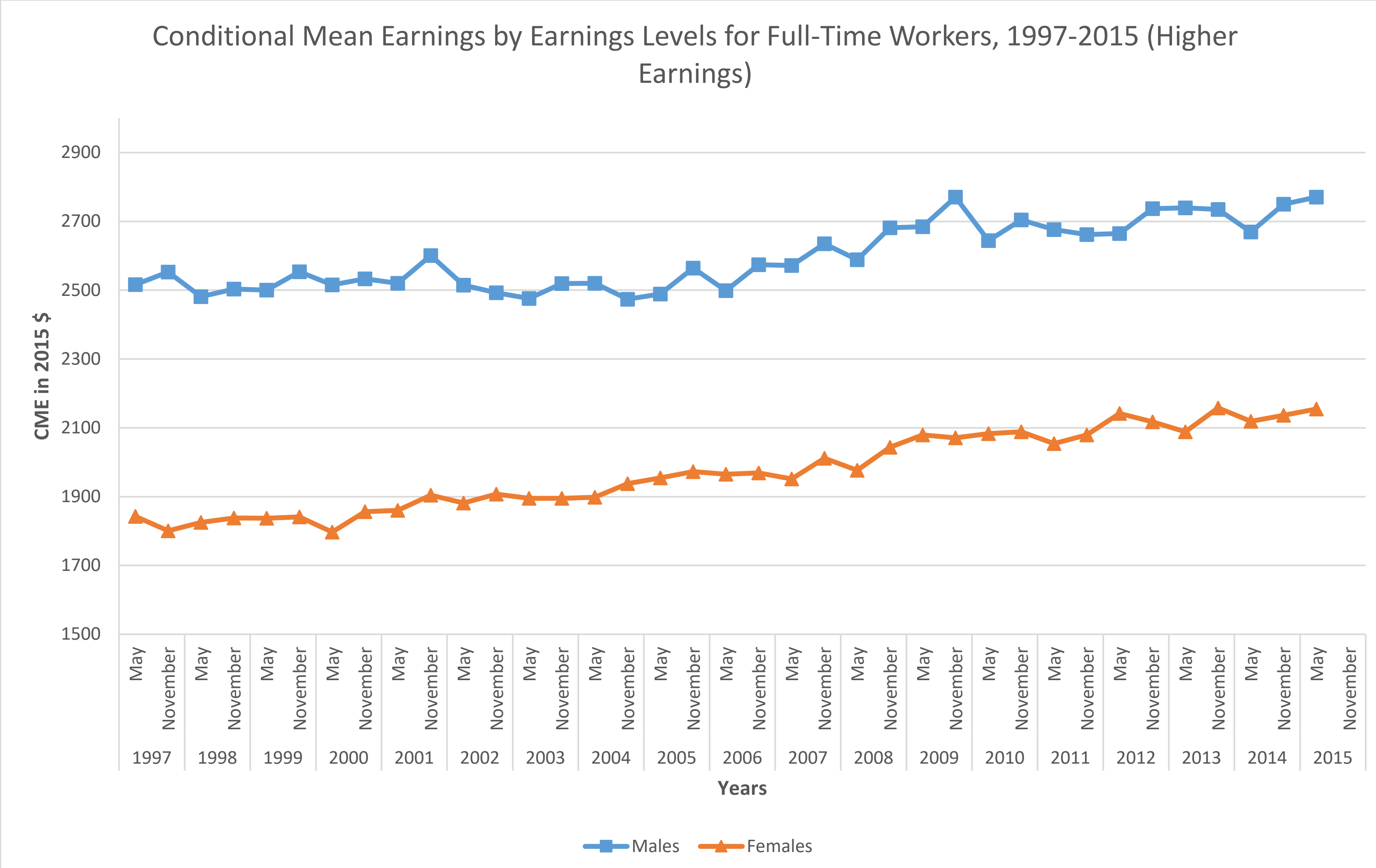
Source: LFS microdata files; calculations by author.

Figure 27



Source: LFS microdata files; calculations by author.

Figure 28



Source: LFS microdata files; calculations by author.

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Appendix A

Appendix Table A1
Summary Statistics on Annual Earnings for Census Estimation Samples for Males
Selective Years 1970-2005
(real 2015 dollars)

	1970	1980	1990	2000	2005
All Workers					
No. obs.	59,123	143,248	234,636	227,828	236,968
Mean earnings	41,963	49,149	48,934	49,976	56,587
Median earnings	37,635	45,478	43,850	42,310	41,650
MC earnings range	18,817-56,451	22,740-68,218	21,926-65,776	21,155-63,465	20,825-62,475
Mean MC earnings	37,947	45,589	44,144	41,670	40,922
Lower earnings cut-off	18,817	22,740	21,926	21,155	20,825
Higher earnings cut-off	75,269	90,956	87,702	84,621	83,300
Full-Time Workers					
No. obs.	28,405	68,614	122,859	121,923	124,231
Mean earnings	55,552	65,238	64,612	65,366	74,474
Median earnings	49,552	59,490	58,468	57,258	58,310
MC earnings range	24,776-74,329	29,745-89,236	29,234-87,702	28,629-85,888	29,155-87,465
Mean MC earnings	47,971	58,713	57,166	55,446	56,151
Lower-earnings cut-off	24,776	29,745	29,234	28,629	29,155
Higher earnings cut-off	99,104	118,981	116,937	114,516	116,620

Note: Based on Census public use microdata files.
Inflation adjustment based on CPI.

Appendix Table A2
Summary Statistics on Annual Earnings for Census Estimation Samples for Females
Selective Years 1970-2005
(real 2015 dollars)

	1970	**1980**	**1990**	**2000**	**2005**
All Workers					
No. obs.	32,164	101,619	196,143	202,491	217,264
Mean earnings	20,245	25,724	29,394	32,689	35,790
Median earnings	18,190	22,572	24,588	26,694	28,560
MC earnings range	9,095-27,286	11,286-33,858	12,294-36,880	13,347-40,041	14,280-42,840
Mean MC earnings	18,322	22,095	24,048	26,306	27,620
Lower earnings cut-off	9,095	11,286	12,294	13,347	14,280
Higher earnings cut-off	36,381	45,144	49,176	53,388	57,120
Full-Time Workers					
No. obs.	8,608	30,653	78,693	87,871	94,693
Mean earnings	32,932	42,065	44,076	47,279	52,076
Median earnings	30,735	39,068	40,603	42,710	44,030
MC earnings range	15,368-46,102	19,534-58,600	20,301-60,904	21,355-64,065	22,015-66,045
Mean MC earnings	29,846	38,278	39,833	41,451	42,528
Lower-earnings cut-off	15,368	19,534	20,301	21,355	22,015
Higher earnings cut-off	61,469	78,135	81,206	85,421	88,060

Note: Based on Census public use microdata files.
Inflation adjustment based on CPI.

Appendix Table A3
Summary Statistics on Weekly Earnings for LFS Estimation Samples for Males
Selective Years 2000-2015
(real 2015 dollars)

	2000	**2005**	**2010**	**2015**
All Workers				
No. obs.	25,511	25,831	26,621	51,680
Mean earnings	968.6	965.6	1031.2	1053.6
Median earnings	906.4	889.2	939.8	957.0
MC earnings range	453.2-1359.6	444.6-1333.8	469.9-1409.7	478.5-1435.5
Mean MC earnings	895.4	864.0	904.9	915.6
Lower earnings cut-off	453.2	444.6	469.9	478.5
Higher earnings cut-off	1812.9	1778.3	1879.5	1913.9
Full-Time Workers				
No. obs.	19,476	19,047	19,268	36,678
Mean earnings	1106.5	1110.5	1185.1	1215.5
Median earnings	1025.3	1021.5	1075.8	1105.4
MC earnings range	512.7-1538.0	510.7-1532.2	537.9-1613.6	552.7-1657.9
Mean MC earnings	990.4	987.2	1031.2	1051.2
Lower-earnings cut-off	512.7	510.7	537.9	552.7
Higher earnings cut-off	2050.7	2043.0	2151.5	2210.6

Note: Based on May Labour Force Surveys.

Appendix Table A4
Summary Statistics on Weekly Earnings for LFS Estimation Samples for Females
Selective Years 2000-2015
(real 2015 dollars)

	2000	**2005**	**2010**	**2015**
All Workers				
No. obs.	23,917	25,414	27,422	51,658
Mean earnings	678.3	707.3	772.4	801.3
Median earnings	614.8	638.3	691.8	711.7
MC earnings range	307.4-922.3	319.2-957.5	345.9-1037.7	355.8-1067.7
Mean MC earnings	590.6	617.2	664.1	684.4
Lower earnings cut-off	307.4	319.2	345.9	355.8
Higher earnings cut-off	1229.7	1276.6	1383.5	1423.5
Full-Time Workers				
No. obs.	14,979	15,842	17,105	32,052
Mean earnings	846.5	880.6	959.9	998.0
Median earnings	769.0	798.1	863.2	881.8
MC earnings range	384.6-1153.6	399.0-1197.1	431.6-1294.9	441.0-1322.8
Mean MC earnings	739.1	761.4	814.5	827.8
Lower earnings cut-off	384.6	399.0	431.6	441.0
Higher earnings cut-off	1538.0	1596.1	1726.3	1763.7

Note: Based on May Labour Force Surveys.

Table A5
Standard Errors on Relative Mean Earnings of Male and Female Workers by Earnings Level, Canada, 1970-2005:
Census Data on Annual Earnings

	Males		Females	
	All Workers	**FTFY Workers**	**All Workers**	**FTFY Workers**
Lower Earnings **(below 50% of median)**				
1970	.00212	.00326	.00222	.00689
1980	.00135	.00205	.00128	.00358
1990	.00096	.00227	.00087	.00134
2000	.00091	.00148	.00092	.00178
2005	.00073	.00122	.00082	.00141
Middle-Class Earnings **(within 50% of median)**				
1970	.00305	.00248	.00584	.00436
1980	.00214	.00160	.00326	.00236
1990	.00177	.00176	.00223	.00144
2000	.00189	.00156	.00203	.00187
2005	.00172	.00149	.00196	.00180
Higher Earnings **(above 200% of median)**				
1970	.01502	.02086	.00914	.03088
1980	.00755	.01242	.00482	.01282
1990	.00522	.01209	.00349	.00844
2000	.00438	.00813	.00352	.00682
2005	.00641	.01583	.00473	.01084

Source: Calculations by author from formulas in section 2, based on Census PUMF files for 1971, 1981, 1991, 2001, and 2006.

Table A6
Standard Errors on Relative Mean Earnings of Male and Female Workers by Earnings Level, Canada, 2000-2015:
LFS Data on Weekly Earnings

	Males		Females	
	All Workers	**FT Workers**	**All Workers**	**FT Workers**
Lower Earnings **(below 50% of median)**				
2000	.00298	.00158	.00266	.00137
2005	.00242	.00106	.00213	.00103
2010	.00210	.00073	.00191	.00078
2015	.00118	.00048	.00121	.00036
Middle-Class Earnings **(within 50% of median)**				
2000	.00337	.00210	.00397	.00247
2005	.00340	.00233	.00373	.00268
2010	.00338	.00262	.00368	.00282
2015	.00253	.00209	.00261	.00236
Higher Earnings **(above 200% of median)**				
2000	.01121	.00713	.00700	.00506
2005	.01106	.00916	.00933	.00976
2010	.01011	.01313	.00895	.01115
2015	.00797	.01294	.00664	.01107

Source: Calculations by author from formulas in section 2, based on May Labour Force Survey PUMF files for indicated years.

Table A7
Standard Errors on Conditional Mean Earnings of Male and Female Workers by Earnings Level, Canada, 1970-2005:
Census Data on Annual Earnings

	Males		**Females**	
	All Workers	**FTFY Workers**	**All Workers**	**FTFY Workers**
Lower Earnings **(below 50% of median)**				
1970	89.0	181.0	45.0	227.0
1980	66.5	133.9	32.9	150.4
1990	46.9	146.5	25.4	58.9
2000	45.6	96.4	30.0	84.3
2005	41.5	90.5	29.2	73.5
Middle-Class Earnings **(within 50% of median)**				
1970	128.2	135.1	118.3	143.6
1980	105.3	104.1	83.9	99.3
1990	86.5	113.9	65.6	63.7
2000	94.3	102.2	66.5	88.2
2005	97.1	110.8	70.3	93.6
Higher Earnings **(above 200% of median)**				
1970	630.5	1159.	185.1	1017.
1980	371.3	810.6	124.1	539.2
1990	255.5	781.1	102.6	372.1
2000	218.7	531.4	114.9	322.5
2005	362.9	1179.	169.4	564.7

Source: Calculations by author from formulas in section 4, based on Census PUMF files for 1971, 1981, 1991, 2001, and 2006.

Table A8
Standard Errors on Conditional Mean Earnings of Male and Female Workers by Earnings Level, Canada, 2000-2015:
LFS Data on Weekly Earnings

	Males		Females	
	All Workers	**FT Workers**	**All Workers**	**FT Workers**
Lower Earnings **(below 50% of median)**				
2000	2.88	1.75	1.80	1.16
2005	2.34	1.18	1.50	0.91
2010	2.17	0.87	1.47	0.75
2015	1.25	0.59	0.97	0.36
Middle-Class Earnings **(within 50% of median)**				
2000	3.26	2.33	2.69	2.09
2005	3.28	2.58	2.64	2.36
2010	3.48	3.10	2.84	2.71
2015	2.66	2.54	2.09	2.35
Higher Earnings **(above 200% of median)**				
2000	10.86	7.89	4.75	4.29
2005	10.68	10.17	6.60	8.59
2010	10.42	15.56	6.91	10.70
2015	8.40	15.73	5.32	11.04

Source: Calculations by author from formulas in section 4, based on May Labour Force Survey PUMF files for indicated years.

Table A9
"t-Ratios" for Time Interval Differences in Relative Mean Earnings of Male and Female Workers by Earnings Level, Canada, 1970-2015

	Males		Females	
	All Workers	**FT Workers**	**All Workers**	**FT Workers**
Lower Earnings **(below 50% of median)**				
1970-80	2.76	3.67	-3.20	-0.70
1980-90	-3.22	-7.50	-0.06	1.11
1990-00	-11.0	-1.61	-12.0	-6.79
2000-05 (SCF)	-1.01	-1.83	0.58	-5.12
2005-10	-0.93	1.18	-2.21	0.11
2010-15	4.27	-3.74	-0.26	5.74
Middle-Class Earnings **(within 50% of median)**				
1970-80	6.23	12.5	-6.88	0.79
1980-90	-9.14	-6.41	-10.3	-2.26
1990-00	-26.4	-15.5	-4.49	-11.4
2000-05 (SCF)	-6.18	-1.98	0.32	-2.35
2005-10	-3.60	-5.36	-2.47	-4.09
2010-15	-2.02	-1.60	-1.21	-5.24
Higher Earnings **(above 200% of median)**				
1970-80	-10.5	-8.16	-4.81	-4.69
1980-90	-2.42	3.51	-7.50	5.87
1990-00	-11.2	-5.56	0.38	-6.96
2000-05 (SCF)	-2.27	-2.51	6.55	8.99
2005-10	-3.25	-0.60	-3.21	-3.44
2010-15	1.00	2.65	-1.78	-0.58

Source: Based on results in Tables A5 and A6.

Table A10
"t-Ratios" for Time Interval Differences in Conditional Mean Earnings of Male and Female Workers by Earnings Level, Canada, 1970-2015

	Males		Females	
	All Workers	**FT Workers**	**All Workers**	**FT Workers**
Lower Earnings **(below 50% of median)**				
1970-80	15.9	16.0	15.9	8.80
1980-90	-3.75	-8.38	16.9	4.67
1990-00	-7.95	-0.47	3.57	1.96
2000-05 (SCF)	-1.23	-1.11	3.91	3.75
2005-10	4.70	20.3	6.06	25.7
2010-15	6.78	7.07	4.07	23.3
Middle-Class Earnings **(within 50% of median)**				
1970-80	46.1	63.0	26.0	48.3
1980-90	-10.6	-10.0	18.4	13.2
1990-00	-19.3	-11.2	24.1	14.9
2000-05 (SCF)	-6.77	-0.95	7.02	7.04
2005-10	8.55	10.9	12.1	14.8
2010-15	2.44	4.97	5.80	3.66
Higher Earnings **(above 200% of median)**				
1970-80	15.7	9.69	55.0	13.8
1980-90	-3.67	2.07	47.0	13.1
1990-00	-3.26	-3.52	51.2	8.31
2000-05 (SCF)	-2.73	-1.81	14.7	16.6
2005-10	6.84	8.37	12.5	9.25
2010-15	4.81	5.75	5.74	4.78

Source: Based on results in Tables A7 and A8.

Appendix B

Statistical Inference for Percentile-Based RMY and CMY Statistics

B.1 RMY Statistics

The problem this appendix addresses is extending the principles of statistical inference – based on the lognormal inference approach – to measures of relative-mean income and conditional mean income which are based on percentile income cut offs. That is, the conditioning income intervals for the lower-earnings (LE), middle-class workers (MC), and higher-earnings (HE) groups are expressed in terms of *percentile values* such as the bottom 20 percent or middle 60 percent of workers, rather than having the income intervals expressed in terms of multiples of the median income level of a distribution. For illustrative purposes, in this appendix, the LE group will consist of workers with income in the lower 100 p_1 percent of the distribution, the HE group consists of the top 100(1 - p_2) percent of the distribution, and the MC group includes the rest of the workers in the middle 100(p_2 - p_1) percent of the distribution. In the case of quintile income data provided by Statistics Canada in their CANSIM distributional series, p_1 = .20, p_2 = .80 and the middle-class group of workers lie in the middle three quintiles or the middle 80 – 20 = 60 percent of the distribution.

For expositional purposes, let us focus on the middle-class income group whose income share is defined by

$$IS_{MC} \equiv \int_{\xi_1}^{\xi_2} \frac{1}{\propto} x f(x)dx \qquad (1)$$

where $f(\cdot)$ is again the income density function, $\propto \equiv E(x)$, and ξ_1 is the p_1'th percentile income cut-off level and ξ_2 is the p_2'th percentile cut-off income value. $\hat{\xi}_1$ and $\hat{\xi}_2$ are the corresponding sample cut-off values, and the sample income share $\widehat{IS}_{MC}$ is calculated as the proportion of total income received by the income recipients with incomes in the sample range $(\hat{\xi}_1, \hat{\xi}_2]$.

In Beach (2016) it is established that – under similar conditions to those underlying the median-based approach used in the main body of the present paper – $\widehat{IS}_{MC}$ is asymptotically normal with (asymptotic) variance given by

$$\text{Asy. Var}(\widehat{IS}_{MC}) = \left[\frac{\xi_1}{\propto} \cdot f(\xi_1)\right]^2 v_{11} \; - 2\left[\frac{\xi_1 \cdot f(\xi_1) \cdot \xi_2 \cdot f(\xi_2)}{\propto^2}\right] v_{12}$$

$$+ \left[\frac{\xi_2}{\propto} \cdot f(\xi_2)\right]^2 v_{22} \qquad (2)$$

where $\begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix} = \begin{bmatrix} \frac{p_1(1-p_1)}{f(\xi_1)^2} & \frac{p_1(1-p_2)}{f(\xi_1) \cdot f(\xi_2)} \\ \frac{p_1(1-p_2)}{f(\xi_1) \cdot f(\xi_2)} & \frac{p_2(1-p_2)}{f(\xi_2)^2} \end{bmatrix}$.

Therefore,

$$\text{Asy. Var}(\widehat{IS}_{MC}) = \left[\frac{\xi_1}{\propto}\right]^2 p_1(1-p_1) \; - 2\left[\frac{\xi_1 \cdot \xi_2}{\propto^2}\right] p_1(1-p_2) \; + \; \left[\frac{\xi_2}{\propto}\right]^2 p_2(1-p_2)\,.$$

Hence $S.E.\left(\widehat{IS}_{MC}\right) = \left[\widehat{\text{Asy. Var}}\left(\widehat{IS}_{MC}\right) / N\right]^{1/2}$ (3)

where N is again the sample size from which IS_{MC} has been calculated.

By construction, the percentile population share, PS_{MC}, is always given by (p_2 - p_1) for the MC group and hence is not random.

It is still the case, though, that the relative-mean income ratio

$$RMY_{MC} \; \equiv \; \frac{E(x \,/\, MC)}{E(x)} = IS_{MC} \,/\, PS_{MC}$$

$$= IS_{MC} \,/\, (p_2 \; - \; p_1)\,, \qquad (4)$$

which is simply a constant proportionality factor multiplied by IS_{MC}. Hence, for $\widehat{RMY}_{MC} \; \equiv \; \widehat{IS}_{MC}$ / (p_2 - p_1), it can simply be seen that $\widehat{RMY}_{MC}$ is also asymptotically normal with

$$\text{Asy. Var}\left(\widehat{RMY}_{MC}\right) = (p_2 - p_1)^{-2} \cdot \text{Asy. Var}\left(\widehat{IS}_{MC}\right), \tag{5}$$

and hence

$$S.E.\left(\widehat{RMY}_{MC}\right) = S.E.\left(\widehat{IS}_{MC}\right) / (p_2 - p_1) . \tag{6}$$

The results for the LE and HE income groups are obtained in similar fashion, but turn out to be simpler in expression. It was also established in Beach (2016) that $\widehat{IS}_{LE}$ and $\widehat{IS}_{HE}$ are each also asymptotically normally distributed with (asymptotic) standard errors

$$S.E.\left(\widehat{IS}_{LE}\right) = p_1(1 - p_1)\left[\frac{\hat{\xi}_1}{\hat{\propto}}\right] / \sqrt{N}$$

and

$$S.E.\left(\widehat{IS}_{HE}\right) = p_2(1 - p_2)\left[\frac{\hat{\xi}_2}{\hat{\propto}}\right] / \sqrt{N} .$$

So it follows that $RMY_{LE} = IS_{LE}/ p_1$ and $RMY_{HE} = IS_{HE}/ (1 - p_2)$. Consequently, $\widehat{RMY}_{LE}$ and $\widehat{RMY}_{HE}$ are also each asymptotically normal with

$$S.E.\left(\widehat{RMY}_{LE}\right) = p_1^{-1} \cdot S.E.\left(\widehat{IS}_{LE}\right)$$

$$= (1 - p_1)\left[\frac{\hat{\xi}_1}{\hat{\propto}}\right] / \sqrt{N} . \tag{7}$$

and

$$S.E.\left(\widehat{RMY}_{HE}\right) = (1 - p_2)^{-1} \cdot S.E.\left(\widehat{IS}_{LE}\right)$$

$$= p_2\left[\frac{\hat{\xi}_2}{\hat{\propto}}\right] / \sqrt{N} . \tag{8}$$

Note, interestingly, that in the case of percentile-based estimates, the terms $f(\xi_1)$ and $f(\xi_2)$ fall out, so that the resulting asymptotic variance and standard error formulas all turn out to be distribution-free – consistent with earlier findings in Beach and Davidson (1983).

B.2 CMY Statistics

With respect to the conditional mean income, CMY_i for $i = LE, MC, HE$, recall from eq. (51) in the text that

$$CMY_i = \propto \cdot \; IS_i \; / \; PS_i$$

where $\propto = E(x)$. These are estimated by calculating the sample average income levels lying within the sample ranges given by $\hat{\xi}_1$ and $\hat{\xi}_2$. From eq. (4) of this appendix, for $i = MC$,

$$CMY_{MC} = \propto \cdot \; IS_{MC} \; / \; (p_2 - p_1).$$

Again, this is a proportionality constant times IS_{MC} . So $\widehat{CMY}_{MC}$ is also asymptotically normally distributed with

$$\text{Asy. Var}\left(\widehat{CMY}_{MC}\right) = \frac{\propto^2}{(p_2 - p_1)^2} \cdot \text{Asy. Var}\left(\widehat{IS}_{MC}\right)$$

and hence

$$\text{S. E.}\left(\widehat{CMY}_{MC}\right) = \left(\frac{\propto}{p_2 - p_1}\right) \cdot \text{S. E.}\left(\widehat{IS}_{MC}\right). \qquad (9)$$

Corresponding results for the LE and HE income groups can be obtained in similar fashion. Again, $\widehat{CMY}_{LE}$ and $\widehat{CMY}_{HE}$ are each asymptotically normal with standard errors

$$S.E.\left(\widehat{CMY}_{LE}\right) = \frac{\hat{\propto}}{p_1} \cdot S.E.\left(\widehat{IS}_{LE}\right) / \sqrt{N}$$

$$= \frac{\hat{\propto}}{p_1}\left[p_1(1 - p_1) \; \frac{\hat{\xi}_1}{\hat{\propto}}\right] / \sqrt{N}$$

$$= (1 - p_1)\hat{\xi}_1 \,/\, \sqrt{N} \qquad (10)$$

and $$S.E.\left(\widehat{CMY}_{HE}\right) = \frac{\hat{\propto}}{1 - p_2}\left[p_2(1 - p_2)\,\frac{\hat{\xi}_2}{\hat{\propto}}\right] / \sqrt{N}$$

$$= p_2\,\hat{\xi}_2 \,/\, \sqrt{N}\,. \qquad (11)$$

Once again, note how the above expressions turn out to be distribution-free as well.