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The Krusell-Smith Algorithm: Are Self-fulfilling Equilibria Likely?

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Abstract

I investigate whether the popular Krusell and Smith algorithm used to solve heterogeneous-agent economies with aggregate uncertainty and incomplete markets is likely to be subject to multiple self-fulfilling equilibria. In a benchmark economy, the parameters representing the equilibrium aggregate law of motion are randomly perturbed 500 times, and are used as the new initial guess to compute the equilibrium with this algorithm. In a sequence of cases, differing only in the magnitude of the perturbations, I do not find evidence of multiple self-fulfilling equilibria. The economic reason behind the result lies in a self-correcting mechanism present in the algorithm: compared to the equilibrium law of motion, a candidate one implying a higher (lower) expected future capital reduces (increases) the equilibrium interest rates, increasing (reducing) the savings of the wealth-rich agents only. These, on the other hand, account for a small fraction of the population and cannot compensate for the opposite change triggered by the wealth-poor agents, who enjoy higher (lower) future wages and increase (reduce) their current consumption. Quantitatively, the change in behavior of the wealth-rich agents has a negligible impact on the determination of the change in the aggregate savings, inducing stability in the algorithm as a by-product.

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1 Introduction

This paper investigates whether the popular Krusell and Smith (KS) algorithm, used to solve heterogeneous-agent economies with aggregate uncertainty and incomplete markets, is likely to be subject to multiple Self-Fulfilling Equilibria (SFE). This possibility arises because the equilibrium Aggregate Law of Motion (ALM) is unknown and needs to be computed through a guess-and-verify iterative procedure. Crucially, the agents' optimal decision rules have to be calculated at each step of this fixed-point problem, but they in turn depend on the ALM being tried. In principle, this process can lead to a complementarity between the guess related to the agents' perception of the evolution of future prices and their implied choices.

This method was first proposed by Krusell and Smith (1998), and it has been successfully applied to a wide variety of problems. Notable examples include the pricing and allocation of risky and safe assets (Krusell and Smith (1997), Pijoan-Mas (2007) and Storesletten et al. (2007)), the magnitude of welfare costs due to business cycles (Castaneda et al. (1998), Mukoyama and Sahin (2006), and Krusell et al. (2009)), fluctuations in frictional labor markets (Gomes et al. (2001) and Nakajima (2012)), the determinants of fiscal policy (Heathcote (2005)), and the analysis of rising wage inequality in a political economy framework (Corbae et al. (2009)).

Where does the possibility of multiple SFE stem from? In the simplest environment, households only choice concerns their savings. If postulating an ALM for capital that is above (below) the equilibrium one leads to more (less) resources being saved by the households in the aggregate, we would be in a situation displaying complementarity between the guessed aggregate dynamics of capital and the resulting saving behavior. This instance could lead to multiple SFE, and it has been acknowledged by Krusell and Smith (2006), who argued for the absence of multiple SFE by analyzing a simple two-period model. Since the uniqueness of the equilibrium is impossible to prove analytically, this paper undertakes a systematic study on the subject with a numerically intensive procedure, considering the full-blown version of their model, with both infinitely-lived agents and preference heterogeneity.

My investigation is complementary to the analysis performed in Young (2005), who assesses the robustness of the KS algorithm along several dimensions. In particular, he argues for the absence of SFE by working with a version of the model with heterogeneous beliefs. Unlike him, I rely on a Monte Carlo analysis of the KS economy. Through appropriately designed perturbation experiments, I do not find evidence supporting the existence of SFE such that the parameters representing the ALM converge to different values, depending on the initial guess. Most replications tend to cluster around two different values of the ALM, and converge to ALM parameters that differ from the equilibrium one, but the discrepancy is always quantitatively negligible. Furthermore, increasing the accuracy of the numerical procedure until hitting a feasibility boundary due to the computational time reduces this difference by an order of magnitude.[1] Although it is difficult to disentangle the gap from the numerical error induced by the discretization of the state space, the sampling variability arising from the simulations and the convergence criteria, this finding makes the numerical error the most likely culprit behind the differences between the postulated equilibrium ALM and the sequence of converged ALM's found in the replications. In terms of substance, even if one is willing to consider these alternative ALM's as different equilibria, the implied differences are always quantitatively minimal.

The findings show that, although the economy features a wealth distribution with a fat right tail, the share of agents increasing their savings in response to ALM's that overpredict the future aggregate capital is well below 0.5%. In particular, agents need to have accumulated no less than 100 and up to 300 times the average income for the negative income effect to start prevailing, leading them to increase their savings. It follows that, in the aggregate, this change is dominated by the reduction in savings of the vast majority of agents, and multiple SFE are unlikely to arise.

The rest of the paper is organized as follows. Section 2 briefly presents the model and the calibration. Section 3 discusses the role of the ALM in the KS model. Section 4 outlines the perturbation experiments. Section 5 describes the main results, while Section 6 concludes. Two Appendices report the complete calibration and discuss in more detail the numerical methods used. Another Appendix presents some additional results and a set of robustness exercises.

2 Model

The model is similar to the set-up in Krusell and Smith (1998) with preference heterogeneity, except for the availability of unemployment benefits for workers without a job. Following den Haan et al. (2010), I assume a budget-balanced Unemployment Insurance (UI) scheme.

I consider a production economy with aggregate shocks in which agents face different employment histories and self-insure by accumulating a single risky asset. A borrowing constraint (b) potentially prevents agents from borrowing the desired amount of resources in periods where they obtain a low income.

Technology: The production side of the economy is modeled as a constant returns to scale technology of the Cobb-Douglas form, which relies on aggregate capital K_t and labor L_t to produce final output $Y_t = z_t K_t^{\alpha} L_t^{1-\alpha}$. The aggregate shock takes only two values: $z_t = \{z_G, z_B\}$, with $z_G = 1.01 > z_B = 0.99$. The aggregate shock follows a symmetric first-order Markov chain, whose transition matrix is reported in Appendix A, and it is such that booms and recessions last the same number of periods. Capital depreciates at the exogenous rate δ and firms hire capital and labor every period from competitive markets. The agents' time endowment is normalized to 1, and total labor services are $L_t = lN_t$,

[1]When run on top-of-the-line workstations, each experiment takes more than two weeks to complete. However, the different experiments can be run in parallel.

namely they are the product of the employment rate N_t and l, the share of the time endowment devoted to market activities. The first order conditions give the expressions for the net real return to capital r_t and the wage rate w_t:

$$r_t = \alpha z_t \left(\frac{L_t}{K_t}\right)^{1-\alpha} - \delta, \tag{1}$$

$$w_t = (1-\alpha)\, z_t \left(\frac{K_t}{L_t}\right)^{\alpha}. \tag{2}$$

Government: The government taxes the employed agents' labor income at rate τ_t to finance a budget-balanced UI scheme. Unemployed agents receive UI benefits equal to a fixed replacement rate ρ of the going labor income. Since labor supply is fixed, and the aggregate unemployment rate can only take two values ($u_t = 0.04$ when $z_t = z_G$ and $u_t = 0.10$ when $z_t = z_B$), the equilibrium tax rate is $\tau_t = \rho(1-N_t)/N_t$, with $N_t = 1 - u_t$. The transition matrix for the employment opportunities is reported in Appendix A.

Households: Agents' preferences are assumed to be represented by a time-separable utility function $U(.)$. Every household $i \in [0,1]$ chooses consumption ($c_{i,t}$) and future asset holdings ($a_{i,t+1}$) to maximize expected discounted utility:

$$\max_{\{c_{i,t},a_{i,t+1}\}_{t=0}^{\infty}} E_0 \sum_{t=0}^{\infty} \beta_{i,t}^{t} \frac{c_{i,t}^{1-\gamma} - 1}{1-\gamma}$$

where E is the expectation operator. $\beta_{i,t} \in (0,1)$ is the agents' discount factor, which varies over time according to the transition matrix reported in Appendix A, and can take three different values, $\beta_{i,t} \in \{\beta_l, \beta_m, \beta_h\}$, with $\beta_l < \beta_m < \beta_h$. Agents can be employed, $s = e$, or unemployed, $s = u$. The employment probabilities follow a first-order Markov process, depend on both the idiosyncratic employment status s and on the aggregate state of the economy z. The related transition matrix is reported in Appendix A. I use recursive methods to solve the model, and the value function associated with this problem is denoted with $V(a, s, \beta, z, K)$. This represents the expected lifetime utility of an agent whose current asset holdings are equal to a, whose current employment status is s, whose current discount factor is β, facing the aggregate shock z, in an economy with K units of aggregate capital. The Bellman equation is:

$$V(a,s,\beta,z,K)=\max_{c,a'}\left\{\frac{c^{1-\gamma}-1}{1-\gamma}+\beta E_{\beta',s',z'|\beta,s,z}V(a',s',\beta',z',K')\right\}$$

$$s.t.$$

$$c+a'=(1+r)a+(1-\tau)wl,\ \text{if } s=e$$

$$c+a'=(1+r)a+\rho wl,\ \text{if } s=u$$

$$c\geq 0,\quad a'\geq b$$

$$\ln K'=\theta_{0,G}+\theta_{1,G}\ln K,\ \text{if } z=z_G \tag{3}$$

$$\ln K'=\theta_{0,B}+\theta_{1,B}\ln K,\ \text{if } z=z_B \tag{4}$$

Agents have to optimally set their consumption/savings plans. They enjoy utility from consumption, and face some uncertain events in the future. Notice that, according to the most parsimonious implementation of the KS algorithm, the relevant state variable in the agents' problem is just aggregate capital K, rather than the whole current endogenous distribution over idiosyncratic states. Hence, the agents forecast future prices relying on the (equilibrium) evolution of the aggregate capital stock, the ALM being specified as the pair of equations (3) and (4), which are commented upon in more detail in the following section.

The calibration of the model's parameters is standard and they are presented in Table 1.

[Table 1 about here]

Calibration: The calibration follows for the most part Krusell and Smith (1998) and den Haan et al. (2010). The only differences pertain to the CRRA parameter γ, the borrowing constraint b, and the discount factors $\{\beta_l,\beta_m,\beta_h\}$. While Krusell and Smith (1998) and den Haan et al. (2010) set $\gamma=1$, I work with $\gamma=2$. Although my choice for this parameter is still well inside the range of available estimates of the Intertemporal Elasticity of Substitution, compared to the log case it implies a relatively stronger income effect. As it will be discussed below, a higher γ increases the likelihood of SFE. Krusell and Smith (1998) and den Haan et al. (2010) consider $b=0$, namely the extreme case of a no-borrowing constraint. I set $b=-1.8$ instead, for the time series average of the share of households in debt to be approximately 10%, which is a more empirically relevant value.[2] As for the discount factors, I rely on the same

[2]The results with the alternative value $b=0$ are both qualitatively and quantitatively very similar to the ones of the baseline calibration, and are reported in an Appendix.

transition matrix specified by Krusell et al. (2009), while I adjust the β's to match a wealth Gini index of 0.8 together with an average quarterly interest rate of 1%.

Numerical Methods: Since there are several techniques used to solve the KS economy, it is worthwhile to present the actual computational methods I rely on. Following the taxonomy of den Haan (2010), I use a hybrid procedure: a projection approach to solve for the individual policy rules, coupled with stochastic simulation techniques to solve for the ALM. In particular, for the household problem I use a time iteration procedure on the set of Euler equations, guessing the future saving functions, and solving for the current ones with a bi-linear interpolation scheme in the (a, K) dimensions.[3] I look for the policy functions such that the residuals of the Euler equations are (close to) zero at the asset grid points. It follows that for all possible combinations of state variables I need to solve a non-linear equation. To get the optimal policy functions, I compute the first order conditions with respect to a' and through the envelope condition I obtain a set of Euler equations, whose unknowns are the policy functions, $a'(a, s, \beta, z, K)$. I start from a set of guesses, $a'(a, s, \beta, z, K)_0$, and keep on iterating until a fixed point is reached, i.e. until two successive iterations satisfy the criterion that the maximum distance between two iterations is less than 10^{-7} at all possible combinations of states. The ALM parameters are considered as converged when the maximum difference between the sum of all their distances between two iterations (in absolute value) is less than 10^{-5}. The aggregate dynamics are computed by simulating a large sample of $30,000$ individuals for $5,000$ periods, with the first $1,000$ periods being discarded as a burn-in. Given the common finding by practitioners in the field that there is a fair amount of non-linearity in aggregate capital, at the simulation stage I perform a polynomial interpolation of the decision rules in the K dimension. Differently, in the individual capital dimension I rely on a linear approximation scheme.

3 The ALM in the Krusell and Smith (1998) model

As for the ALM, the system (3) and (4) specifies its functional form. Following Krusell and Smith (1998), and most of the papers thereafter, I use a log-linear specification.

Let Θ denote the vector of four parameters $\theta_{j,z}$ representing the ALM, with Θ^* referring to their equilibrium values. Θ^* is obtained by guessing a Θ^g, solving and simulating the model under this guess, computing an update $\Theta^{g'}$ as the parameter estimates of OLS regressions on the simulated data, and repeating

[3]This method is fairly similar to the one thoroughly described in Maliar et al. (2010) and proved to be more stable than the relatively common value function iteration scheme with cubic spline interpolation, used for example by Krusell and Smith (1998), Krusell et al. (2009) and Young (2010). For more details, see the Appendix and chapter 17 in Judd (1998).

these steps until the four parameters in Θ converge. For the economy under study, the values of Θ^* are reported in the system (5).[4] Notice that z_t stands for the time-t aggregate shock, while K_t stands for aggregate capital, and that there are two parameters (an intercept θ_{0,z_t} and a slope θ_{1,z_t}) per aggregate shock.

$$\begin{cases} \ln K_{t+1} = \theta^*_{0,G} + \theta^*_{1,G} \ln K_t = 0.072901 + 0.971442 \times \ln K_t, & \text{for } z_t = z_G = 1.01 \\ \ln K_{t+1} = \theta^*_{0,B} + \theta^*_{1,B} \ln K_t = 0.064744 + 0.972626 \times \ln K_t, & \text{for } z_t = z_B = 0.99 \end{cases} \tag{5}$$

Can this algorithm display SFE? Figure 1 shows the differential response of savings for selected household types, under two different specifications of the ALM. The solid line plots the saving function for the equilibrium ALM, while the dashed line for an alternative ALM, obtained by increasing $\theta^*_{0,G}$ and $\theta^*_{0,B}$ until the implied forecasted aggregate capital is approximately 1% higher than its equilibrium counterpart.[5]

[Figure 1 about here]

It is clear from the figure that asset-rich and asset-poor households react differently to the perturbation of the equilibrium ALM. In particular, rich agents *increase* their savings, while poor ones *reduce* them. The onset of a complementarity between the ALM and individual savings is indeed a possibility, and it crucially depends on both the strength of the individual responses and the shape of the wealth distribution. This is why I focus on the KS economy with preference heterogeneity. Only in this case is the economy able to match a wealth Gini index of 0.8. Not only this is a desirable feature in a model of endogenous wealth accumulation, but I am also making sure that the model delivers an empirically plausible mass of wealth-rich individuals, affecting the likelihood of the above mentioned complementarity.

To further investigate this issue, I conduct a sequence of experiments where the four parameters in Θ^* are perturbed randomly. Each $\theta^*_{j,z}$ is multiplied by the realization of a random variate, drawn from independent uniform distributions, whose supports are set to different ranges in the sequence of experiments. The range of the perturbations is progressively increased, from $\pm 1\%$ to $\pm 25\%$, to allow for a deteriorating quality of the initial guess. Finally, the ALM is updated until it converges again and the procedure is repeated several times, with different initial perturbations.

[4] It goes without saying that the computed Θ^* could be only one of the potentially many equilibria, and that from the perspective of the perturbation experiment it just represents a candidate solution. With some abuse of language, sometimes I am going to refer to Θ^* as the equilibrium ALM, and denote it as ALM*.

[5] All the saving functions are for employed agents ($s = e$), during a boom ($z = z_G$), and with an aggregate capital close to its time average computed in the simulations ($K = 12$). The saving functions for other combinations of state variables are qualitatively similar to the ones reported.

Loosely speaking, this Monte Carlo procedure mimics the actual steps that researchers follow in their search for the equilibrium ALM.[6] As some authors point out, a good initial guess for the ALM is often crucial to the success of the procedure. A sensible choice is represented by the ALM computed for the corresponding economy with complete markets. Here I take a different route, to subject the KS algorithm to a thorough stress test. In this regard, before presenting the details of the Monte Carlo procedure, it is worthwhile discussing some general aspects of the perturbation experiment, and how it relates to the literature.

In order to implement the perturbation experiment, it is necessary to identify a candidate equilibrium ALM. This is done in a preliminary step, by solving the model and obtaining Θ^*. This solution, compared to the sequence of solutions that will be obtained in the perturbation experiments, does not satisfy any additional requirements. By their nature, perturbations are a local concept, and can only be performed around some specific candidate solution, namely the ALM* obtained from one initial run of the KS algorithm. Although ALM* is by no means exceptional, in the presence of multiple SFE the outcome of the Monte Carlo experiments should consists of ALM parameters clustered around several quantitatively distinct values of Θ^*.

Perhaps, the paper closest to mine is Giusto (2014). Even though a perturbation approach is shared by both contributions, the ultimate goal of the analysis, the underlying motivation, and its actual implementation are remarkably different. Where Giusto (2014) addresses the issue of stability of the equilibrium under learning, studying the implications of the latter for wealth inequality and aggregate dynamics, my contribution aims at quantifying potentially many SFE. Consequently, his perturbations represent an accurate approximation of a Jacobian, while mine do not, because they are intentionally designed to lead to initial guesses that are far from Θ^* (to capture the fact that a researcher might start the quest for the equilibrium ALM with a poor guess) and to span a very wide parameter region. Finally, the benchmark KS model without preference heterogeneity he focuses on is less prone to SFE, because in the baseline KS economy wealth concentration is very low, leading to a negligible mass of agents having a response in their savings going in the same direction as the forecasted change in K_{t+1}.

4 The Perturbation Experiment

The procedure used to check for the existence of SFE is:

1. For a given calibration, solve the benchmark economy and store the vector

[6] Several other aspects of the computational framework are usually fine-tuned with a trial and error approach: the choice of the grids, Horvath (2012), the specification of the number of moments needed to accurately forecast the future endogenous state variables, Young (2005), and the functional forms of the forecasting rules, Storesletten et al. (2007). As these additional features are less controversial, I focus on the parameters of the ALM.

of four parameters $\Theta^* = \left[\theta^*_{0,G}, \theta^*_{1,G}; \theta^*_{0,B}, \theta^*_{1,B}\right]$ representing the equilibrium ALM.

2. Choose a grid $X = \{x_1, x_2, ..., x_n\}$ for the perturbation factor x.
3. Set $x = x_1$.
4. Perturb the four parameters by drawing four random variates from a uniform distribution with support $[-x\%, +x\%]$.
5. Check that the new candidate ALM satisfies a requirement of non-explosive dynamics. If not, discard the perturbation.
6. Given the guesses $\Theta^g = \left[\theta^g_{0,G}, \theta^g_{1,G}; \theta^g_{0,B}, \theta^g_{1,B}\right]$, solve the households problem, simulate the economy and update the ALM parameters with a weighted average between the current guess and the parameters resulting from state-dependent OLS regressions on the simulated data.
7. Iterate until convergence of each of the four parameters in Θ and store them.
8. Repeat the procedure 500 times.
9. Move to the next case for x and redo the whole sequence of steps.

In a first batch of experiments, I set $n = 4$ and $X = \{1, 2, 3, 4\}$. In particular, in these experiments all the parameters in Θ^* share the common perturbation factor x.

In another batch of experiments, I still set $n = 4$, but I change the perturbation scheme. For three out of the four parameters the perturbation factor is set to a very small value $x = 0.1$, so that the initial guesses for these parameters are always extremely close to their equilibrium ALM values. Differently, the support of the perturbation factor for the remaining parameter is set to a very large value.[7] This corresponds to $[-10\%, +10\%]$ for $\theta^*_{1,G}$ and $\theta^*_{1,B}$, and to $[-25\%, +25\%]$ for $\theta^*_{0,G}$ and $\theta^*_{0,B}$.[8]

The outcomes of these procedures are distributions of the converged parameters, obtained by perturbing in a systematic way the equilibrium ones.

[7]This does not imply that only the markedly perturbed parameter varies in these experiments, as in the iterations towards the ALM convergence the regressions on simulated data induce changes in the other parameters as well.

[8]Some experimentation showed that wider supports resulted in several cases either being discarded for not meeting the non-explosive dynamics restriction, or leading to a collapse in the ALM after some iterations. These instances do not represent an issue for the perturbation experiment, as a researcher would throw them away and try the model's solution with a different guess.

5 Results

This section first examines the characteristics of the distributions of the converged ALM parameters, then it discusses why the results do not support the existence of multiple SFE, casting the analysis in terms of income, substitution and human wealth effects. Finally it provides a quantitative assessment of the individual wealth thresholds for the income effect to start dominating. A detailed explanation of the economic mechanisms at work is presented, linking the KS algorithm to an economic analysis of the agents saving behavior.

5.1 SFE are not likely

Tables 2 and 3 report a set of statistics of the distributions of the converged ALM, after having randomly perturbed Θ^* 500 times. Table 2 considers the four sets of experiments where all four parameters are perturbed together, while Table 3 considers another four cases where only one parameter is perturbed by a sizable amount, while the others are de-facto kept at their equilibrium values.

[Table 2 about here]

The statistics reported are the minimum, maximum, mean, median and standard deviation for each converged ALM parameter in the sequence of 500 perturbations. By inspecting their values, it is apparent that there is no evidence of multiple SFE. The range of the converged parameters is always tiny, and each parameter differs from its counterpart in Θ^* by 10^{-5}, at worse. In particular, the equilibrium value for the first parameter is $\theta^*_{0,G} = 0.0729005$, while its widest range combining the four experiments is $[0.0728934, 0.0729091]$. Similarly, $\theta^*_{1,G} = 0.9714424$, while its widest range is $[0.9714390, 0.9714451]$, $\theta^*_{0,B} = 0.0647444$, while its widest range is $[0.0647389, 0.0647549]$, and $\theta^*_{1,B} = 0.9726256$, while its widest range is $[0.9726213, 0.9726277]$.[9]

Also when considering even poorer guesses, with one of the parameters being perturbed wildly, the values in Table 3 do not alter the picture: all the parameters converge in a neighborhood of the equilibrium ones, and the related ranges are only marginally wider.

[Table 3 about here]

The same results can be appreciated visually from the kernel density estimates of the distributions of the converged parameters, reported in Figure 2. Interestingly, the densities are found to be bimodal. The plots show that each converged parameter tends to cluster around two distinct values. There are two possible interpretations for this outcome: a) these are genuinely two separate

[9] As for the parameters converging to the boundaries of their support, for a subset of them I restart the procedure, using their values as the new initial guesses. Since they converge to values closer to Θ^*, this constitutes further evidence that these cases are most likely due to the numerical approximation.

equilibria, or b) the clustering is induced by the numerical error. Given the microscopic size of the gap between the two values, it is safe to speculate that these differences are induced by the convergence criteria. In particular, using a finer grid for individual capital together with tighter convergence criteria for both the policy functions and the ALM, led on average to a decrease in the interquartile range by a factor of 5, and to a decrease in the interval between the 10th and 90th percentiles by a factor of 6.[10] Under the SFE interpretation, these values should have remained fairly constant, because the gap between the two modes shouldn't have shrunk considerably. Nevertheless, even if these were two distinct SFE, quantitatively the discrepancy is so minimal that it doesn't have any discernible effect on the outcomes of interest. The business cycles statistics, correlations among endogenous variables, time series behavior of prices, Gini coefficients of wealth, percentages of households in debt, and ergodic distributions of aggregate capital are all virtually identical to the ones obtained with Θ^*.[11]

[**Figure 2 about here**]

It is worthwhile to consider the economic reason behind the absence of multiple SFE. Fundamentally, this result lies in a self-correcting mechanism present in the algorithm. Compared to the equilibrium ALM, a candidate one implying a higher (lower) expected future capital reduces (increases) the equilibrium interest rates, increasing (reducing) the savings of the wealthy agents only. These, on the other hand, account for a small fraction of the population and cannot compensate for the opposite change triggered by the poor agents, who enjoy higher (lower) future wages and increase (reduce) their current consumption. Quantitatively, the change in behavior of the wealth-rich agents has a negligible impact on the determination of the change in the aggregate savings, inducing stability in the algorithm as a by-product.

Given the simple market structure that is typically assumed in this class of models, with competitive markets there is a known relationship between the value of aggregate capital and the equilibrium prices, namely equations (1) and (2). This is one of the reasons underlying the effectiveness of the algorithm: for the agents to accurately predict the future prices, instead of using the whole endogenous distribution over the state variables (an infinite dimensional object), they use only a finite number of its moments (typically just the mean), contributing to deliver the celebrated approximate aggregation result.

Mechanically, postulating parameters for the ALM that are above the equilibrium ones leads the agents to believe that more capital will be available in all future periods. Consequently, a higher future capital reduces the future interest rates and increases the future wages. As usual, these changes trigger three

[10]For example, in the first experiment, the values for $\theta^*_{0,G}$ interquartile range changed from 2.3×10^{-5} to 4.6×10^{-6}, while the 10th-90th interval changed from 3.8×10^{-5} to 5.3×10^{-6}.

[11]Some experimentation with different calibrations (e.g., log preferences) led to similar results. It is worth mentioning that a systematic study wasn't always possible in those alternative cases, because of the tendency for the ALM to diverge or collapse with some perturbations.

different effects affecting the intertemporal motive of savings: a human wealth effect, through increased wages from equation (2), and income and substitution effects, through decreased interest rates from equation (1). Furthermore, there is a complex response of precautionary savings: higher wages and UI benefits make the borrowing constraint less likely to be binding, while lower interest rates make the already accumulated wealth a less effective instrument to smooth consumption in the bad states of the world. The change in precautionary savings can go either way, but it is typically found to be quantitatively small.[12] Whether the individuals increase or decrease their savings (compared to their behavior under the equilibrium ALM) depends on their accumulated wealth, the fraction of income they obtain from capital and the relative change in prices.

All agents experience the three effects mentioned above, and the overall response of their savings depends on which ones dominate (assuming that the net effect on precautionary savings is always unimportant). As argued already, in principle there is indeed the possibility of multiple SFE. Because of consumption smoothing, wealthy individuals increase their savings, as lower interest rates decrease their future income, leading them to save more in the current period. Only for this class of agents the negative income effect more than compensates the sum of the human wealth and substitution effects. In contrast, poor individuals will enjoy higher wages and unemployment benefits in the future, a positive human wealth effect, which together with the substitution effect drives their savings down. For this class of agents, the sum of the human wealth and substitution effects more than offsets the negative income effect. Theoretically, it is hard to state whether in the aggregate the increased savings of the first group will more than compensate the decreased savings of the second one, namely to sign unambiguously the relative strengths of the human wealth, income and substitution effects. What makes this hard is finding the threshold value for accumulated wealth such that the negative income effect starts dominating, together with the related mass of agents that are above them. Furthermore, there are several such thresholds, one for each possible combination of state variables (with the exception of individual wealth). These are presented in the next subsection, also showing how they are affected by changes in two key parameters, the CRRA coefficient γ and the borrowing constraint b. Quantitatively it turns out that, for a plausible calibration of the model, these thresholds are more than 100 times higher than the average aggregate income, its value being 1.13. Since the share of agents holding such high wealth levels is well below 0.5%, multiple SFE are not likely to arise in this model. These considerations should suggest that in the KS economy there is a self-correcting mechanism with respect to wrong guesses in the ALM, because the human wealth effect, the income effect for agents in debt, and the substitution effect, move the aggregate savings in the opposite direction.

[12] Just like in Krusell and Smith (1998), I find that the aggregate capital (averaged over time) is higher in the incomplete markets economy compared to its complete markets counterpart by approximately 2%. This represents a measure of the importance of precautionary savings in the aggregate wealth. Moreover, in the economy with the no-borrowing assumption, the capital stock increases by only a further 0.32%.

5.2 When does the Income Effect dominate?

In order to quantify the relative strength of the negative income effect, I consider the response of the saving functions to a change in the ALM for all household types in all possible exogenous states. The first crucial property of the saving functions under the two specifications of the ALM is that they cross only once. Hence, these intersections represent the thresholds for individual capital such that the negative income effect is stronger than the other (combined) effects previously discussed.

[Figure 3 about here]

In particular, Figure 3 plots the intersections between the policy functions computed under the equilibrium ALM and under the same 1% perturbation considered in Figure 1. Namely, the four panels display the thresholds where the income effect starts dominating, plotted as a function of aggregate capital, for all household types, and combinations of idiosyncratic and aggregate shocks. Each panel focuses on a specific (s, z) pair, showing the thresholds for all preference types. As it is apparent from the figure, these thresholds for the prevalence of the income effect are very similar in the four panels. The employment status and the aggregate shock have a relatively minor role in shaping them, while the discount factor and the aggregate capital have a more pronounced effect. The thresholds show a clear decreasing behavior in the former and an increasing one in the latter. At the average aggregate capital, the least patient households have thresholds that are approximately 26% higher than the corresponding figures for the most patient ones. The effect of aggregate capital is fairly linear, with an average slope that varies between 0.89 and 0.92, depending on the preference type, and combination of exogenous states.

As the figure suggests, it is improbable for the income effect to dominate in the aggregate, making the onset of a complementarity between the postulated ALM and aggregate savings not plausible. The most intuitive way of explaining this conclusion is as follows. Along an equilibrium, aggregate capital fluctuates around its long run average of 12. Under the equilibrium ALM, the aggregate shocks do not move the aggregate capital very far away from this value, the minimum value in the simulation being 7.3% lower than the long-run average, and the maximum one being 8.9% higher. Hence, from the figure we appreciate that the corresponding income effect thresholds are between 187 and 236, depending on the household type. By inspecting the simulated panel data, agents need to have wealth holdings that are greater than 145 to belong to the top 1%. Moreover in a number of cross sections I considered, the actual percentage of households above their corresponding threshold is between 0.32% and 0.35%. Since they account for a negligible fraction of the population, and the response of their savings is not very conspicuous, it is highly unlikely for SFE to occur.

[Figure 4 about here]

Additional results related to the thresholds' response to some parameter changes are provided. In particular, changing the borrowing limit from $b = -1.8$ to $b = 0$ doesn't lead to large changes in the thresholds, whose variation is between 0.1% and 1.1%. As expected, changes in the CRRA parameter have a first order effect, because they influence the consumption smoothing motive. As reported in Figure 4, a fairly small drop in the CRRA, from $\gamma = 2$ to $\gamma = 1.75$, brings about large responses in the thresholds. At the long-run average for aggregate capital, the increases in the individual assets needed for the income effect to prevail are between 53 and 73. Similar to the benchmark economy, in a number of cross sections I considered, the percentage of households above their corresponding threshold in this case is between 0.21% and 0.25%. It goes without saying that larger drops in the CRRA lead to even stronger responses and to lower shares of households with such large wealth holdings. As a final remark, preference homogeneity would decrease the magnitude of the negative income effect, making the occurence of SFE even less likely.

6 Concluding remarks

Several algorithms tackling economies with heterogeneous agents and aggregate shocks have been recently developed, and their relative performance is discussed in den Haan (2010). The simplicity of the KS procedure, together with its successful implementation in many (and diverse) applications, make it often the method of choice. This paper has showed that a potential threat to this methodology, multiple SFE, does not appear to be a relevant problem for a canonical version of the incomplete markets model with aggregate shocks and preference heterogeneity. However, it is not straightforward that this result will hold in substantially more complicated models, with several endogenous variables appearing in the specification of the ALM. Researchers applying this method should provide further evidence on the absence of multiple SFE: considering large systematic perturbations to the equilibrium ALM, as done here, is now feasible for a large set of models solved with the KS algorithm.

References

Castaneda, A., Diaz-Gimenez, J., and Rios-Rull, J.V. (1998). Exploring the income distribution business cycle dynamics. *Journal of Monetary Economics*, Vol. 42, 93-130.

Corbae, D., D'Erasmo, P., and Kuruscu, B. (2009). Politico-economic consequences of rising wage inequality. *Journal of Monetary Economics*, Vol. 56, 43-61.

Giusto, A. (2014). Adaptive learning and distributional dynamics in an incomplete markets model. *Journal of Economic Dynamics & Control*, Vol. 40, 317-333.

Gomes, J., Greenwood, J., and Rebelo, S. (2001). Equilibrium Unemployment. *Journal of Monetary Economics*, Vol. 48, 109-152.

den Haan, W. (2010). Comparison of solutions to the incomplete markets model with aggregate uncertainty. *Journal of Economic Dynamics & Control*, Vol. 34, 4-27.

den Haan, W., Judd, K., and Juillard, M. (2010). Computational suite of models with heterogeneous agents: Incomplete markets and aggregate uncertainty. *Journal of Economic Dynamics & Control*, Vol. 34, 1-3.

Heathcote, J. (2005). Fiscal Policy with Heterogeneous Agents and Incomplete Markets. *Review of Economic Studies*, Vol. 72, 161-188.

Horvath, M. (2012). Computational accuracy and distributional analysis in models with incomplete markets and aggregate uncertainty. *Economics Letters*, Vol. 117, 276-279.

Judd, K. (1998). Numerical Methods in Economics. MIT Press, Cambridge (MA).

Krusell, P., Mukoyama, T., Sahin, A., and Smith, A. (2009). Revisiting the welfare effects of eliminating business cycles. *Review of Economic Dynamics*, Vol. 12, 393-404.

Krusell, P., and Smith, A. (2006). Quantitative Macroeconomic Models with Heterogeneous Agents, in: Blundell, R., Newey, W., and Persson, T. (Eds), Advances in Economics and Econometrics: Theory and Applications. Econometric Society Monographs 41, Cambridge University Press, 298-340.

Krusell, P., and Smith, A. (1998). Income and Wealth Heterogeneity in the Macroeconomy. *Journal of Political Economy*, Vol. 106, 867-896.

Krusell, P., and Smith, A. (1997). Income and Wealth Heterogeneity, Portfolio Choice, and Equilibrium Asset Returns. *Macroeconomic Dynamics*, Vol. 1, 387-422.

Maliar, L., Maliar, S., and Valli, F. (2010). Solving the incomplete markets model with aggregate uncertainty using the Krusell-Smith algorithm. *Journal of Economic Dynamics & Control*, Vol. 34, 42-49.

Mukoyama, T., and Sahin, A. (2006). Costs of business cycles for unskilled workers. *Journal of Monetary Economics*, Vol. 53, 2179-2193.

Nakajima, M. (2012). Business Cycles In The Equilibrium Model Of Labor Market Search And Self-Insurance. *International Economic Review*, Vol. 53, 399-432.

Pijoan-Mas, J. (2007). Pricing Risk in Economies with Heterogeneous Agents and Incomplete Markets. *Journal of the European Economic Association*, Vol. 5, 987-1015.

Press, W., Teukolsky, S., Vetterling, W., and Flannery, B. (2002). Numerical recipes in Fortran 90. Cambridge University Press, Cambridge (UK).

Storesletten, K., Telmer, C., and Yaron, A. (2007). Asset pricing with idiosyncratic risk and overlapping generations. *Review of Economic Dynamics*, Vol. 10, 519-548.

Young, E. (2010). Solving the incomplete markets model with aggregate uncertainty using the Krusell-Smith algorithm and non-stochastic simulations. *Journal of Economic Dynamics & Control*, Vol. 34, 36-41.

Young, E. (2005). Approximate aggregation. Working paper, University of Virginia.

Parameter	*Value*	*Target*
β - *Rate of time preference*	$\{0.9858, 0.9894, 0.9930\}$	*Average wealth Gini* ≈ 0.8, *average* $r \approx 1\%$
γ - *CRRA*	2.0	*Micro estimates on the IES*
δ - *Capital depreciation rate*	0.025	*Average investment share of output* $\approx 26\%$
α - *Capital share*	0.36	*Capital share of output* $= 36\%$
b - *Borrowing limit*	-1.8	*Average share of households in debt* $\approx 10\%$
l - *Labor supply*	0.3271	*Share of market time* (*time endowment* $= 1$)
ρ - *UI replacement rate*	0.40	*Average UI replacement rate*

Table 1: Calibration.

Equilibrium ALM	*Min*	*Max*	*Mean*	*Median*	*S.d.×1000*
Perturbation Range: ±1%					
$\theta^*_{0,G}$	0.072893	0.072909	0.072901	0.072901	0.00254
$\theta^*_{1,G}$	0.971439	0.971445	0.971442	0.971442	0.00099
$\theta^*_{0,B}$	0.064741	0.064755	0.064748	0.064749	0.00420
$\theta^*_{1,B}$	0.972621	0.972627	0.972624	0.972624	0.00172
Perturbation Range: ±2%					
$\theta^*_{0,G}$	0.072894	0.072909	0.072901	0.072901	0.00244
$\theta^*_{1,G}$	0.971439	0.971445	0.971442	0.971442	0.00095
$\theta^*_{0,B}$	0.064739	0.064754	0.064747	0.064747	0.00425
$\theta^*_{1,B}$	0.972622	0.972628	0.972624	0.972624	0.00174
Perturbation Range: ±3%					
$\theta^*_{0,G}$	0.072895	0.072909	0.072901	0.072900	0.00242
$\theta^*_{1,G}$	0.971439	0.971444	0.971442	0.971443	0.00094
both $\theta^*_{0,B}$	0.064742	0.064754	0.064747	0.064744	0.00434
$\theta^*_{1,B}$	0.972622	0.972627	0.972624	0.972626	0.00178
Perturbation Range: ±4%					
$\theta^*_{0,G}$	0.072896	0.072906	0.072901	0.072900	0.00238
$\theta^*_{1,G}$	0.971440	0.971444	0.971442	0.971443	0.00093
$\theta^*_{0,B}$	0.064742	0.064753	0.064747	0.064744	0.00421
$\theta^*_{1,B}$	0.972622	0.972627	0.972625	0.972626	0.00173

Table 2: Results - 500 perturbations, per perturbation range; all ALM parameters are perturbed by independent random draws from a uniform distribution. The Equilibrium ALM is: $\theta^*_{0,G} = 0.0729005$, $\theta^*_{1,G} = 0.9714424$, $\theta^*_{0,G} = 0.0647444$, $\theta^*_{0,G} = 0.9726256$.

Equilibrium ALM	*Min*	*Max*	*Mean*	*Median*	*S.d.×1000*
Perturb. Range for $\theta_{0,G}$*:* $\pm 25\%$					
$\theta^*_{0,G}$	0.072893	0.072911	0.072901	0.072901	0.00212
$\theta^*_{1,G}$	0.971438	0.971445	0.971442	0.971442	0.00082
$\theta^*_{0,B}$	0.064742	0.064757	0.064748	0.064749	0.00413
$\theta^*_{1,B}$	0.972621	0.972627	0.972624	0.972624	0.00169
Perturb. Range for $\theta_{1,G}$*:* $\pm 10\%$					
$\theta^*_{0,G}$	0.072892	0.072906	0.072900	0.072899	0.00149
$\theta^*_{1,G}$	0.971440	0.971446	0.971443	0.971443	0.00058
$\theta^*_{0,B}$	0.064739	0.064753	0.064745	0.064744	0.00303
$\theta^*_{1,B}$	0.972622	0.972628	0.972626	0.972626	0.00124
Perturb. Range for $\theta_{0,B}$*:* $\pm 25\%$					
$\theta^*_{0,G}$	0.072894	0.072909	0.072901	0.072899	0.00298
$\theta^*_{1,G}$	0.971439	0.971445	0.971442	0.971443	0.00117
$\theta^*_{0,B}$	0.064740	0.064753	0.064748	0.064744	0.00403
$\theta^*_{1,B}$	0.972622	0.972627	0.972624	0.972626	0.00165
Perturb. Range for $\theta_{1,B}$*:* $\pm 10\%$					
$\theta^*_{0,G}$	0.072892	0.072910	0.072903	0.072904	0.00235
$\theta^*_{1,G}$	0.971439	0.971446	0.971442	0.971441	0.00092
$\theta^*_{0,B}$	0.064740	0.064763	0.064750	0.064752	0.00358
$\theta^*_{1,B}$	0.972618	0.972627	0.972623	0.972623	0.00146

Table 3: Results - 500 perturbations, per perturbation range; only one ALM parameter is perturbed by a random draw from a uniform distribution.The Equilibrium ALM is: $\theta^*_{0,G} = 0.0729005$, $\theta^*_{1,G} = 0.9714424$, $\theta^*_{0,G} = 0.0647444$, $\theta^*_{0,G} = 0.9726256$.

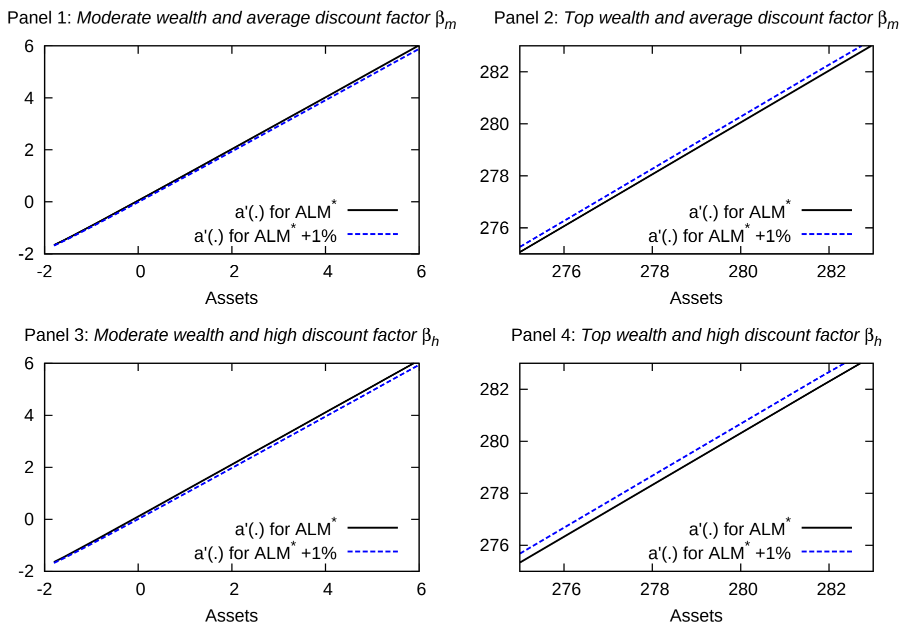

Figure 1: Differential response of savings for selected household types, Equilibrium ALM (solid line) Vs. Perturbation = 1% (dashed line). $\text{ALM}^* + 1\%$ stands for a perturbation to the equilibrium ALM^* such that the forecasted aggregate capital is 1% higher than under ALM^*.

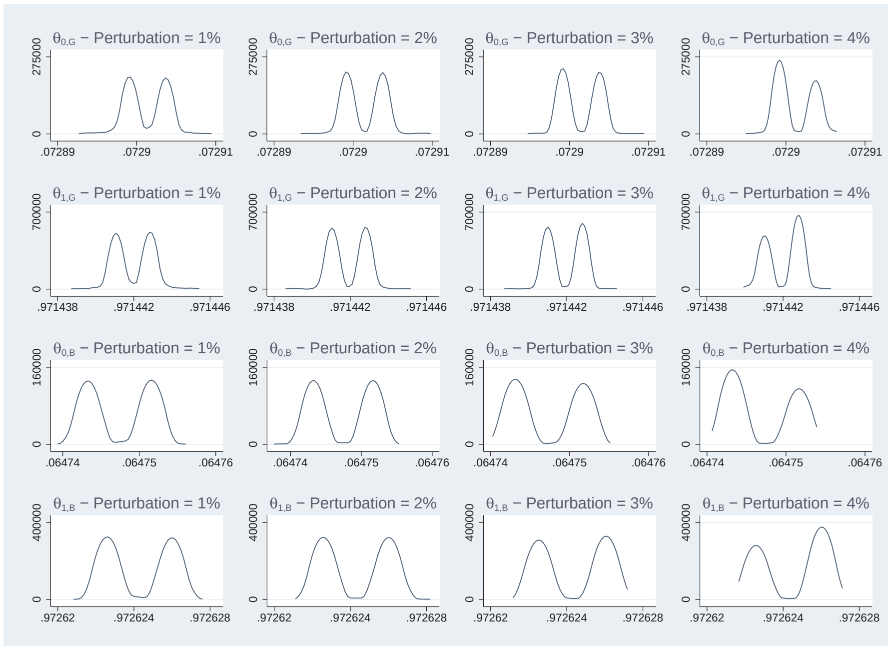

Figure 2: Kernel density estimates of the 500 converged parameters Θ^*. Each column refers to the four ALM parameters in a specific perturbation experiment, while each row refers to a specific ALM parameter in the sequence of perturbation experiments.

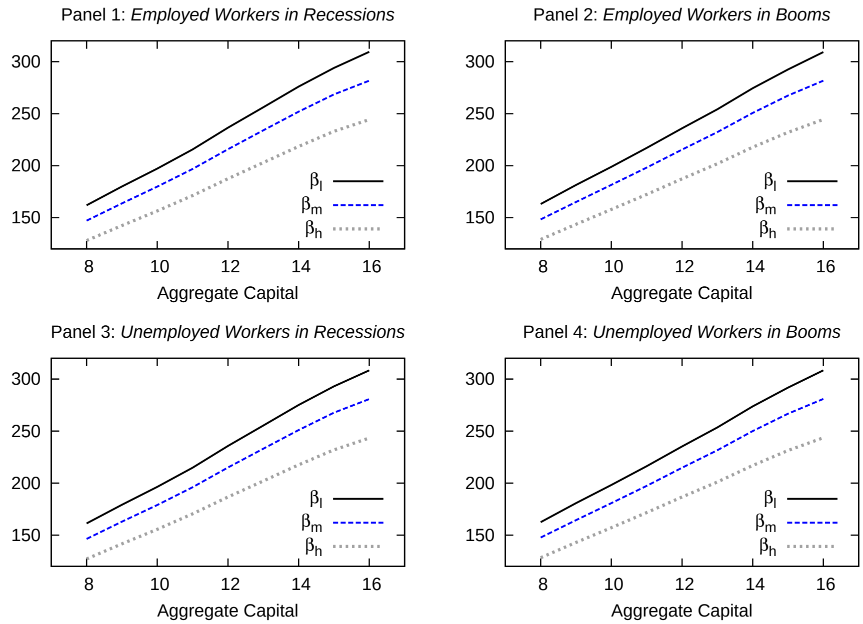

Figure 3: Individual capital thresholds such that the negative income effect starts dominating.

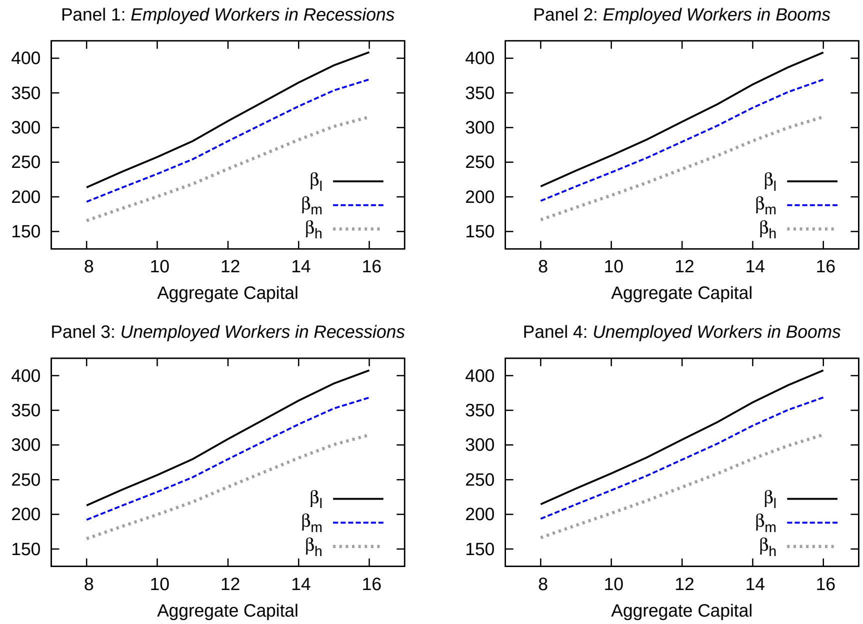

Figure 4: Response of individual capital thresholds to a change in the CRRA parameter: $\gamma = 1.75$.

Appendix A - The Complete Calibration

The transition matrix of the individual shocks, conditional on the aggregate ones, is (each entry refers to the probability $\pi_{z,z';s,s'}$):

$$\pi(z, z', s, s') = \begin{bmatrix} \pi_{G,G;u,u} & \pi_{G,G;u,e} & \pi_{G,B;u,u} & \pi_{G,B;u,e} \\ \pi_{G,G;e,u} & \pi_{G,G;e,e} & \pi_{G,B;e,u} & \pi_{G,B;e,e} \\ \pi_{B,G;u,u} & \pi_{B,G;u,e} & \pi_{B,B;u,u} & \pi_{B,B;u,e} \\ \pi_{B,G;e,u} & \pi_{B,G;e,e} & \pi_{B,B;e,u} & \pi_{B,B;e,e} \end{bmatrix}$$

$$= \begin{bmatrix} 0.292 & 0.583 & 0.094 & 0.031 \\ 0.024 & 0.851 & 0.009 & 0.116 \\ 0.031 & 0.094 & 0.525 & 0.350 \\ 0.002 & 0.123 & 0.039 & 0.836 \end{bmatrix}$$

The transition matrix for the preference heterogeneity is:

$$\pi(\beta, \beta') = \begin{bmatrix} 0.995 & 0.005 & 0.000 \\ 0.000625 & 0.99875 & 0.000625 \\ 0.000 & 0.005 & 0.995 \end{bmatrix}$$

The transition matrix of the aggregate shocks is:

$$\pi(z, z') = \begin{bmatrix} 0.875 & 0.125 \\ 0.125 & 0.875 \end{bmatrix}$$

Appendix B - Computation

- All codes were written in the FORTRAN 95 language, relying on the Intel Fortran Compiler, build 11.1.048 (with the IMSL library). They were compiled selecting the O3 option (maximize speed), and without automatic parallelization. They were executed on two 64-bit workstations, running Windows 7 Professional Edition (either natively or as a virtual machine in CentOS 6.5), with an Intel $i7-2600k$ Quad-core processor clocked at 4.6 Ghz or an Intel Xeon $E5-2687Wv2$ Octo-core processor clocked at 3.4 Ghz.
- On either machine, the replications take more than 15 days to complete. Notice that 500 equilibria have to be computed, and typically from 13 to 19 iterations on the ALM are needed to find each equilibrium.
- In the actual solution of the model I need to discretize the continuous state variables a and K (the employment status s, the preference heterogeneity β, and the aggregate productivity shock z are already discrete). For the household assets a I rely on an unevenly spaced grid, with the distance between two consecutive points increasing geometrically. This is done to allow for a high precision of the policy rules at low values of a, where the change in curvature is more pronounced. In order to keep the computational burden manageable, I use 75 grid points on the household assets space, the lowest value being the borrowing constraint b and the highest one being a large value $a_{\max} = 400$. In the aggregate capital dimension K, I use an evenly spaced grid over the $[3, 27]$ interval. I use 25 points, which are far more than the typical 4-6. However, in the iterative process on the ALM parameters and for extreme initial guesses, the simulations do visit regions of the state space that are very far from the support of the ergodic equilibrium distribution, causing convergence issues when using a coarse grid. Finally, the polynomial approximation at the simulation stage in the K dimension is of degree 24, and it is implemented with the routine polint in Press et al. (2002).
- The main differences between my Euler equation procedure and the one outlined in Maliar et al. (2010) pertain to what is considered the unknown of the Euler equations. In their formulation, the unknown is future wealth entering only in the formula for current consumption. In mine, the unknown is future wealth entering in the formulas for both current and future consumption. It follows that, unlike me, they never have to deal with the solution of a non-linear equation. This feature also leads to differences in how the borrowing constraint is dealt with. Finally, I potentially allow for extrapolation, although I make sure that in the simulations of the benchmark economy there are no agents close to the upper bound for individual wealth.

- More formally, the Euler equation approach I rely on is such that, given the current guess for the policy function $a'(a, s, \beta, z, K)_n$, at each point in the state space the unknown I need to solve for is a'_{n+1}:

$$
\begin{aligned}
&\left[(1+r)\,a + y_s - a'_{n+1}\right]^{-\gamma} \geq \\
&\beta E_{\beta',s',z'|\beta,s,z}\left[(1+r')\,a'_{n+1} + y_{s'} - a'\left(a'_{n+1}, s', \beta', z', K'\right)_n\right]^{-\gamma} \\
&\text{where} \\
&y_s = (1-\tau)\,wl,\ \text{ if } s = e; \\
&y_{s'} = (1-\tau')\,w'l,\ \text{ if } s' = e; \\
&y_s = \rho wl,\ \text{ if } s = u; \\
&y_{s'} = \rho w'l,\ \text{ if } s' = u.
\end{aligned}
$$

- It is now understood that I keep on iterating until a fixed point is reached, i.e. until two successive iterations satisfy:
$\underset{a}{Sup}\,|a'(a, s, \beta, z, K)_{n+1} - a'(a, s, \beta, z, K)_n| < 10^{-7}$, $\forall s, \forall\beta, \forall z, \forall K$.

- An alternative, and perhaps more intuitive, way of describing the solution method is to consider it as an application of weighted residuals/finite elements methods. The weighting function is the Dirac delta function, which involves (degenerate) weights of 1 at the grid points and of zero at all other values in an element. Since the basis functions are assumed to be linear, the (global) solution for the saving functions is represented by a number of coefficients equal to the number of grid points in the individual wealth dimension, multiplied by the number of points for the other state variables. It follows that the optimal policy functions are exact (up to the convergence criterion) at the grid points.

Appendix C - Additional Results

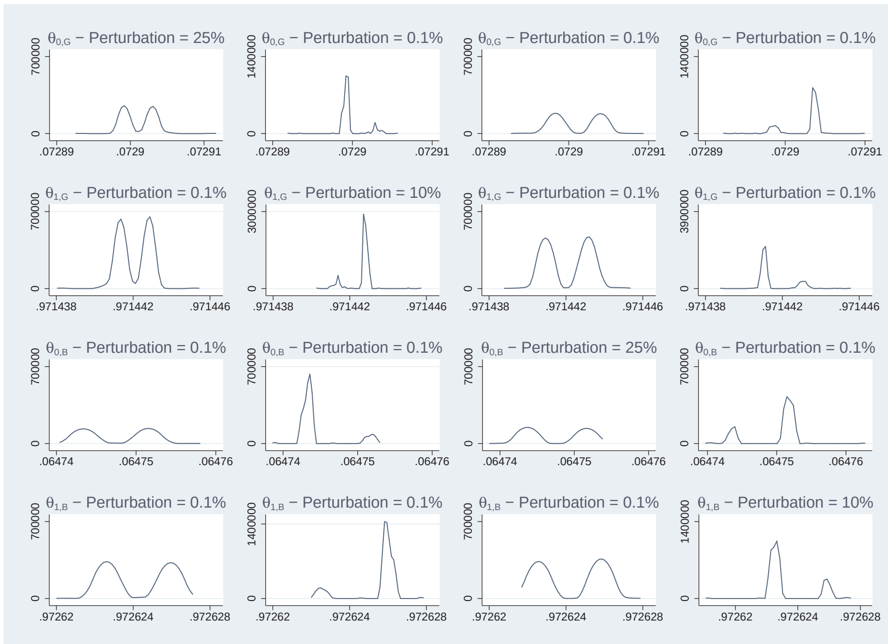

Figure 5: Kernel density estimates of the 500 converged parameters Θ^*. Each column refers to the four ALM parameters in a specific perturbation experiment, while each row refers to a specific ALM parameter in the sequence of perturbation experiments.

Equilibrium ALM	*Min*	*Max*	*Mean*	*Median*	*S.d.×1000*
Perturbation Range: ±1%					
$\theta^*_{0,G}$	0.073146	0.073161	0.073152	0.073153	0.00237
$\theta^*_{1,G}$	0.971366	0.971372	0.971369	0.971369	0.00091
$\theta^*_{0,B}$	0.065527	0.065539	0.065533	0.065535	0.00395
$\theta^*_{1,B}$	0.972345	0.972350	0.972348	0.972347	0.00395
Perturbation Range: ±2%					
$\theta^*_{0,G}$	0.073146	0.073159	0.073152	0.073153	0.00230
$\theta^*_{1,G}$	0.971367	0.971372	0.971369	0.971369	0.00089
$\theta^*_{0,B}$	0.065527	0.065538	0.065532	0.065531	0.00403
$\theta^*_{1,B}$	0.972346	0.972350	0.972348	0.972348	0.00165
Perturbation Range: ±3%					
$\theta^*_{0,G}$	0.073148	0.073159	0.073152	0.073151	0.00226
$\theta^*_{1,G}$	0.971367	0.971371	0.971369	0.971370	0.00087
$\theta^*_{0,B}$	0.065527	0.065537	0.065532	0.065529	0.00401
$\theta^*_{1,B}$	0.972346	0.972350	0.972348	0.972349	0.00165
Perturbation Range: ±4%					
$\theta^*_{0,G}$	0.073147	0.073158	0.073152	0.073151	0.00227
$\theta^*_{1,G}$	0.971367	0.971371	0.971369	0.971370	0.00087
$\theta^*_{0,B}$	0.065527	0.065538	0.065532	0.065529	0.00403
$\theta^*_{1,B}$	0.972346	0.972350	0.972348	0.972349	0.00166

Table 4: Robustness ($b = 0$) - 500 perturbations, per perturbation range; all ALM parameters are perturbed by independent random draws from a uniform distribution. The Equilibrium ALM is: $\theta^*_{0,G} = 0.07315561$, $\theta^*_{1,G} = 0.97136796$, $\theta^*_{0,G} = 0.06553407$, $\theta^*_{0,G} = 0.97234717$.

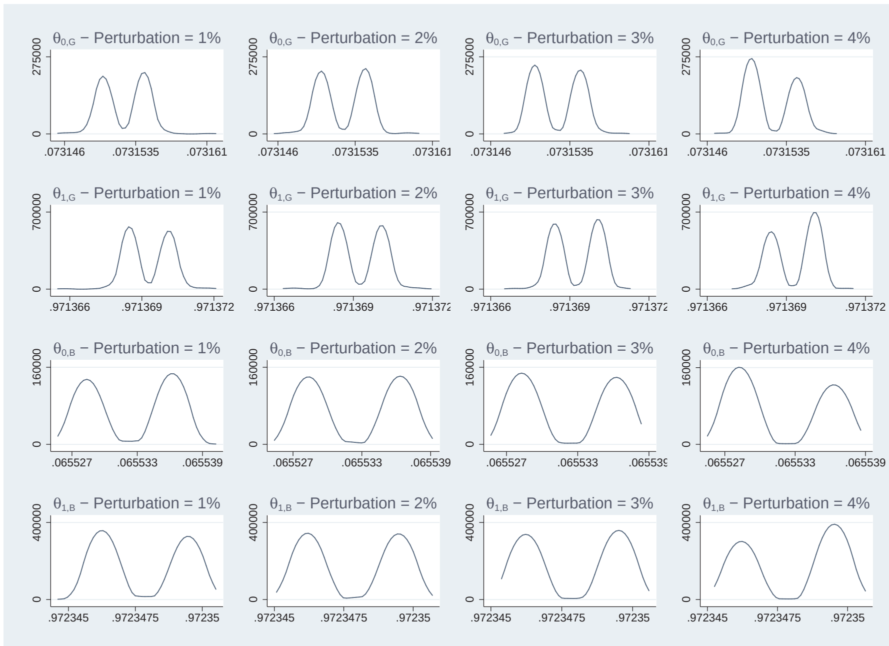

Figure 6: Robustness ($b = 0$) - Kernel density estimates of the 500 converged parameters Θ^*. Each column refers to the four ALM parameters in a specific perturbation experiment, while each row refers to a specific ALM parameter in the sequence of perturbation experiments.

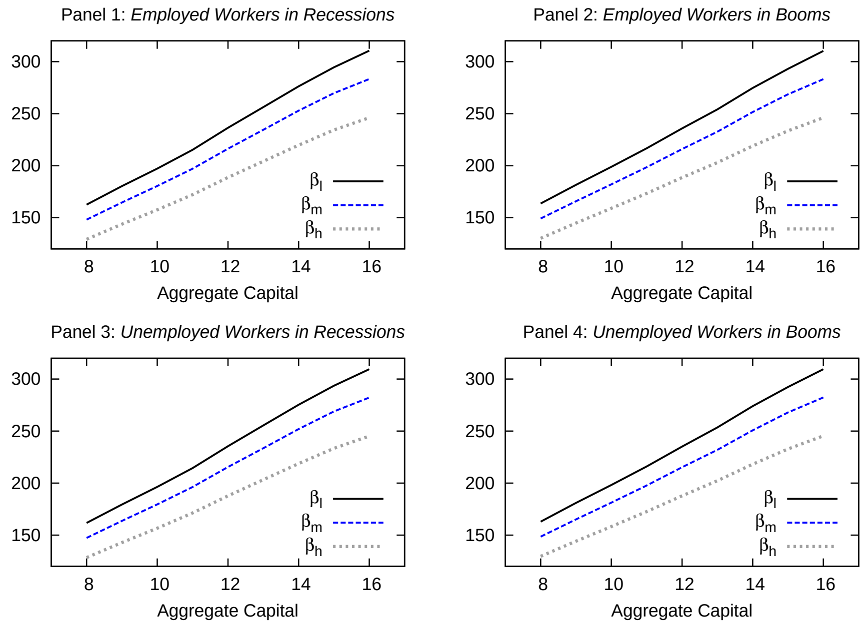

Figure 7: Robustness ($b = 0$) - Individual capital thresholds such that the negative income effect starts dominating.