Queen's Economics Department Working Paper No. 1301

Externalities, Social Value and Word of Mouth: Notions of Public Economics on Networks

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1-2013

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January 8^{th} 2013

Abstract

I examine an environment where advertisers can "seed" word-of-mouth advertising by providing initial information about a product to specific users of a social network. Discussion over a social network generates spillover effects for firms when consumers can use the social network to inform each other about products. When a firm can exploit a social network's structure, it can increase its sales. However, when the network formation process is costly, firms free-ride on such costs at the expense of agents on the network. If agents can form coalitions, I show that they can recoup the value of this externality by charging a toll.

When users actively modify the information, generating word-of-mouth advertising about a product provides a "social value." This social value stems from the discussions that agents have about the product, without any intervention. Since this process occurs regardless of the firm's actions, the firm cannot capture such valuation. The opinion leaders, or highly regarded agents on the network, play a key role in the formation of this social value.

1 Introduction

Word-of-mouth communication is a topic widely discussed when it comes the economics of social networks. Some papers cover the endogenous formation of communications networks (Galeotti, Ghiglino & Squintani 2009), others are interested in harnessing the process from a marketing or sales perspective (Godes & Mayzlin, 2001), while others examine the impact of word-of-mouth on the behavior of agents on the network (Schiraldi & Liu, 2010). However, very little is said on contagion from the standpoint of public economics.

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In this paper, I develop a model of word-of-mouth communication to discuss the notion of network spillover and the notion of social value of a network. Common ideas and discussions are non-rivalrous, in the sense that once an idea is disseminated to a neighborhood of the network, nothing prevents users who can access the idea to reproduce it at will. As such, the network acts as a technology that allows for the dissemination of ideas.

Anyone with a Twitter or Facebook account can see this process of information sharing at work. Topics range from the latest Justin Bieber song to information about earthquakes (Sakaki, Okazaki & Matsuo, 2010). For instance, roughly 6000 Justin Bieber fans showed-up on a rumor of a concert spread on twitter (see Ryland 2012) In short, seeding one idea on the network might go a long way through a process of replication: that is, in essence, the contagion spillover.

The natural idea, from a sales point of view, is to try to exploit this phenomenon to increase sales. For instance, a company might want to target a group of users and let them evaluate a product to generate some word-of-mouth advertising. By selecting the right group, the company can use the network to generate a cascade of positive recommendations that can increase sales. This cascade of adoption might be caused by herding (as in Banerjee, 1992), by an informative signal about the quality of the product (Cambell 2005, Vettas 1997) or simply be the result of truthful communication (Galeotti and Goyal, 2009). The consistent feature of these models is that regardless of the *motives* of the agents on the network, or their behavior (Bayesian updating, randomly talking, or being fooled), they do *communicate* with each other. If what drives this communication is known to the company, it can be exploited to increase profits. The company, however, profits from such communication for free, and this is where the free rider problem occurs. I show below under which conditions this problem is important and how it can be solved.

If conversations between agents influence how consumers evaluate or perceive a product of the company, the company will have some interest in these changes of perceptions as well. In this case, the process is not only about disseminating an idea, but also about ensuring the right conversations occur to generate a positive value.

However, human nature means that agents will discuss ideas even if the company performs no action. The conversation might be different in nature, but it can be taken as granted that some conversation will happen. Such communication has some value and is, in essence, the social value generated by the network. Because this exchange of information does not depend on the actions of the company, it cannot be captured in any fashion by the company. If the discussion is about the quality of the company's product, this conversation has certainly some value.

In section 2, I discuss the existing literature on network externalities. In section 3, I present a simple model of word-of-mouth communication over a social network. In sections 4 and 5, I exploit the model to discuss its implications from a standpoint of externalities and social value. A brief conclusion follows.

2 Literature

To my knowledge, there are no academic articles on the social value of word-of-mouth communication, nor on the externalities it induces. Some of the academic articles dealing with network externalities refer to the adoption of a technology (see Katz & Shapiro, 1985 or Farrell & Saloner for an exposition). This area of literature focuses on the adoption of a product, where the value of the product depends on the number of people using it. Such articles are however of little help in the context of word of mouth because the externality in word-of-mouth is a consequence of discussion rather than the value of the product. The latter can have a value independent of the number of adopters but still lead to word-of-mouth communication. In the context of communication, the contagion effect stems from non-rivalry[1] of ideas rather than the adoption of the idea itself.

However, there are parallels to be made between word-of-mouth communication and the congestion literature (see Kelly, 2008 for a good introduction). In discussing about a product, agents on a network let the information produced circulate for free. As such, companies profit from this information flow although they bear no cost in the process of link formation. Such problem of flow has been studied in the design of information network or roads. In such cases, the externality is negative: an increase in the flow (traffic) leads to longer waiting times. The standard approach to address this externality is to introduce a toll, which influences road users to take the socially optimal route.

In the context of word-of-mouth communication, the externality is positive if the product being discussed is valuable: a firm can profit from the process of link creation between agents on a network to generate a contagion of ideas, or spillovers. Hence, the externality is different in nature. The idea of tolls can however still be applied to show how agents can address the free rider problem. This is discussed further below.

3 Model

The Network

There is a directed network $G(V, E)$ with $|V|$ agents linked together by a set of $|E|$ directed edges. A particular agent is denoted by and index $i \in V$, while a link between agents $i, j \in V$ is denoted by e_{ij}. Since edges are directed, e_{ij} represents the direction of information traveling from $j \rightarrow i$. An example is provided in Figure 1. The set $V = \{0, 1, 2, 3, 4, 5, 6\}$ contains seven agents linked by the set of edges $E = \{e_{01}, e_{20}, e_{50}, e_{30}, e_{13}, e_{41}, e_{52}, e_{53}, e_{36}, e_{46}, e_{65}\}$.

This network topology is assumed to be fixed and the outcome of a network formation process. Various economic models can explain link formation. Some assume the formation

[1] A product is non-rival if its consumption does not prevent somebody else from consuming it as well. An idea has such a property.

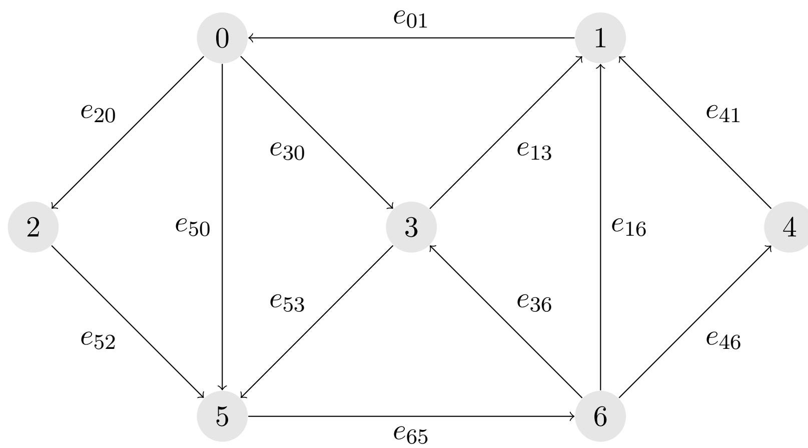

Figure 1: **An Example of A Network**

is a consequence of random draws of a given distribution (see the Erdös-Renyi framework in Jackson, 2009) or a random rewiring of edges of a known network structure (see Watts and Strogratz, 1998). There are other processes of link formation based on the allocation of scarce resources in building costly links between agents. These allocations might be based on a trade-off between personal biases on the state of the world and the willingness to learn the truth (Galeotti, Ghiglino & Squintani 2009), or *homophily*, the tendency for people with similar preferences to be linked together (Boucher, 2012). Whatever the reason for the form of the network, it is taken as granted. For the discussion about the externality, it will be useful to assume that link formation is costly. This can be thought of as the cost to maintain a relationship or the additional time required to search through a larger set of information.

A convenient definition is the notion of the neighborhood of sources of an agent i: η_i. This represents the set of agents from which agent i can choose to listen to. For example, in Figure 1, $\eta_1 = \{3, 4, 6\}$ since agent 1 has agents 3, 4 and 6 as sources of information. Agents exchange a signal $s_{i,t} \in \mathbb{R}$ on the network. This signal can take on various meanings: it can represent the quality of a product, the level of (dis)utility of a message or simply a truthful message about the state of the world. In all cases, the signal has some value to the agents and to the company. This will be discussed further below. This signal might change in time and might be different from one agent to another. I will denote $s_t \in \mathbb{R}^{|V|}$ the vector of signals distributed by each agent at time t.

3.1 The Agents

The next step is to define the behavior of agents regarding the information they have. Agents are mathematically modeled as three objects: a neighborhood η_i, a set of personal perceptions $b_{ji} \in \mathbb{R}$ and a choice function $f : \{s_j : j \in \eta_i\} \to s^*_j$. The notion of neighborhood has already been discussed: these are the possible sources of information for a given agent. When these are specified for every agent, the network structure is fully known. The notion of personal perception pertains to how agent i values information received from agent j. The higher the value of b_{ji}, the more highly agent i thinks of agent j. One way to think of this is personal preferences over sources. These numbers could change over time, as with a dynamic reputation function, but I hold them fixed for simplicity. The choice function represents how an agent will choose a source of information given its perceptions about other agents and the signals it receives. In this paper, I will use the following model of agents :

$$s_{i,t+1} = \max_{j \in \eta_i} (s_{j,t} + b_{ij}) \tag{1}$$

The motivation for this choice stems from numerous examples in the literature:

1. Agents learn about the quality of a product only from their neighbors (as in Campbell, 2012). Such quality is uncertain and can be high or low. They have a unit demand for it when the quality is high and no demand for it when it is low. Neighbors can observe the quality of the product when someone in the neighborhood has learned it. Hence, the vector of signals can be thought of as a binary vector where each element $s_{i,t}$ is either one or zero. When the value is one, the signal means "the product is of high quality," while when it is zero it means "we do not know about the quality of the product." In this context, the signaling process $s_{i,t+1} = \max_{j \in \eta_i}(s_{j,t} + 0)$ represents the evolution of the beliefs about the product over time.

2. The information can be thought as a vector v drawn from a vector space of concepts. These concepts might be "liberty" and "equality" (a two-dimensional vector space) and a particular vector represents a level of support for those concepts. Agents have some given preferences $\tilde{v}$ over those concepts (e.g. what they believes the right level of liberty and equality) and they weight each concept according to a diagonal matrix W, each element of the diagonal being a weight for a concept. The distance function $(v-\tilde{v})'W(v-\tilde{v})$ represents a measure of homophily, or the distance between the message and what the agent prefers (see Currarini, Jackson & Pin for further discussion on homophily). Hence, an agent chooses its messages based on $s_{i,t+1} = \max_{j \in \eta_i}(-(v_j - \tilde{v})'W(v_j - \tilde{v}) + b_{ij})$. That is, they choose the message that is the closest to their own preferences, given the bias they have for individuals (b_{ij}).

3. The information can simply be the perceived quality of a given product. Hence, s_{jt} represents the quality level communicated by agent j and b_{ij} represents the correction agent i applies to agent j's perceptions of quality. Once the signal is corrected for perceptions, agent i sends his own message of perceived quality: $s_{i,t+1} = \max_{j \in \eta_i}(s_{j,t} + b_{ij})$.

This behavior is sufficient to discuss the economic nature of word-of-mouth communication over a network. There is also a practical reason for choosing this functional form: it is tractable on large-scale networks. Social networks are, by design, large. Facebook has 900 million users, Twitter has 500 million users, and both are growing. The number of links between these agents is much larger. Although companies or scientists might be interested in subsets of these networks, they *still* represent large objects. Thus, the models used to represent them must be tractable to provide meaningful answers to practitioners. The model above can be described as a linear system, very similar to the linear system of equations $As_t = s_{t+1}$ in what is called a *tropical algebra* (or max-plus semirings). It is thus very tractable (Bouchard St-Amant, 2012). All of the examples above share the same constraint: information only flows through the network structure. In other words, agents are myopic; an agent cannot get more information than what his sources tell him. In the discussion on externalities, I will show that this assumption is critical for the firm to "free-ride" on the network structure. If agents are able to understand how critical they are to the spillover effect, they can solve the free rider problem by forming coalitions. This would, however, require a knowledge of the network structure. orming coalitions. This requires however a knowledge of the network structure.

3.2 The Company

There is one company that seeks to have information distributed about its product. It derives value from the fact that consumers are informed by word of mouth. I assume the company discounts time at the rate $\beta \in [0, 1)$, meaning that messages received earlier are better. Finally, I assume that the number of periods T is large enough to let the information reach all agents on the network. For simplicity, I assume that the number of periods is infinite, although this assumption does not affect much of my results Hence, the company values :

$$I(s_0) = p \sum_{t=0}^{\infty} \beta^t \sum_{i \in V} s_{i,t}, \tag{2}$$

where p is some exogenous price representing the market value for such word-of-mouth exposure. This formula captures, in essence, the value derived from the word-of-mouth discussion about the company's product. For instance, the company could value $h(s_{i,t})$ where h is some probability distribution of buying, given the signal. However, the main idea is embodied in I: more information is better, and sooner is better. This assumes that the information reveals something positive about the product. As noted by Goyal and Galeotti (2009), if agents perceive the product to be of poor quality, communication is harmful from the firm's standpoint. So is assumed throughout the paper that signals convey positive information about a product.

The firm prefers word-of-mouth advertising to standard advertising because information that comes from acquaintances has more impact thnt direct advertising (Bond et al., 2012). Hence, the company seeks to "seed" its advertising message only once, to a particular group of users on the network, and allow word-of-mouth communication to disseminate its message further. The firm can choose the seed s_0 from a compact strategy space S. For each $s_0 \in S$, the company must pay a cost $c(s_0)$. This cost could represent what has to be paid to convince the seeded user to disseminate the information, or simply the cost of producing the signal (advertising costs). Thus, the firm seeks to maximize:

$$\begin{aligned} s_0^* &= \arg\max_{s_0 \in S} I(s_0) - c(s_0) \\ \text{s.t. } s_{i,t+1} &= \max_{j \in \eta_i}(s_{j,t} + b_{ij}) \ \forall i \in V \end{aligned} \tag{3}$$

Hence, the firm seeks to maximize exposure given the reaction function of agents. To stick with the examples above, this problem might represent:

1. A firm that sells a product of good quality that wants to get the word out through word of mouth. It must use a "money burning" signal to seed some initial agents on the network while other agents learn quality through word of mouth. The strategy space is the set of binary vectors with only $k \leq |V|$ seeds in it. The cost of any vector is the sum of its components.

2. A political party that seeks to design a popular message with the name of its candidate in it. It thus seeks the optimal vector v^* that will maximize diffusion given the homophilic preferences of agents. The cost of producing such vector could be linear in the distance to zero: $c(s_0) = v'v$ and the strategy space is defined implicitly as any v^* such that profits are non-negative.

3. A firm wants to exploit the perceptions of agents on the network to *spin* the perceived quality of a product. It might also seek herding to increase sales. The cost of producing such signal is quadratic in quality $\left(c(s_0) = \sum_{i \in V} \frac{s_{i,0}^2}{2}\right)$ and the strategy space is defined implicitly by the set of vectors with positive profits.

The following statement can then be proven:

Theorem 1. *Given a graph $G(V,E)$ and the behavior of agents described in (1), then (3) admits at least one optimal solution $s_0^*(G)$.*

Proof. It is sufficient to notice that $I(s_0)$ is continuous in s_0 and that S is a compact set. It therefore has at least one maximum. □

In the next sections, I will work with $s_0^*(G)$ and $I(s_0^*(G))$ to discuss the relevant concepts.

Note that I leave aside all the technical considerations to *find* this solution. For instance, the first problem is known to be the set-covering problem (see Cormen, 2009), which is cursed with dimensionality (NP-hard). For a generic value of k, it is intractable on large-scale networks[2].

4 The Contagion Spillovers

4.1 The Baseline Case

In this section, I assume that the personal perceptions about other agents (b_{ij}) are equal to zero (much of the analysis remains the same when this is not the case). As I am more interested in explaining the spillover effect and the externalities rather than a particular form of solution, I proceed without these perceptions to simplify the exposition.

As a baseline, I first use the case where $E = \emptyset$. In this case, there is no word-of-mouth discussion at all on the network, as agents are isolated. In this case, the firm's decision is quite simple: it will invest in seeding a node if it can derive some profit out of this note (if $s_i > c(s_i)$ for some $s_i \in S$). If agents are identical, this condition will be the only one. This leads to the following lemma:

Lemma 1. *If $E = \emptyset$, the company seeds the network only if $s_i > c(s_i)$ for any agent.*

As there is no transmission of the signal over the network, this can be thought of as the company informing a consumer directly about the product. As the company pays all costs of such seeding, the firm bears the whole cost of advertising. There is little to discuss about this case, but I use it as a benchmark in the following section.

4.2 Word-of-Mouth Externalities and Spillovers

When $E \neq \emptyset$, the company can exploit the network structure to increase profits. If producing links is costless to agents, the increase in profits is done at no expense. Hence, all the costs of producing the signal is captured by the firm and the additional profits over the benchmark case are a result of a better technology (a spillover). However, if producing links is costly to agents, the firm profits from such links at the expense of those who paid for it. There is a free rider problem.

[2] It is, however, tractable when $k = 1$, which is the optimal solution to the problem if β is "close enough" to one.

The free rider problem exists if agents are unaware of the structure of the network and cannot form coalitions. If they can form coalitions, agents could charge the company a toll for disseminating its information. In this case, the optimal toll is such that on aggregate, the firm is indifferent between the baseline case and the word-of-mouth campaign: all the increase in the firm's profits can be captured by agents on the network. As the name "toll" suggest, this analysis is similar to the congestion literature (Kelly, 2008).

To discuss this formally, I start by defining how the solution can be expressed in terms of choices of agents:

Definition 1 (Choices)**.** *Let $k_{jt} \in \mathbb{N}$ denote the number of times the signal $s^*_{j,0}$ is chosen as the optimal signal by all agents at time t. Then, for any signal $s^*_{j,0}$, the value generated is given by:*

$$I(s^*_{j0}) = p \sum_{t=0}^{\infty} \beta^t k_{jt} s^*_{j,0}.$$

The optimal solution thus accounts for the discounted number of times each seed j is being selected. In the baseline case, $k_{j,0} = 1 \; \forall j$ and $k_{j,t} = 0 \; \forall t > 0$, but when the network has some edges, it takes different values for t greater than zero. What this shows is that there are now spillovers associated with the same seed and there is thus some latitude in increasing profits. I formalize this idea in the proposition below:

Proposition 1 (Profits non-decreasing in links)**.** *Let $G(V, E)$ be a graph with its associated profits $\Pi(s^*_0(G))$ and consider $G'(V', E')$ such that $V' = V, E \subset E'$. Then, $\Pi(s^*_0(G')) \geq \Pi(s^*_0(G))$.*

Proof. It is sufficient to notice that $\Pi(s^*_0(G))$ is still available under G' and thus, adding links can only have a non-decreasing effect on the optimum. □

This proposition simply states that the company can increase its reach when there are more links for word-of-mouth discussion. The seed of one agent has a spillover effect from the agent to some neighbors who choose his signal and thus, increases exposure.

When the cost of links are supported by agents, the company profits from a positive externality. The solution to this problem is for agents to introduce a toll on a given link. However, this requires an understanding of the alternative paths the information can take. So for the following discussion, it is assumed that agents on the network know the structure of the network. The following two propositions show that this solves the free rider problem.

Lemma 2. *Let $G(V, E)$ be graph and C be a coalition of agents able to remove a set of links $e(C) \subseteq E$, generating a graph $G'(V, E \setminus e(C))$. Then, the maximum toll the coalition can charge for the links in $e(C)$ is given by:*

$$T(C) = \Pi(s^*_0(G)) - \Pi(s^*_0(G')) \geq 0$$

Proof. The proof is straightforward: with such a toll, the company is indifferent between the links being removed on the network or paying the toll, as it leads to the same payoff. If the payoff is bigger, the company will prefer to avoid paying the toll. If it is lower, the agents make less profit. □

The next proposition summarizes the result:

Proposition 2. *Let $C_1, C_2 \dots C_i, \dots C_k$ be a sequence of coalitions covering E. Consider the sequence of graphs generated by the sequence:*

$$G(V,E), G(V, E \setminus e(C_1)), G(V, E \setminus e(C_1) \cup e(C_2)) \dots G\left(V, E \setminus \bigcup_{j=1}^{i} e(C_i)\right) \dots G(V, \emptyset).$$

Then, any such sequence of coalitions solves the free rider problem.

Proof. For simplicity, let G_i the graph $G\left(V, E \setminus \bigcup_{j=1}^{i} e(C_i)\right)$. Then, by the previous lemma, the toll charged by C_i is given by:

$$T(C_i) = \Pi(s_0^*(G_i)) - \Pi(s_0^*(G_{i-1})).$$

In particular, for $i = k$, the toll is given by:

$$T(C_k) = \Pi(s_0^*(G_i)) - \Pi(s_0^*(G(V, \emptyset)),$$

that is, all the additional value above the baseline case. Summing over all tolls yield:

$$\begin{aligned}\sum_{i=1}^{k} T(C_i) &= \sum_{i=1}^{k} \Pi(s_0^*(G_i)) - \Pi(s_0^*(G_{i-1})) \\ &= \Pi(s_0^*(G(V,E)) - \Pi(s_0^*(G(V,\emptyset)),\end{aligned}$$

which is the whole spillover effect. □

If agents are able to form coalitions and charge a toll, they can sequentially break-up the spillover effect induced by the word-of-mouth on the network. They are aware of their word-of-mouth value and threaten the company not to share anything if they do not get paid. When this occurs, the company becomes indifferent between using the network and opting for the baseline case as the efforts of coalitions capture all the additional profits. As agents charge a number arbitrarily close to the maximal toll, they induce the company to use the network and capture the effect. In this case, the free rider problem no longer occurs. What this also means is that word-of-mouth advertising is a profitable strategy in an environment where agents have no means to form coalitions or if the efforts required to form a coalition are too high.

For instance, professional news organizations might have the means to cooperate and thus internalize some of the free-riding (convergence), but blogs might find it more profitable to let the traffic flow through their websites rather than spend efforts to organize.

I do not address how these coalitions could be formed, if at all, and what their motives could be. In particular, I have said nothing on how the structure (or the order) of these coalitions can influence the distribution of the profits among agents. These are potential areas for future research.

5 The Social Value of Word-of-Mouth Advertising

A company might have some interest in shaping messages about its product on a social network. In doing so, the firm hopes to use the network to its own advantage not only to increase diffusion, but also to change the perceived quality of the product. This can be done only if the network performs some active modifications of the "seed" the initial message the advertiser sends to certain agents on the network. If the seed is shared from one user to another without modifications, the signal remains identical so the only consideration is diffusion. Thus, in order to have a meaningful discussion, I assume that the perceptions about other users are no longer zero (i.e. $b_{ij} \neq 0$). This means that although one user j might send a signal of zero value about the product, a source i thinks j's valuation of the product is incorrect, so i corrects the signal to what i believes to be the true value, $0 + b_{ij}$. These perceptions can be positive or negative. The main assumption behind these numbers is that how we see our sources of information influences our perceptions of the products. Again, these numbers need not be fixed, but are kept so in the context of this article for tractability.

This introduces an additional dimension in the firm's optimization problem: there is now a trade-off between dispersion and credibility. One agent might have a high number of sources but his sources might think poorly of him. So by introducing heterogenous b_{ij}s in the model, the firm changes his optimal solution s_0^* to exploit these perceptions optimally.

In the context of this paper, the interesting part is what happens when the company does not seed any message to the network (or if they seed a zero vector). Such a solution might not be optimal, but shows what happens in terms of valuation. After one period, every agent i chooses the message from the most highly perceived source in his neighborhood and shares it with his neighbors:

$$s_{i,t+1} = \max_{j \in \eta_i}(0 + b_{ij}).$$

In the next period, the signals, or the valuations about the product, will be changed by the perceived value between agents. Summing these numbers over time according to the income function I will yield the "natural" value generated by the network. I define this as the social value of the product:

Definition 2 (Social Value)**.** *Let $G(V, E)$ be a network of agents responding to signals according to (1) and let $\vec{0} \in \mathbb{R}^{|V|}$ be the zero vector. Then, the social value of the product on the network is given by $I(\vec{0})$.*

By construction, this value is independent of the actions of the company. It therefore cannot be captured or internalized. To characterize the social value, a bit of investment in the mathematics of difference equations is required. I will state in the theorem below and refer to a treaty on tropical algebras for the proof.

Theorem 2. *Let $G(V, E)$ be a graph of agents and let $G(s_t) = s_{t+1}$ be the law of motion induced by the behavior of agents summarized in (1). Assume that this graph as at least one cycle*[3]*. Then:*

1. *There exists a time $t^* < \infty$, a vector $\tilde{s}$ and constant $\lambda \in \mathbb{R}$ such that $G(\tilde{s}) = \tilde{s} + \lambda \imath$ for all $t > t^*$, where $\imath$ is the unit vector.*
2. *Denote the composition function by:*

$$G^n(s) = \underbrace{G \circ G \circ \cdots \circ G(s)}_{n \text{ times}},$$

then $\tilde{s}$ is given by $G^{t^*}(\vec{0}) - \lambda \imath t^*$.

3. *Consider the average value of perceptions b_{ij} over a given cycle on the network*[4]*. Then λ is the highest such average amongst all cycles in the graph.*

Proof. See Bacelli, Cohen, Olsder, Geert and Quadrat, 1992 . □

This theorem explains the dynamics of signal selection over the network when the zero vector is seeded to agents. After a given finite time t^*, the network enters a periodic regime where agents choose the same sources of information over and over. As such, the vector of signals remains identical, up to an increase of λ. This value λ is given by the highest average perception of agents that lie on a cycle.

What this means is that after t^*, highly perceived agents are responsible for the dynamics of the network. Agents on the cycle listen to each other, generating at every step an increase in the perceived value of λ. As they have are the most well regarded, all

[3]A cycle is a repetitionless path of edges whose starting point is also its endpoint. For instance, in Figure 1, the path e_{30}, e_{13}, e_{01} is a cycle.

[4]Using the example in the last footnote, the average value if the cycle $\{e_{30}, e_{13}, e_{10}\}$ is given by $(b_{30} + b_{13} + b_{10})/3$.

other agents listen to them either directly by choosing them as a source, or indirectly by choosing a source that listens to them. After t^*, they generate the whole stream of value. This theorem allows for the following statement:

Proposition 3 (Social value)**.** *Let $G(V, E)$ be graph that admits at least one cycle of agents and let agents behave as in (1). Then, the social value can be decomposed in the following three terms:*

$$I(\vec{0}) = \underbrace{p\frac{\beta^{t^*+1}}{1-\beta}\sum_{i\in V}\tilde{s}_i}_{\textit{Long-run value}} + \underbrace{p\lambda\frac{\beta|V|}{(1-\beta)^2}}_{\textit{Opinion leaders' value}} + \underbrace{p\sum_{t=0}^{t^*}\beta^t\left(\sum_{i\in V}\left[G^t(\vec{0})\right]_i - \lambda t\right)}_{\textit{Short-run value}} \tag{4}$$

Proof. After t^*, the vector is given by:

$$p\sum_{t=t^*+1}^{\infty}\beta^t\sum_{i\in V}(\tilde{s}_i + \lambda t) = p\frac{\beta^{t^*+1}}{1-\beta}\sum_{i\in V}\tilde{s}_i + p\lambda|V|\sum_{t=t^*+1}^{\infty}\beta^t t.$$

Before t^*, it is sufficient to substract λt from the summation to obtain:

$$p\sum_{t=0}^{t^*}\beta^t\sum_{i\in V}\left[G^t(\vec{0})\right]_i = p\lambda|V|\sum_{t=0}^{t^*}\beta^t t + p\sum_{t=0}^{t^*}\beta^t\left(\sum_{i\in V}\left[G^t(\vec{0})\right]_i - \lambda t\right)$$

Combining the two elements leads to the desired answer. □

The last proposition shows that the social value of word-of-mouth advertising on the network can be decomposed into three components. The first is the long-run value of the choices induced by perceptions. As explained in the previous paragraphs, these choices reflect direct or indirect connections to the cycle of highly perceived agents. It measures the value of this choice, net of the increase λ at each period.

The second term is the value of the influence of highly perceived agents. As they generate an increase of λ at every step of the word-of-mouth process, it is measured as the discounted value of this increase.

The last term pertains to the transitionary dynamics from $t = 0$ to t^* and is explicitly dependent on the structure of the network. This is the short-run social value of the work-of-mouth advertising. This also illustrates that time plays a role in determining the value. If the number of periods to allow word-of-mouth communication is smaller than t^*, then only the last term and the relevant share of the second term matters in the social value.

It also shows that depending on how the company values time (through β), some components might be more valuable than others. In particular, if β is close to one, the value of influential persons is much higher, through the factor $(1-\beta)^{-2}$. Intuitively, all agents connect to these agents in the long run, so if the company is patient, they are the relevant persons to influence the discussion about the product. There are however two exceptions that I discuss in the following proposition.

Corrolary 1. *Let $G(V, E)$ be a graph with agents behaving as in (1). Then:*

1. *If the highest average on all cycles is zero $\lambda = 0$, the value of opinion leaders is zero.*
2. *If there is no cycle on the graph, then there is no long-run value of word-of-mouth.*

Proof. Both points are direct consequences of the previous theorem. ☐

One can substitute $\lambda = 0$ in the previous formula to find the result. As for the second point, the intuition is that if there is no way information can get back to its original starting point, there is no way a fixed point of the form $G(s) = s + \lambda$ can occur.

Hence, letting agents discuss for a long enough period is critical in the creation of social value. If there is no cycle, the flow of information on the network stops and the long-run value is lost. This is relevant in situations where agents do not repeat information they have already shared.

6 Conclusion

I have discussed how the notion of externalities and social value apply in the context of word-of-mouth communication. I have shown that a connected network can be used to increase sales. In a context where information is valuable, the more interconnected a network is, the more companies have flexibility to increases their profits through word-of-mouth advertising. If the network technology is costless, this can be seen as a simple improvement of the technology firms have at their disposition. However, if users on a network bear a cost to creating links, a free rider problem occurs as the firm benefits freely from the word-of-mouth advertising while it is costly for agents on the network. If agents can form a coalition and charge a "toll" for sharing information about a firm's product, then any sequence of coalitions that covers the whole set of agents will capture the value of the externality. The distributional aspects of such a sequence of coalitions, or how each agent could capture some of the externality, remains a question open to further research.

If agents perform some active modifications of the information disseminated on the network, they generate a social value of information. In the context of a firm seeking to spread information about its product, this social value can be thought as the valuation that agents form about a product by talking to each other, without any intervention. Such a valuation can be positive, or negative, depending on how information is modified. Since this process occurs regardless of the action of the company, the firm has no way to capture this valuation.The opinion leaders, or highly regarded agents on the network, play a key role in the formation of such social value. As these agents are highly perceived, all other agents link directly or indirectly to these agents and they become central players in the modification of information.

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