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Subjective Evaluations with Performance Feedback

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Abstract

This paper models two key roles of subjective performance evaluations: their incentive role and their feedback role. The paper shows that the feedback role makes subjective pay feasible even without repeated interaction, as long as there exists some verifiable measure of performance. It also shows that while subjective pay is helpful, it cannot achieve full efficiency. However, fully efficient incentives are achievable if the firm can commit to a forced distribution of evaluations and employs a continuum of workers. With a small number of workers, a forced distribution is valuable only if the verifiable measure is poor.

Keywords: Subjective Evaluations, Performance Feedback, Optimal Incentive Contracts

JEL Classification: D82, D86, M52

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1. Introduction

Most workers are regularly evaluated by their superiors. Such an evaluation typically includes the superior's subjective judgement about the worker's performance — for example, Gibbs et al (2007) document the use of subjective performance evaluations in the compensation packages of auto dealership managers; Levin (2003) cites survey evidence of subjective performance pay in law firms; and Eccles and Crane (1988) describe how the compensation of investment bankers depends on such subjective measures as the quality of their deals and customer satisfaction. Even the pay of the CEOs often depends on subjective assessments by the firms' boards of directors (Bushman et al, 1996; Hayes and Schaefer, 1997).

Performance evaluations usually serve multiple goals, but two of the most important ones are provision of incentives and performance feedback. For example, Cleveland et al (1989) report that 69% of their survey respondents considered salary administration and 53% considered performance feedback to be among the three main purposes of performance appraisals. The incentive role of subjective evaluations has been studied extensively in the economics literature (e.g. MacLeod and Malcomson, 1989; Baker et al, 1994; Levin, 2003), but their feedback role, although recognized (Milgrom and Roberts, 1992, p. 407; Prendergast, 2002), has been largely missing from formal models.[1]

This paper spotlights the feedback role of evaluations by incorporating it into a principal-agent model in which subjective evaluations both motivate workers and provide them with information about their productive abilities. The paper shows how the feedback and the incentive roles of evaluations interact with each other and that absent reputational concerns, performance feedback is key to making the incentive part of the evaluations operational.

A well recognized problem with subjective pay, which seriously undermines its incentive effects, is that supervisors are tempted to underreport workers' performance in order to save on labor costs. Gibbons (2005) argues that this reneging/commitment problem is at least as important for understanding real world contracts as the more frequently studied trade-off between incentives and insurance. Accordingly, the previous research on the topic has focused primarily on how firms can make subjective pay functional despite this trust problem. Most of this work follows Bull (1987) in emphasizing the supervisors' reputational concerns in infinitely repeated games and the received wisdom is that the reneging problems make stationary contracts with subjective pay ineffective (Levin, 2003; Prendergast, 2002).

Using a two-period model, I show that the feedback role of performance assessments

[1] One exception is Suvorov and van de Ven (2009), discussed in greater detail below. The effects of feedback have also been studied in several tournament models; these models, however, are not about subjective evaluations.

mitigates the reneging problem and makes subjective incentive schemes feasible even without infinite interaction. Central to this conclusion are two ideas. The first is that ability and effort are complementary in production. Although in some jobs (say, janitorial jobs) ability may have no bearing on the productivity of one's effort, most jobs are likely to exhibit some degree of complementarity between effort and ability. The second central idea is that feedback from a supervisor allows a worker to update his belief about his productive ability. Because under complementarity the worker's subsequent performance depends on this belief, the employer has an incentive to give the worker a good evaluation, in order to boost his effort. A properly designed reward scheme then balances the supervisor's desire to inflate the worker's self-assessment against her temptation to save on labor costs by under-reporting. This makes honest evaluations feasible without resorting to infinite interaction arguments.

The model's contracting framework builds on the observation that a worker's contribution to firm value is frequently complex and hard to capture by an objective measure. Consequently, when objective measures are imperfect and could lead to dysfunctional behavior, firms complement them with subjective schemes in which a worker's salary, bonus, or promotion depend upon his superior's perception of his performance. This point has been recognized by many writers (e.g., Baker et al., 1994; Prendergast, 1999) and can be traced at least to Alchian and Demsetz's (1972) classic theory of the firm.[2] It is also consistent with the evidence on incentive systems in auto dealerships provided by Gibbs et al (2007).

In line with the above, I assume that an objective (verifiable) performance measure is available, but imperfect. The exact source of contracting imperfections is not important for the paper's main conclusions, but for concreteness, I assume that they are due to multitasking problems similar to those studied in Feltham and Xie (1994), Datar et al (2001), and Baker (2002). Specifically, the objective measure distorts an agent's allocation of effort across tasks because it aggregates his individual efforts in a manner that differs from his contribution to the firm value. In this setting, subjective evaluations are useful because they are based on an undistorted measure; in fact, the principal would ideally like all of the agent's incentives to derive from subjective pay. I show that this is in general not possible, as the principal's freedom to design the contract is constrained by the need to ensure that the evaluations are truthful. Nevertheless, at least some incentives derived from subjective pay are always feasible, as long as the objective measure is not completely worthless.

An optimal contract in this environment arises from a mechanism design problem in which

[2] Alchian and Demsetz argue that the lack of good objective measures for workers' individual contributions is the very reason firms exist. The firm, in their view, is a device that allows some individuals to specialize in observing workers' performance and in rewarding them according to their marginal contribution to joint output.

the principal faces her own, rather than the agent's, truthtelling constraint. Consequently, the contract is not shaped by the standard trade-off between rent extraction and allocation efficiency. Rather, the trade-off is between the efficiency of the incentives provided by the objective measure in the second period and the efficiency of the subjective pay in the first period. In particular, for the subjective evaluations to provide any incentives, the second period objective contract must necessarily be distorted away from the optimal form it would have in the absence of subjective pay. This trade-off limits the usefulness of subjective pay, preventing the optimal contract from achieving full efficiency in the first period.

The situation is different when firms can pre-commit to a specific distribution of evaluations. Such "forced distributions" are common in real world firms (one well known example is GE's "vitality curve"), but their purpose is not well understood. I show that, similar to the benefit of tournaments pointed out by Malcomson (1984), the advantage of a forced distribution is that it relaxes the principal's truthtelling constraint by making the size of the wage bill independent of individual evaluations. This eliminates the above tradeoff and allows the principal to achieve full efficiency in the first period by completely replacing the objective measure with subjective pay.

The truthtelling benefit of forced distributions does not come for free, however. A forced distribution limits the amount of information that the evaluations convey about the workers' productive capacities, which impedes the workers' ability to properly tailor their second period efforts. Crucially, this constraint gets more stifling the smaller is the number of workers. The number of workers is therefore of central importance in the choice between subjective evaluations with and without a forced distribution: When the workforce is large, a forced distribution can closely approximate the true distribution of the workers' productivities and is therefore very informative. In this case, the efficiency gain from improved first period incentives outweighs the loss from the misallocation of the second period effort.

In contrast, when the number of workers is small, the choice between the two subjective schemes depends on the quality of the *objective* measure. If the objective measure is good, the main benefit of subjective evaluations is to inform the workers about their productive abilities, which favors subjective evaluations without a forced distribution. If the objective measure is poor, then it provides very inefficient incentives even if the workers are fully informed about their abilities. In this case, the main goal is to strengthen the workers' incentives, which is best achieved via evaluations with a forced distribution.

Related literature. In its focus on the interaction between the incentive effects of objective and subjective measures, this paper is related to Baker et al (1994), who were the first to formally model such an interaction. In Baker et al, however, subjective pay is sustained through infinite interaction in a repeated game, whereas I present a finite horizon model. Furthermore, the feedback function of subjective evaluations, so prominent in the current model, plays no role in their analysis.

Another seminal contribution to the theory of subjective evaluations under repeated interaction is Levin (2003). Levin shows that in his model it is optimal for the principal and the agent to govern their relationship through a termination contract, in which a poor performance evaluation is followed by the two parties dissolving the relationship.

MacLeod (2003) has generalized the logic of repeated game models by demonstrating that subjective pay schemes can be feasible even without infinite interaction if workers can punish a deviation from the implicit contract by imposing on the employer some type of socially wasteful cost, say, through quitting or sabotage at the firm. The optimal contract then trades off ex post socially wasteful conflict against ex ante performance incentives. This model was further developed by Fuchs (2007), who extended it to a more dynamic environment, and by Rajan and Reichelstein (2009), who introduced in it objective measures of performance.

The present paper complements the MacLeod/Fuchs/Rajan/Reichelstein theory by examining an alternative mechanism for sustaining subjective evaluations that does not require ex post destruction of surplus. Further, it points to the availability of an objective measure as a crucial determinant of feasibility of subjective pay and to the number of workers as a determinant of whether the subjective pay scheme will include a forced distribution. Also, where Fuchs concludes that in his setting it is optimal for incentive purposes not to give the agent interim feedback about his performance, the current paper provides a framework in which interim feedback is vital. This accords well with the evidence that companies cite feedback to workers as one of the main reasons for using subjective evaluations.

Rajan and Reichelstein also examine a setting with two agents. The optimal scheme in this part of their analysis does not require surplus destruction and resembles a forced distribution of evaluations studied in the second part of this paper. However, unlike in the present model, a forced distribution is necessary for Rajan and Reichelstein's scheme to work (in the absence of surplus destruction) and hence the question whether or not the firm will find it optimal to use a forced distribution cannot be answered within their framework.

The effects of a principal's feedback on an agent's effort have recently been studied by Aoyagi (2007), Goltsman and Mukherjee (2011), and Ederer (2010) in the context of multi-stage tournaments, and by Suvorov and van de Ven (2009) in the context of subjective evaluations. Suvorov and van de Ven's analysis is closely related to the first part of the present paper. They, too, show that informative subjective evaluations are feasible if the principal has private information about the agent's ability. They, however, do not allow for objective measures of performance and instead assume that the agent has intrinsic motivation to provide effort. Consequently, their model is not suitable for studying the interaction between subjective and objective measures of performance central to this paper. Furthermore, they confine their attention to a single agent setting and do not study forced distributions.

In the multi-stage tournament models of Aoyagi (2007) and others, feedback does not always affect effort in a desirable way and the main question is whether the agents provide more effort with or without information revelation.[3] In the present paper, feedback is always useful. More to the point, this tournament literature is not about subjective evaluations, as it assumes that the feedback is contractible, which eliminates the problem of inducing the principal to reveal her information truthfully.

Finally, the model of this paper is also formally related to Hermalin (1998) and Benabou and Tirole (2003). These two papers share with the present model the feature that a principal uses her private information about the production process to influence agents' incentives. More specifically, Hermalin shows how a leader of a team can use a contract with side payments to credibly communicate her superior information to other team members and Benabou and Tirole study how a principal can adjust employment policy, such as contingent pay, to manage an agent's self-confidence. Despite these common features, the present paper focuses on issues that do not arise in the above two papers, such as the incentive effects of subjective pay, the interplay between subjective and objective measures of performance, and the usefulness of forced distributions of messages.

The plan for the rest of the paper is as follows. Section 2 describes the model and the main assumptions. Section 3 contains an analysis of the case without a forced distribution of evaluations. It provides conditions under which subjective pay is feasible, a result regarding the efficiency of subjective pay schemes, and a characterization of the optimal contract. This section also discusses the role of commitment for the feasibility of subjective pay. Section 4 allows for evaluations with a forced distribution and compares the benefits and disadvantages of the two types of subjective schemes. Section 5 concludes.

2. The Model

Production technology. A principal (she) supervises an agent/worker (he) over two periods, $t = 1, 2$. The worker's output in period t is $y_t \in \{0, 1\}$. The probability of high output $y_t = 1$ is given by $q_t = a\mathbf{e}_t \cdot \mathbf{f}$, where $a \in \mathbb{R}_+$ is the worker's innate time-invariant ability, $\mathbf{e}_t = (e_{1t}, e_{2t}, ..., e_{Kt}) \in \mathbb{R}_+^K$ is his K-dimensional, $K \geq 2$, vector of efforts provided in period t, and $\mathbf{f} = (f_1, f_2, ..., f_K) \in \mathbb{R}_+^K$ is the vector of marginal contributions of the worker's efforts to firm value. As noted in the Introduction, a key feature of this specification is that ability

[3] An early paper that addresses this question in a single-agent setting is Lizzeri et al (2003).

and effort are complements in the production function.

The worker's ability is initially unknown. Both the worker and the principal only know that the ability is drawn from an interval $[0, \bar{a}]$ according to a distribution function $H(a)$ with density $h(a)$, which is positive and twice differentiable at each a.[4]

Performance measures. Neither the worker's expected contribution to firm value, q_t, nor its realization, y_t, are contractible. Instead, the worker's incentives come from two alternative sources:

Objective measures. First, there are contractible but imperfect measures of the worker's performance, $z_t \in \{0,1\}$, $t = 1, 2$. The probability that $z_t = 1$ is $p_t = a\mathbf{e}_t \cdot \mathbf{g}$, where $\mathbf{g} = (g_1, g_2, ..., g_K) \in \mathbb{R}_+^K$ captures the marginal impact of the worker's efforts on z_1 and z_2.

The measures z_t are imperfect in the sense that $\mathbf{g} \neq \alpha\mathbf{f}$ for any constant α. This makes it impossible for a contract based solely on z_t to induce the efforts that maximize the firm's value. The degree of distortion of the objective measure will be captured by the angle between $\mathbf{g}$ and $\mathbf{f}$ denoted by θ and defined by $\cos\theta = \frac{\mathbf{f} \cdot \mathbf{g}}{\|\mathbf{f}\|\|\mathbf{g}\|}$, where $\|\mathbf{f}\|$ and $\|\mathbf{g}\|$ are the lengths of the vectors $\mathbf{f}$ and $\mathbf{g}$ respectively, that is, $\|\mathbf{f}\| = \sqrt{\sum_{k=1}^K f_k^2}$ and $\|\mathbf{g}\| = \sqrt{\sum_{k=1}^K g_k^2}$. To ensure that p_t and q_t can be interpreted as probabilities, assume $\max\{\|\mathbf{f}\|, \|\mathbf{g}\|\} \leq 1/\bar{a}$.

Without loss of generality, the focus will be on performance measures such that $\cos\theta \geq 0$ — an undistorted measure would have $\cos\theta = 1$ and the smaller is $\cos\theta$, the more distorted is the measure.

Subjective measures. At the end of period t, the principal privately observes the worker's expected contribution to the firm's period-t value, $q_t = a\mathbf{e}_t \cdot \mathbf{f}$.[5] This specification captures the idea, long present in the economics literature, that by the nature of her job, a supervisor has superior information about the worker's contribution to firm value: "The employer, by virtue of monitoring many inputs, acquires special superior information about their productive talents" (Alchian and Demsetz, 1972, p. 793). Alternatively, one could think of a as the quality of the principal's project and q_t as her private signal about this quality.

To ease the exposition, it will be assumed that z_1, z_2, y_1, and y_2 only become observable at the end of period 2. If z_1 and/or y_1 were observable at the end of period 1, the worker would use them to update his belief about his ability, but this would not allow him to fully infer a. Since the analysis will not depend on the exact functional form of the worker's prior belief $H(a)$, such an updating plays no substantive role and can be ignored.

[4] It would be straightforward to adapt the model so that a represents human capital that the worker develops during the first period.

[5] This stark informational structure is adopted for its simplicity. At the cost of complicating the analysis, both the principal and the worker could receive imperfect signals about the worker's ability as long as the worker's signal is not a sufficient statistic for the principal's signal with respect to a.

Subjective evaluations and contracting. After privately observing the worker's first period performance, the supervisor provides him with a subjective evaluation, which consists of a message $m \in [0, \bar{a}]$ about the worker's ability a.[6] This message is contractible, so that the worker's wage, w, can be written as $w = w(z_1, z_2, m)$. It will be convenient to write the general contract in terms of a base salary $s(m)$ and bonuses $b_1(m)$, $b_2(m)$, and $b_3(m)$, all of which can depend on m. The worker receives $b_1(m)$ if $z_1 = 1$, $b_2(m)$ if $z_2 = 1$, and $b_3(m)$ if $z_1 = z_2 = 1$, while the salary $s(m)$ is independent of z_1 and z_2.

In the first part of the paper, m will be the only contractible part of the subjective evaluation scheme. The second part of the paper will consider subjective evaluations with a forced distribution, where not only m but also the resulting distribution of m is contractible. The possibility that m is not contractible will be discussed in Subsection 3.5.

Preferences. Both the principal and the worker are risk neutral and do not discount future income. The principal's goal is to maximize the firm's expected profit. The worker's per period reservation utility from not working is normalized to zero and his lifetime utility from being employed by the firm is $w - \Psi(\mathbf{e}_1) - \Psi(\mathbf{e}_2)$, where $\Psi(\mathbf{e}_t) = \sum_{k=1}^{K} \psi(e_{kt}) = \sum_{k=1}^{K} e_{kt}^2/2$ is his disutility from effort in period $t = 1, 2$. The worker's participation constraint only needs to be satisfied at the beginning of the relationship, when the contract is signed.

Timing. At the beginning of the first period, the principal and the worker sign a contract that specifies the wage function $w(z_1, z_2, m)$. Subsequently, the worker chooses his first period effort levels, $\mathbf{e}_1$. At the end of the first period, the principal observes the worker's input q_1 and provides the performance evaluation m. At the beginning of the second period, the worker updates his belief about his own ability and exerts second period efforts $\mathbf{e}_2$. At the end of the second period, z_1 and z_2 are observed and the worker is paid $w(z_1, z_2, m)$.

3. The Analysis

3.1. Two benchmarks

To understand the nature of the optimization problem faced by the principal, it is helpful to start with a brief analysis of two benchmark cases.

Symmetric information. In the first benchmark, the agent receives the same information about his performance as the principal. In this case subjective evaluations do not play any meaningful role and the agent's incentives depend solely on the objective measures z_1 and

[6] Alternatively, the message could be about the agent's first period contribution q_1. The current formulation simplifies the exposition.

z_2. The principal's problem is then to choose a message independent contract (s, b_1, b_2, b_3) so as to maximize the expected total surplus[7]

$$E_a[q_1 - \Psi(\mathbf{e}_1) + q_2 - \Psi(\mathbf{e}_2)],$$

subject to the agent's incentive compatibility constraints for the two periods

$$\begin{aligned}
\mathbf{e}_2 &= \arg\max_{\mathbf{e}_2'} \left(b_2 + b_3 p_1\right) p_2' - \Psi(\mathbf{e}_2'); \\
\mathbf{e}_1 &= \arg\max_{\mathbf{e}_1'} E_a \left[s + b_1 p_1' + \left(b_2 + b_3 p_1'\right) p_2 - \Psi(\mathbf{e}_2) - \Psi(\mathbf{e}_1')\right],
\end{aligned}$$

where $p_1' = a\mathbf{e}_1' \cdot \mathbf{g}$ and $p_2' = a\mathbf{e}_2' \cdot \mathbf{g}$.

Since conditional on the worker's ability a the realizations of z_1 and z_2 are independent of each other, one can without loss of generality set $b_3 = 0$ and treat the incentive problems in the two periods as two separate problems. Replacing the agent's IC constraints with their respective first order conditions, the principal's first period problem is then to choose b_1 so as to maximize $E_a[a\mathbf{e}_1 \cdot \mathbf{f} - \Psi(\mathbf{e}_1)]$ subject to $E(a) b_1 g_k = \psi'(e_{k1})$, and her second period problem is to maximize $a\mathbf{e}_2 \cdot \mathbf{f} - \Psi(\mathbf{e}_2)$ subject to $a b_2 g_k = \psi'(e_{k2})$, $k = 1, 2, ..., K$. The only difference between these two problems is that in period 2 a is publicly known, whereas in period 1 only the distribution of a is known. In either case, though, the optimal bonus is the same:

$$b_1 = b_2 = b^{SB} = \frac{\|\mathbf{f}\|}{\|\mathbf{g}\|} \cos\theta,$$

where the superscript SB indicates that the solution represents a second-best contract.

As discussed in Baker (2002), $\cos\theta$ in b^{SB} captures the degree of congruence between the performance measure and the firm value (the closer is $\cos\theta$ to 1, the better is the objective measure), while the term $\frac{\|\mathbf{f}\|}{\|\mathbf{g}\|}$ reflects scaling (i.e., it accounts for the fact that $\mathbf{f}$ and $\mathbf{g}$ can have different lengths). The benchmark bonus b^{SB} will prove useful later, in characterizing the optimal contract. For future reference, note that b^{SB} does not depend on a.

To summarize, without an informational asymmetry, the problem collapses into a standard problem familiar from the literature on multitasking.

Perfect objective measures. In the second benchmark of interest, instead of assuming that the agent observes the principal's information, assume that the measure z_t is perfectly

[7] As usual, the problem reduces to surplus maximization after the agent's individual rationality constraint is substituted into the principal's expected profit function.

aligned with y_t, so that $\cos\theta = 1$. In this case the optimal contract does entail subjective evaluation m, but s, b_1, b_2, and b_3 are again set to be independent of m. Under such a contract, the principal is willing to reveal her private information truthfully. The evaluations then do not have any incentive effect, but in this ideal case incentives from subjective pay are not needed, because a perfect objective measure can ensure the first-best outcome. In particular, the agent's incentive problems in the two periods can again be viewed as independent of each other and solved separately. Analogous to the solution obtained in the previous benchmark, the optimal contract is obtained as $b_1 = b_2 = b^{FB} = \frac{\|\mathbf{f}\|}{\|\mathbf{g}\|}$. To see that this achieves the first best, note that when the vectors $\mathbf{f}$ and $\mathbf{g}$ are of the same length, this solution reduces to the standard first-best contract for risk-neutral agents, $b^{FB} = 1$.

In what follows, the objective measure is imperfect and cannot provide the first-best incentives. Additional incentives from subjective evaluations, in which the agent's pay depends on $\mathbf{e}_1$ through m, are therefore valuable. But such a subjective scheme cannot be arbitrary — it has to be incentive compatible for the principal. As we will see, this will be made possible by the fact that the evaluations also affect the agent's *second* period effort, $\mathbf{e}_2$.

Thus, when the objective measure is distorted, the agent's incentive problems in the two periods are no longer independent of each other. Rather, they are connected through the principal's truthtelling constraint and have to be solved simultaneously. I will now turn to the analysis of this problem.

3.2. Feasibility of subjective evaluations with incentive effects

It is clear that truthful evaluations always affect the agent's second period effort (as long as it is positive), by affecting his belief about his ability. But is it possible for subjective evaluations to also have first-period incentive effects? The analysis will start by addressing this question.

3.2.1. The worker's problem

Working backwards from the second period, let $x(m)$ be the worker's posterior belief about his expected ability based on his subjective evaluation m. The worker's second period problem is then to choose $\mathbf{e}_2$ so as to maximize $\beta(m)x(m)\mathbf{e}_2 \cdot \mathbf{g} - \Psi(\mathbf{e}_2)$, where $\beta(m) \equiv b_2(m) + b_3(m)p_1$. One can think of $\beta(m)$ as a "composite bonus," but it is important to bear in mind that it depends on p_1 and hence on $\mathbf{e}_1$. The worker's second period efforts $e_{k2}(\beta, x)$ are then determined by the first order conditions

$$\beta(m)x(m)g_k = \psi'(e_{k2}), \quad k = 1, 2, ..., K.$$

The principal does not observe the worker's first period efforts, but she makes a conjecture about them, $\tilde{\mathbf{e}}_1$. She then uses her observation of $q_1 = a\mathbf{e}_1 \cdot \mathbf{f}$ to infer the worker's ability as $a = \frac{q_1}{\tilde{\mathbf{e}}_1 \cdot \mathbf{f}}$. Focusing on truth-telling and fully separating contracts, the principal's equilibrium message will be $m = \frac{q_1}{\tilde{\mathbf{e}}_1 \cdot \mathbf{f}}$, which will allow the worker to infer his ability via $x(m) = m\frac{\tilde{\mathbf{e}}_1 \cdot \mathbf{f}}{\mathbf{e}_1 \cdot \mathbf{f}}$.[8,9] In equilibrium, the principal's conjecture will be correct, $\tilde{\mathbf{e}}_1 = \mathbf{e}_1$. The first set of incentive compatibility constraints for the principal's optimization problem is thus obtained as

$$\beta(m)mg_k = \psi'(e_{k2}), \quad k = 1, 2, ..., K. \tag{ICW$_2$}$$

The thing to notice here is that, holding $\beta(m)$ fixed, each component of the worker's vector of efforts increases in his belief $x(m)$ and hence in the principal's evaluation m.

In period 1, the worker chooses his efforts so as to maximize his expected lifetime utility

$$E_a[s + b_1(m)a\mathbf{e}_1 \cdot \mathbf{g} + \beta(m)a\mathbf{e}_2 \cdot \mathbf{g} - \Psi(\mathbf{e}_2)] - \Psi(\mathbf{e}_1),$$

taking into account the effect of $\mathbf{e}_1$ on the principal's report m. In particular, for any first period effort vector $\hat{\mathbf{e}}_1$, the worker expects the evaluation $m(\hat{\mathbf{e}}_1) = \frac{a\hat{\mathbf{e}}_1 \cdot \mathbf{f}}{\tilde{\mathbf{e}}_1 \cdot \mathbf{f}}$. This yields the worker's first-period incentive compatibility constraint

$$\mathbf{e}_1 \in \arg\max_{\hat{\mathbf{e}}_1} E_a\left[s(\frac{a\hat{\mathbf{e}}_1 \cdot \mathbf{f}}{\tilde{\mathbf{e}}_1 \cdot \mathbf{f}}) + b_1(\frac{a\hat{\mathbf{e}}_1 \cdot \mathbf{f}}{\tilde{\mathbf{e}}_1 \cdot \mathbf{f}})a\hat{\mathbf{e}}_1 \cdot \mathbf{g} + \beta(\frac{a\hat{\mathbf{e}}_1 \cdot \mathbf{f}}{\tilde{\mathbf{e}}_1 \cdot \mathbf{f}})a\mathbf{e}_2 \cdot \mathbf{g} - \Psi(\mathbf{e}_2)\right] - \Psi(\hat{\mathbf{e}}_1) \tag{ICW$_1$}$$

3.2.2. The principal's problem

The principal's message m maximizes her expected second period profit subject to the worker's incentive compatibility constraint (ICW$_2$). Combined with the requirement of truthtelling, this yields the following incentive compatibility constraint for the principal:

$$a \in \arg\max_m a\mathbf{e}_2 \cdot \mathbf{f} - \beta(m)a\mathbf{e}_2 \cdot \mathbf{g} - s(m) - b_1(m)a\mathbf{e}_1 \cdot \mathbf{g}. \tag{ICP}$$

[8] The possibility of pooling will be discussed in Section 3.4.2.

[9] Although for an arbitrary m the belief $x(m)$ depends on $\mathbf{e}_1$, I do not indicate this dependence in notation, as under truthful reporting, $x(m)$ is independent of $\mathbf{e}_1$. That is, the agent cannot fool himself by providing more (or less) effort.

The principal's general problem is then to maximize the total surplus from the employment relationship according to the following program:

$$\text{(P)} \qquad \max_{s(m),b_1(m),b_2(m),b_3(m)} E_a[a\mathbf{e}_1\cdot\mathbf{f} - \Psi(\mathbf{e}_1) + a\mathbf{e}_2\cdot\mathbf{f} - \Psi(\mathbf{e}_2)]$$

subject to (ICW_1), (ICW_2), and (ICP).

I will say that subjective evaluations provide incentives when the marginal effect of the agent's first-period effort on his expected pay is weakly (and at least for some effort levels strictly) higher when his pay depends on m than when it does not. Recalling that θ denotes the angle between $\mathbf{f}$ and $\mathbf{g}$, the first result provides conditions under which subjective evaluations with incentive effects are feasible.

Proposition 1.

(i) *If* $\cos\theta = 0$, *no subjective evaluation scheme with incentive effects is feasible.*

(ii) *If* $\cos\theta > 0$, *then a subjective scheme that is both truthful and provides incentives for first period effort is feasible.*

Proof: All proofs are in Appendix A.

In most of the existing literature, subjective evaluations with incentive effects are feasible only if the principal and the agent interact repeatedly (e.g., Baker et al, 1994; Levin, 2003) or if agents can take ex post inefficient actions that destroy surplus (MacLeod, 2003). Proposition 1 shows that neither repeated interaction nor surplus destruction are needed for subjective evaluations to have incentive effects.

The logic behind the result is related to the idea of countervailing incentives in Lewis and Sappington (1989). Specifically, in contrast to most of the earlier models of subjective pay, the subjective measure of the worker's performance depends not only on the worker's actions, but also on his underlying type (ability). This presents the principal with two opposing temptations. On the one hand, she wants to give the worker a bad evaluation in order to save on the wage bill. This is the standard consideration, extensively studied in the previous literature. On the other hand, the principal knows that the worker's second period effort increases in m and this tempts her to boost the worker's self-assessment through a good evaluation. A truthtelling wage scheme then balances these two temptations in such a way that they offset each other.

Critically, the second effect is only present if the worker's output in period 2 depends on m. For a subjective scheme to work, it is therefore important that the principal has access to some objective measure, no matter how poor. Otherwise, the worker provides no valuable effort in period 2 and there is no point trying to influence his belief. In such a case, evaluations can be truthful only if they do not affect the worker's pay, which means they cannot have any incentive effect. This is why the result in Proposition 1 depends on $\cos\theta$.

As mentioned earlier, the larger is $\cos\theta$, the less distorted is the objective measure. At one extreme, $\cos\theta = 1$ and the measure z_t is perfectly aligned with y_t. As discussed in the analysis of this benchmark case, the first-best outcome is feasible in this ideal situation and is achieved by a contract in which the agent's pay does not dependent on the evaluation he receives. At the other extreme, $\cos\theta = 0$ and the performance measure elicits no valuable effort. Consequently, (ICP) can hold only if the worker's wage is independent of m, which leads to part (i) in the proposition.

3.3. Limits on efficiency

Constraint (ICW_2) makes it clear that full efficiency cannot be achieved in the second period. This is because $\mathbf{e}_2$ is induced only through the objective measure z_2, which provides distorted incentives: actual efforts are proportional to $\mathbf{g}$, whereas efficient efforts are proportional to $\mathbf{f}$. The best the principal can do in period 2 is to set β equal to the benchmark second best bonus $b^{SB} = \frac{\|\mathbf{f}\|}{\|\mathbf{g}\|}\cos\theta$.

What about the first period efforts? The benefit of subjective evaluations is that in period 1 incentives from the distorted measure z_1 are at least partly replaced by incentives from the undistorted measure m. Does this imply that the optimal contract will elicit the first best vector of efforts $\mathbf{e}_1^{FB}$?[10] As the next result shows, the answer is No.

Proposition 2. *The optimal contract elicits vectors of efforts $\mathbf{e}_1^*$ and $\mathbf{e}_2^*$ such that $\mathbf{e}_1^* \neq \mathbf{e}_1^{FB}$ and $\mathbf{e}_2^* \neq \mathbf{e}_2^{FB}$.*

Even though a subjective scheme that elicits the efficient efforts in period 1 might be feasible, Proposition 2 says that the principal will not find such a scheme optimal. This result may seem surprising because at the time of contracting there is no informational asymmetry and the principal can hold the worker down to his reservation utility. But the reasoning is simple: Because $\mathbf{e}_1^{FB}$ does not depend on $\mathbf{g}$, any scheme that elicits $\mathbf{e}_1^{FB}$ requires that the contract is independent of z_1. The principal is thus left with two measures, m and z_1, which do not give her enough degrees of freedom to solve the three agency problems she faces: the worker's two moral hazard problems and her own truth-telling problem. Put

[10] Because a is not known when $\mathbf{e}_1$ is chosen, vector $\mathbf{e}_1^{FB}$ is defined by $\psi'(e_{k1}^{FB}) = E(a)f_k$. At the end of period 1, the evaluations reveal a to the worker, so $\mathbf{e}_2^{FB}$ is defined by $\psi'(e_{k2}^{FB}(a)) = af_k$.

differently, the principal has two functions she can control, $s(m)$ and $\beta(m)$, tied down by two constraints, (ICP) and $\mathbf{e}_1 = \mathbf{e}_1^{FB}$. The functions that satisfy these two constraints do not in general optimize the worker's second period incentives. Consequently, starting from $\mathbf{e}_1 = \mathbf{e}_1^{FB}$, a small change in the contract that moves $\beta(m)$ towards b^{SB} increases the principal's overall payoff, as it generates a first order improvement in period 2 incentives, but only a second order loss due to period 1 deviation from $\mathbf{e}_1^{FB}$.

Proposition 2 will prove useful in Section 4, where the current setting is compared with a setting in which the principal commits to a specific distribution of subjective evaluations.

3.4. Optimal contract

In general, the worker's first period efforts depend on b_1 and b_3 directly and on all of s, b_1, b_2, and b_3 indirectly, through the effects of $\mathbf{e}_1$ on the evaluation m. Furthermore, $\mathbf{e}_2$ can depend on $\mathbf{e}_1$ through β. A wage scheme that allows for all of s, b_1, b_2, and b_3 to depend on m therefore has complicated effects on both $\mathbf{e}_1$ and $\mathbf{e}_2$. Fortunately, Lemma 1 below simplifies the problem significantly. It shows that if the optimal contract is piecewise differentiable (assumed throughout the rest of this section[11]) the only parts of the contract that need to be allowed to depend on m are $s(m)$ and $b_2(m)$. Specifically, consider the following augmented version of problem (P):

$$\text{(P')} \qquad \max_{b_1, b_2(m), \mathbf{e}_1} E_a \left[a\mathbf{e}_1 {\cdot} \mathbf{f} - \Psi(\mathbf{e}_1) + \|\mathbf{g}\|^2 a^2 b_2(a) \left[b^{SB} - \frac{b_2(a)}{2} \right] \right]$$

subject to

$$\mathbf{e}_{1k} = E_a \left[ab_1 g_k + \frac{\|\mathbf{g}\|^2 f_k}{\mathbf{e}_1 {\cdot} \mathbf{f}} a^2 \left[b^{SB} - b_2(a) \right] \left[b_2(a) + ab_2'(a) \right] \right]. \qquad \text{(ICW}_1\text{')}$$

Problem (P') was obtained from problem (P) by (i) setting $b_3(m) = 0$ and $b_1(m) = b_1$, where b_1 is a constant, (ii) substituting (ICW$_2$) into the objective function and into the remaining constraints, and (iii) substituting $s'(m)$ from the first order condition for (ICP) into the first order condition for (ICW$_1$) (see the proof of Lemma 1 for details). Steps (ii) and (iii) are self-explanatory. The logic behind setting $b_3(m) = 0$ is that the agent's incentive problems in the two periods are tied only through the principal's truthtelling constraint (ICP) — otherwise,

[11] Piecewise differentiability is a common assumption in optimal control problems. Although in many mechanism design problems the optimal contract can be shown to be monotonic, which ensures that it is differentiable almost everywhere, in the current setting monotonicity of the optimal contract cannot be ascertained ex ante.

they are independent of each other, as discussed in the analysis of the benchmarks. This independence means that conditioning the agent's bonus on the joint realizations of the objective measures across the two periods is of no help in improving his incentives. Finally, b_1 can be set constant without loss of generality because $s(m)$ is enough to capture any incentive effects m can have on the agent's first period efforts.

Lemma 1. *Suppose b_1^*, $b_2^*(.)$, $\mathbf{e}_1^*$, and $\mathbf{e}_2^*$ solve the amended problem (P'). Suppose also $ab_2^*(a)\left[b^{SB} - b_2^*(a)\right]$ is non-decreasing in a. Then b_1^*, $b_2^*(.)$, $\mathbf{e}_1^*$, and $\mathbf{e}_2^*$ solve the original problem (P).*

The following assumption will be sufficient to guarantee that the condition in the lemma is satisfied, so that the solution to the simplified problem (P') also solves (P). Let $\varepsilon(a) \equiv \frac{ah'(a)}{h(a)}$ be the elasticity of the density function $h(.)$ at a.

ASSUMPTION 1. $|\varepsilon(a)| \leq M$ and $|\varepsilon'(a)| \leq M$ for each a, where M is positive but small.

Assumption 1 requires that the density function $h(a)$ does not vary "too much." This assumption is satisfied, for example, by the uniform distribution.

The optimal control program (P') is formally analyzed in Appendix B. The analysis yields the following characterization of the optimal bonuses b_1^* and $b_2^*(m)$.

Proposition 3. *Under Assumption 1, the optimal contract sets (i) $b_1^* > 0$ and (ii) $0 < b_2^*(m) \leq b^{SB}$ for all m. Specifically, $b_2^*(m) = b^{SB}$ for $m = \bar{a}$ and*

$$b_2^*(m) = b^{SB}\left[1 - \frac{\lambda}{1 - \lambda - \lambda\varepsilon(m)}\right] < b^{SB} \quad \text{for } m < \bar{a}, \tag{1}$$

where $\lambda \in (0, \frac{1}{2})$ is a constant.

Proposition 3 reveals that in the presence of subjective evaluations the principal optimally weakens the agent's formal second period incentives: $b_2^* < b^{SB}$ almost everywhere. This is not because subjective pay has second period incentive effects that substitute for formal contracts. Rather, the purpose of weakening the second period formal contract is to ensure that the evaluations induce *first period* effort. To see this, recall that b^{SB} maximizes the second period surplus; hence, there is no further benefit to strengthening the second period incentives when $b_2 = b^{SB}$. If b_2 were equal to b^{SB}, the principal would therefore have no desire to induce a higher $\mathbf{e}_2$ through a good evaluation and the only way to ensure truthful evaluation would be to make $s(m)$ independent of m. But then the evaluations would provide no incentives. Thus, to ensure that the evaluations have incentive effects, the principal scales back the second period formal contract. This makes it desirable for her to boost $\mathbf{e}_2$ by providing good evaluations, which in turn allows $s(m)$ to depend on m and thus provide first period incentives. However, since incentives in period 2 come solely from b_2, it is not optimal to scale b_2 all the way back to zero. Consequently, $b_2^* > 0$.

Observe that the optimal contract retains some formal incentives also in the first period, $b_1^* > 0$, even though subjective evaluations, being undistorted, provide more efficient incentives. The logic is similar to that behind Proposition 2: The first period incentive effects of subjective evaluations come at the cost of further muting second period effort. Because this is costly, first period incentives are supplemented by incentives from the objective measure.

Finally, note that, under Assumption 1, $e_{k2}^*(m) = mb_2^*(m)g_k$ increases in m. A potentially testable implication is that good evaluations are followed by good performance and bad evaluations are followed by poor performance.

3.4.1. Optimal contract under uniform distribution

To gain further insights into the economic forces that govern the relationship between formal contracts and subjective evaluations, it is instructive to consider the case where the distribution of abilities is uniform. When $H(.)$ is uniform, $\varepsilon(a) = 0$ and the optimal bonus in (1) becomes $b_2^* = b^{SB}\frac{1-2\lambda}{1-\lambda}$. Thus, b_2^* is independent of m in this case (except at $m = \bar{a}$) and the optimal contract is effectively separated into a subjective part, consisting of $s(m)$, and an objective part, consisting of the fixed bonuses b_1 and b_2. This makes it possible to characterize the optimal subjective scheme $s(.)$.

With b_1 and b_2 constant, the principal's truthtelling problem reduces to

$$q_1 \in \arg\max_m E_a[a\mathbf{e}_2 \cdot \mathbf{f} - b_2 a\mathbf{e}_2 \cdot \mathbf{g} | q_1] - s(m), \tag{ICP}$$

subject to $$\psi'(e_{k2}^*) = e_{k2}^* = b_2 m g_k \ , \quad k = 1, 2, ..., K, \tag{ICW$_2$}$$

for which the first order condition yields

$$s'(m) = ab_2 \left(\mathbf{g} \cdot \mathbf{f} - b_2 \left\|\mathbf{g}\right\|^2\right). \tag{2}$$

The proof of Proposition 4 below verifies that (2) describes the principal's optimum. Imposing truthtelling then yields the differential equation that implicitly defines the optimal subjective scheme $s^*(m)$:

$$s^{*\prime}(m) = mb_2 \left(\mathbf{g} \cdot \mathbf{f} - b_2 \left\|\mathbf{g}\right\|^2\right).$$

This leads to the following result.

Proposition 4. *When $H(.)$ is uniform, it is optimal to set*

$$b_2^* = b^{SB}\frac{1-2\lambda}{1-\lambda} \text{ for } m < \bar{a}, \quad b_2^* = b^{SB} \text{ for } m = \bar{a}, \quad \text{and} \tag{3a}$$

$$s^*(m) = \frac{m^2}{2}\|\mathbf{g}\|^2 b_2^*\left(b^{SB} - b_2^*\right) + D, \tag{3b}$$

where λ and D are constants. Moreover, $b^{SB}/2 < b_2^* < b^{SB}$ for $m < \bar{a}$.

Proposition 4 provides several insights into the economics of the model. First, (3b) reveals that to be truthful, the wage scheme $s(m)$ must be not only increasing but also strictly convex in the subjective evaluation m. This skewness of the subjective pay reflects the production complementarities between ability and effort. Intuitively, the more productive is the worker, the bigger is the principal's potential gain from misleading him about his ability through a good evaluation. To balance this temptation, the "price" for increasing the evaluation must be higher for higher ability workers, which leads to convexity of the pay scheme.

Second, Proposition 4 confirms more directly the result of Proposition 1, which says that subjective evaluations can have incentive effects only if the objective measure is not useless. In the absence of an objective measure ($b_2 = 0$), telling the worker that he is a high type entails no benefit to the principal. In this case, evaluations can be truthful only if they do not affect the worker's pay, as can be seen from (3b), which reduces to $s^*(m) = D$. Such a scheme, however, has no incentive value. More generally, (3b) shows that the first period incentives from subjective pay are directly proportional to the term $b_2^*\left(b^{SB} - b_2^*\right)$, whose maximum is attained when $b_2 = b^{SB}/2$ and is equal to $\left(b^{SB}\right)^2/4$. The maximum possible subjective incentives therefore increase in the quality of the objective measure as captured by $\cos\theta$. Thus, when the objective measure is poor and provides weak incentives, the subjective part of the scheme also provides weak incentives.

Finally, the expression for $s^*(m)$ further illustrates the point that the second period formal contract must be weakened compared to its constrained efficient level if subjective evaluations are to induce effort. In particular, if it were $b_2 = b^{SB}$, then $s^*(m)$ would again be constant and hence without incentive effects.

3.4.2. Optimality of separation

The above analysis was conducted under the assumption that the contract entails no pooling, i.e., the agent always learns his ability precisely. It can be readily seen that from the point of view of second period efficiency, pooling is never desirable, as it prevents the agent from tailoring his effort to his ability. In principle, though, it might be that pooling improves the first period incentive effects of the subjective part of the contract. If this were the case, the overall desirability of full separation would depend on the tradeoff between the incentive gain from pooling and the second period loss due to inefficient allocation of effort. A complete characterization of the optimal pooling contract and its comparison with a fully separating contract is a daunting task, not undertaken here. However, relatively simple logic shows that if the objective measure is sufficiently good, then even if an optimal contract were to require some pooling, the measure of types that are pooled would approach zero.

Proposition 5. *For any $\rho > 0$, there exists a $\bar{c} < 1$ such that for $\cos\theta \geq \bar{c}$, the measure of agents whose evaluations are pooled under the optimal contract is less than ρ.*

The logic behind Proposition 5 is that when $\cos\theta$ is close to one, the separating contract provides incentives that are already quite efficient, so that even if there were additional efficiency gains from pooling, they would have to be small. By the same token, the effort level in the second period is large and therefore the cost of misallocating effort through pooling is also large. Taken together, these two arguments imply that extensive pooling cannot be optimal when the objective measure is sufficiently good.

3.5. No commitment

The assumption that the firm can make the worker's pay contractually contingent on the evaluations is not unrealistic. Subjective evaluation schemes are often well defined in advance and adherence to such schemes might be verifiable. But even if commitment of this sort were not possible, the subjective evaluation scheme characterized above could still work. Without commitment, the setting is formally a signaling game and as such can have multiple equilibria. For the purposes of this analysis, the most interesting among them is a separating Perfect Bayesian Equilibrium (PBE), in which the principal reveals her information truthfully.

To see that such a separating PBE exists, suppose that at the end of period one, the principal can pay the worker a wage s (to which she cannot commit) in addition to giving him an evaluation m. Assume also that $s \geq 0$, i.e., at this stage the agent cannot be forced to transfer money to the principal.[12] Then even without committing to it ex ante, the principal may have an incentive to convey her information by paying more to higher ability workers, as long as the workers interpret this signal correctly. Consider for simplicity the uniform distribution case (so that b_2 is optimally constant) and suppose that upon being

[12] In the previous section, it was assumed that all of the wages are paid at the end of the second period, but it would be without loss of generality to allow a part of the wage that only depends on m to be paid at the end of the first period.

paid a wage s, the agent's belief x is given by $x = m$ if $s = s^{*-1}(m)$ for some $m \in [0, \bar{a}]$ and by $x = 0$ otherwise, where $s^*(m)$ is as in (3b). Then Proposition 4 tells us that if faced with a worker of ability a, the principal prefers the evaluation $m = a$ and wage $s^*(a)$ to any other evaluation $m = a' \in [0, \bar{a}]$ and wage $s^*(a')$. The only deviation one therefore needs to worry about is where the principal decides to pay a wage that is not in the range of $s^*(.)$. This, however, can be prevented by setting $D = s^*(0) = 0$ — any deviation from $s^*(a)$ then necessarily involves paying the agent more than $s^*(\bar{a})$.[13] Given the specified beliefs, this is dominated by paying $s^*(\bar{a})$. Thus, the above beliefs, together with the wage scheme (3b) and $D = 0$, support a separating PBE of this signalling game.

4. Forced distributions

Some companies, for example GE, Intel, Ford, Goodyear, EDS, and others (Lawler, 2003), adopt forced distributions of subjective evaluations (FDSE), where they commit to a pre-specified distribution of evaluations. This section provides justification for such practice. It also shows that whether an FDSE improves upon the subjective evaluation scheme studied in the previous section depends critically on the quality of the objective measures z_1 and z_2 and on the number of workers the firm employs. Accordingly, I proceed by exploring two alternative settings: In the first one, the firm employs a continuum of workers; in the second, the firm employs a finite number of workers.

4.1. Continuum of workers

Under a forced distribution, the firm's total wage bill associated with the evaluations is always constant, whether the evaluations are truthful or not. Misreporting therefore affects the firm's profit only through the effects it has on the workers' actions.[14]

4.1.1. Truthtelling

The main benefit of an FDSE is that it allows the principal to eliminate the truthtelling constraint (ICP). To see this, suppose the firm employs a measure one of agents whose

[13] Setting $D = 0$ is always feasible: While in the previous analysis D was lumped together with the agent's base salary and hence determined by his participation constraint, conceptually, these two wage components can be separated. The base salary is then specified in the initial contract so as to meet the agent's participation constraint when $D = 0$.

[14] In this respect, an FDSE game is similar to a cheap talk game, and, as is common in cheap talk games, has multiple equilibria, including a babbling equilibrium in which evaluations are completely uninformative. Note, however, that an FDSE is formally not a cheap talk game because the individual workers' payoffs depend directly on the messages.

abilities are drawn independently from $[0, \bar{a}]$ according to the cumulative distribution $H(a)$ with density $h(a)$. Suppose also that the firm pre-commits to an FDSE under which a fraction $h(m)$ of the workers get evaluation m. Then the principal has no incentive to misreport because misreporting does not affect the wage bill, but hurts her second period expected profit by preventing the workers from tailoring their efforts to their abilities.

To make this argument formally, suppose the principal's evaluation strategy upon inferring that a worker has ability a is to report $m \in [0, \bar{a}]$ according to the probability density function $\sigma(m|a)$. An agent with evaluation m then forms a posterior belief $h(a|m)$ about the distribution of his true ability according to

$$h(a|m) = \frac{\sigma(m|a)h(a)}{\int_0^{\bar{a}} \sigma(m|\tau)h(\tau)d\tau} = \frac{\sigma(m|a)h(a)}{h(m)}, \tag{4}$$

where the second equality exploited that the evaluations must adhere to the forced distribution. The worker's expected ability conditional on evaluation m, $x(m)$, is then

$$x(m) = \int_0^{\bar{a}} ah(a|m)da, \tag{5}$$

and his optimal second period efforts $\mathbf{e}_2(m)$ are given by the first order condition

$$e_{2k}(m) = \beta(m)x(m)g_k \ , \quad k = 1, 2, ..., K.$$

Ignoring $s(m)$, the firm's second period expected profit from a worker of ability a is therefore

$$\begin{aligned} E_\sigma \pi_2(a) &= \int_0^{\bar{a}} [a\mathbf{e}_2 \cdot \mathbf{f} - \beta(m)a\mathbf{e}_2 \cdot \mathbf{g}] \, \sigma(m|a)dm \\ &= \int_0^{\bar{a}} a \left\| \mathbf{g} \right\|^2 x(m)\beta(m) \left[b^{SB} - \beta(m) \right] \sigma(m|a)dm, \end{aligned}$$

so that its total expected second period profit is

$$E\pi_2 \equiv E_a E_\sigma \pi_2(a) = \int_0^{\bar{a}} \int_0^{\bar{a}} a \left\| \mathbf{g} \right\|^2 x(m)\beta(m) \left[b^{SB} - \beta(m) \right] \sigma(m|a)h(a)dmda. \tag{6}$$

As will become apparent shortly, $\beta(m)$ can be set constant here w.l.o.g. Thus, let $\beta(m) =$

b_2, where $b_2 \leq b^{SB}$ is a constant. Using (4) and (5), $E\pi_2$ can then be written as

$$
\begin{aligned}
E\pi_2 &= \|\mathbf{g}\|^2 b_2 \left(b^{SB} - b_2\right) \int_0^{\bar{a}} \left[\int_0^{\bar{a}} ah(a|m)da\right] x(m)h(m)dm \\
&= \|\mathbf{g}\|^2 b_2 \left(b^{SB} - b_2\right) E_m \left[x(m)\right]^2 .
\end{aligned}
$$

Now, any improvement in the informativeness of the principal's reporting strategy σ, in particular a switch to truthful reporting, induces a mean-preserving spread of the agents' posteriors x (Marschak and Miyasawa, 1968). This increases the principal's expected second period profit, because $E\pi_2$ is an expectation of a convex function of the posteriors. The principal's profit is therefore maximized under truthful evaluations. That an equilibrium with truthful evaluations indeed exists is shown in the proof to Proposition 6 below.

4.1.2. First period incentives

Observe that the principal's incentives to provide truthful evaluations under the FDSE depend neither on $b_1(m)$ nor on the exact shape of $s(m)$; the only constraint on the contract is that $b_2 \leq b^{SB}$. The principal can therefore choose $s(.)$, b_1, and $b_2 \leq b^{SB}$ so as to optimize the workers' incentives. In the second period, the best she can do is to set $b_2 = b^{SB}$. I will now show that in the first period, it is possible to achieve the efficient vector of efforts $\mathbf{e}_1^{FB}$.

Because $\mathbf{e}_1^{FB}$ does not depend on $\mathbf{g}$, set $b_1(m) = 0$ for all m. Given that b_2 is constant, $\mathbf{e}_1$ then depends solely on $s(m)$. Now, recall that when evaluations are truthful, a worker who provides effort $\hat{\mathbf{e}}_1$ expects his evaluation to be $m = \frac{a\hat{\mathbf{e}}_1 \cdot \mathbf{f}}{\tilde{\mathbf{e}}_1 \cdot \mathbf{f}}$, where $\tilde{\mathbf{e}}_1$ is the principal's conjecture about $\mathbf{e}_1$. The first best outcome in period 1 is therefore obtained by setting $s(m) = m\mathbf{e}_1^{FB} \cdot \mathbf{f}$. The worker's first period expected utility is thus $E_a \left[a\hat{\mathbf{e}}_1 \cdot \mathbf{f} \frac{\mathbf{e}_1^{FB} \cdot \mathbf{f}}{\tilde{\mathbf{e}}_1 \cdot \mathbf{f}} - \Psi(\mathbf{e}_1)\right]$, so that his optimal efforts are given by the first order condition

$$
\psi'(e_{1k}) = E(a) f_k \frac{\mathbf{e}_1^{FB} \cdot \mathbf{f}}{\tilde{\mathbf{e}}_1 \cdot \mathbf{f}} \ , \quad k = 1, 2, ..., K.
$$

The equilibrium requirement $\tilde{\mathbf{e}}_1 = \mathbf{e}_1$ then yields $\mathbf{e}_1 = \mathbf{e}_1^{FB}$ as an equilibrium outcome.

Proposition 6. *Suppose the firm employs a continuum of workers. There always exists a forced distribution of subjective evaluations (FDSE) that induces a Perfect Bayesian Equilibrium in which the evaluations are truthful. The optimal contract sets $b_1 = b_3 = 0$, $b_2 = b^{SB}$, and $s(m) = m\mathbf{e}_1^{FB} \cdot \mathbf{f}$ and achieves the first best outcome in $t = 1$ and the second best outcome in $t = 2$. This contract strictly dominates the optimal contract without FDSE.*

Proposition 6 shows that when the firm employs a large number of workers, a forced distribution improves efficiency by ensuring that the subjective pay scheme is incentive compatible. The subjective scheme can then be used solely to shape the workers' first period incentives. This has two effects on the optimal contract. First, it allows the firm to provide fully efficient incentives in period 1 by completely removing from the contract the distortive objective measure z_1 and replacing it with the undistorted incentives from subjective evaluations. Second, it eliminates the dependence of $s(.)$ on $\beta(.)$, thus freeing $\beta(.)$ to be used solely for the purpose of second period incentives, which improves efficiency in period 2.

4.2. Finite number of workers

Now suppose the firm employs $n \geq 2$ workers, where $n < \infty$. In this case, it is not possible for a forced distribution to replicate the true distribution of abilities $H(.)$. Nevertheless, an FDSE again relaxes the principal's truthtelling constraint and, as will be shown shortly, allows for the first best to be achieved in period 1. Hence, $\beta(m)$ will again be optimally set equal to b^{SB}. Also as before, $\mathbf{e}_1 = \mathbf{e}_1^{FB}$ requires $b_1(m) = 0$.

An FDSE then entails (i) n possible evaluations, $m_1 \leq m_2 \leq ... \leq m_n$, (ii) the corresponding salaries $s_j \equiv s(m_j)$, $j = 1, 2, ..., n$, and (iii) a commitment by the firm to assign each evaluation to exactly one worker.[15] Clearly, for n finite, this scheme cannot fully reveal the workers' true abilities. However, arguments similar to those behind Proposition 6 imply that the principal will assess the workers truthfully in the sense that she will assign evaluation m_n to the highest ability worker, m_{n-1} to the second highest, and so on.[16]

To see that one can find a salary scheme $\{s_j\}_{j=1}^{n}$ that elicits $\mathbf{e}_1^{FB}$, suppose the FDSE entails only two possible salaries, s^H and $s^L < s^H$, and define $\triangle s \equiv s^H - s^L$. Let salary s^L be attached to the first r lowest evaluations $m_1, m_2, ..., m_r$, and salary s^H to the evaluations $m_{r+1}, ..., m_n$. Consider a worker i of ability a and denote by $H_{(r)}(a)$ the c.d.f. of the event that at least r workers other than i have abilities less than a and let $h_{(r)}(a)$ be the corresponding density function.[17] Then if worker i provides effort $\mathbf{e}_1^i$ and anticipates that all the other

[15] This allows for the possibility that multiple workers get the same evaluation. For example, if $m_1 = m_2 = ... = m_n$, then all workers effectively receive the same evaluation. In this particular case, the evaluations do not convey any information.

[16] Two or more workers having the same ability is a zero probability event and will be ignored.

[17] In other words, $H_{(r)}(a)$ is the cdf of the rth order statistic: $H_{(r)}(a) = \sum_{i=r}^{n-1} \binom{n-1}{i} H^i(a)[1 - H(a)]^{n-1-i}$.

workers provide the efforts $\mathbf{e}_1^{FB}$, he expects the high salary s^H with probability

$$\Pr\{m_i \geq m_r\} = \begin{cases} \int_0^{\bar{a}\frac{\mathbf{e}_1^i\cdot\mathbf{f}}{\mathbf{e}_1^{FB}\cdot\mathbf{f}}} \left[1 - H\left(\frac{a\mathbf{e}_1^{FB}\cdot\mathbf{f}}{\mathbf{e}_1^i\cdot\mathbf{f}}\right)\right] h_{(r)}da & \text{for } \mathbf{e}_1^i\cdot\mathbf{f} \leq \mathbf{e}_1^{FB}\cdot\mathbf{f} \\ \int_0^{\bar{a}} \left[1 - H\left(\frac{a\mathbf{e}_1^{FB}\cdot\mathbf{f}}{\mathbf{e}_1^i\cdot\mathbf{f}}\right)\right] h_{(r)}da & \text{for } \mathbf{e}_1^i\cdot\mathbf{f} \geq \mathbf{e}_1^{FB}\cdot\mathbf{f}. \end{cases}$$

Worker i's problem is then to maximize $s^L + \triangle s \Pr\{m_i \geq m_r\} - \Psi(\mathbf{e}_1^i)$, which yields the first order condition

$$\psi'(e_{1k}^i) = \begin{cases} \triangle s \int_0^{\bar{a}\frac{\mathbf{e}_1^i\cdot\mathbf{f}}{\mathbf{e}_1^{FB}\cdot\mathbf{f}}} a\frac{f_k\mathbf{e}_1^{FB}\cdot\mathbf{f}}{\left(\mathbf{e}_1^i\cdot\mathbf{f}\right)^2} h\left(\frac{a\mathbf{e}_1^{FB}\cdot\mathbf{f}}{\mathbf{e}_1^i\cdot\mathbf{f}}\right) h(a)da & \text{for } \mathbf{e}_1^i\cdot\mathbf{f} \leq \mathbf{e}_1^{FB}\cdot\mathbf{f} \\ \triangle s \int_0^{\bar{a}} a\frac{f_k\mathbf{e}_1^{FB}\cdot\mathbf{f}}{\left(\mathbf{e}_1^i\cdot\mathbf{f}\right)^2} h\left(\frac{a\mathbf{e}_1^{FB}\cdot\mathbf{f}}{\mathbf{e}_1^i\cdot\mathbf{f}}\right) h(a)da & \text{for } \mathbf{e}_1^i\cdot\mathbf{f} \geq \mathbf{e}_1^{FB}\cdot\mathbf{f}. \end{cases}$$

Setting $\mathbf{e}_1^i = \mathbf{e}_1^{FB}$, we see that $e_{1k}^i = e_{1k}^{FB}$ if

$$\triangle s \frac{f_k}{\mathbf{e}_1^{FB}\cdot\mathbf{f}} \int_0^{\bar{a}} ah(a)h_{(r)}(a)da = E(a)f_k,$$

which is achieved by $\triangle s = \frac{E(a)\mathbf{e}_1^{FB}\cdot\mathbf{f}}{\int_0^{\bar{a}} ah(a)h_{(r)}(a)da}$.

Proposition 7 below summarizes the analysis of the case with a finite number of workers.

Proposition 7. *Suppose the firm employs $n \geq 2$ workers. There exists an FDSE scheme that induces a Perfect Bayesian Equilibrium in which the evaluations are truthful in the sense that the highest ability worker receives the highest evaluation, the second highest ability worker receives the second highest evaluation, and so on. An optimal contract sets $b_1 = b_3 = 0$, $b_2 = b^{SB}$, and entails two salary levels, s^L and $s^H = s^L + \frac{E(a)\mathbf{e}_1^{FB}\cdot\mathbf{f}}{\int_0^{\bar{a}} ah(a)h_{(r)}(a)da}$. This contract achieves full efficiency in $t = 1$ and provides the second-best level of incentives in $t = 2$.*

The above analysis suggests that the main difference between the cases with n workers and a continuum of workers (and, similarly, between the cases with n workers and $n' > n$ workers) is in how precise is the information the workers have about their abilities in the second period. When there is a continuum of workers, each worker learns his exact ability. When the number of workers is finite, the workers' information remains coarse in the second period, because they only learn their rank out of n workers. This coarseness of beliefs is a source of a second period inefficiency, as it prevents the workers from fully tailoring their efforts to their abilities. However, the information content of the evaluations increases with the number of workers, since it is more informative to know how one ranks among $n + 1$ workers than to know how one ranks among n workers. In particular, as $n \to \infty$, each worker's estimate of his ability converges to his true ability a. These observations, combined with the last claim in Proposition 6, lead to the following result.

Proposition 8. *The (per worker) efficiency of FDSE increases with the number of workers. Furthermore, for any given $H(.)$ and $\cos\theta$, there exists an $n^* \geq 2$ such that for all $n \geq n^*$, FDSE dominates subjective evaluations without a forced distribution.*

One implication of the above proposition that is worth noting is that if formal performance appraisal systems are costly to administer, then, all else equal, larger companies should have an advantage in adopting them. This is consistent with the evidence that larger organizations are more likely to use performance appraisal than smaller organizations (Murphy and Cleveland, 1995, p. 4).

4.3. The role of the contractible measures

As shown above, FDSE is always optimal if the firm employs sufficiently many workers, but this leaves open the question whether subjective evaluations without a forced distribution can be optimal when the number of workers is small. Proposition 8 implies that to answer this question, it is enough to consider $n = 2$. Thus, for the remainder of this section, I will concentrate on a setting with two workers.

The advantage of FDSE is that it allows the firm to achieve full efficiency in $t = 1$ and to set $\beta(m) = b^{SB}$ in $t = 2$. The downside is that the agents' information about their abilities remains coarse in the second period, which distorts almost every agent's effort choice from the level appropriate for his ability. When the number of workers is small, this trade-off determines whether the firm prefers subjective evaluations with or without a forced distribution.

Proposition 9. *When $n = 2$, FDSE is optimal if $\cos\theta \leq c^*$, whereas subjective evaluations without a forced distribution are optimal if $\cos\theta \geq c^{**}$, where $0 < c^* \leq c^{**} < 1$.*

Proposition 9 says that the relative benefits of FDSE depend on the quality of the objective measure z. This result is intuitive. When the objective measure is poor ($\cos\theta \leq c^*$), a contract based solely on this measure provides poor incentives. The additional incentives from subjective evaluations are therefore highly valuable. This favors FDSE, as FDSE induces fully efficient effort in the first period and hence improves efficiency substantially.

Moreover, because the second period incentives are severely distorted (as $\cos\theta$ is small), giving the workers more precise information about their abilities would not do much to improve efficiency in this period. This limits the efficiency loss from adopting FDSE.

In contrast, when the objective measure is good ($\cos\theta \geq c^{**}$), the contract provides relatively efficient incentives in both periods even without subjective pay. The main benefit of subjective evaluations is then in informing the workers about their abilities, which is better achieved through evaluations without a forced distribution. Thus, when the contractible measure is good, subjective evaluations without a forced distribution are optimal.

5. Conclusion

Firms that use subjective performance evaluations typically use them with multiple goals in mind. Economists have traditionally focused on the incentive effects of subjective evaluations, mostly overlooking their other functions. This paper brings to forefront the feedback role of evaluations, which appears to be of equal, if not greater, importance to real world firms as their incentive role. In the model, the feedback and the incentive roles of subjective evaluations are complementary in the optimal contract: when both are present, subjective evaluations are feasible where they could not be sustained otherwise.

The feedback from the evaluations improves efficiency by informing workers about their abilities, which allows them to better choose their optimal actions. Because higher ability workers optimally provide more effort, the principal has a motivation to give good evaluations, which makes truthful evaluations possible. The paper shows that truthful subjective evaluations are always feasible if there exists some, albeit imperfect, verifiable measure of performance. However, the need to ensure that the evaluations are truthful means that the optimal contract never fully replaces the imperfect objective measure with subjective pay. Instead, subjective and objective pay are intertwined in the optimal contract, and the contract's exact shape depends upon the quality of the objective measure. In particular, the strength of the incentives from subjective pay is limited by the quality of the objective measure — when the objective measure is poor, subjective evaluations can only have weak incentive effects.

The paper also explains the benefits and the costs of a forced distribution of evaluations, that is, of the ability to commit to a specific distribution to which the evaluations must adhere. It shows that a forced distribution of subjective evaluations is better than a subjective scheme without a forced distribution when the number of employees is sufficiently large or when the objective performance measure is poor.

Although it expands the view of subjective evaluations beyond that in traditional economic models, the model of this paper is far from capturing the variety of purposes for which subjective appraisals are used in practice. Building a more comprehensive economic model of performance evaluations that would incorporate additional reasons real world firms find performance evaluations useful (such as improved job matching or ensuring the employees' ongoing development) could be a fruitful topic for future research.

A. Appendix A: Proofs

Proof of Proposition 1. (i) The principal's period 2 expected revenue from a worker of ability a is $ETR_2 = a\mathbf{e}_2{\cdot}\mathbf{f}$. Using $e_{k2} = \beta(m)mg_k$ from (ICW$_2$) yields $ETR_2 = am\beta(m)\mathbf{g}\cdot\mathbf{f} = am\beta(m)\left\|\mathbf{g}\right\|\left\|\mathbf{f}\right\|\cos\theta$. If $\cos\theta = 0$, then $ETR_2 = 0$, so that truthtelling requires

$$[b_2(q_1) + b_3(q_1)a\mathbf{e}_1{\cdot}\mathbf{g}]\, a\mathbf{e}_2{\cdot}\mathbf{g} + s(q_1) + b_1(q_1)p_1 \leq [b_2(q_1') + b_3(q_1')a\mathbf{e}_1{\cdot}\mathbf{g}]\, a\mathbf{e}_2'{\cdot}\mathbf{g} + s(q_1') + b_1(q_1')p_1 \tag{A1}$$

for all a and a', where $q_1 = a\mathbf{e}_1{\cdot}\mathbf{f}$ and $q_1' = a'\mathbf{e}_1{\cdot}\mathbf{f}$.

Now, for subjective pay to provide incentives, the marginal effect of the agent's effort on his expected pay must be (at least for some effort levels) higher when his pay depends on m than when it does not. Formally, consider two first period effort vectors $\mathbf{e}_1$ and $\mathbf{e}_1'' \leq \mathbf{e}_1$, i.e. $e_{1k}'' \leq e_{1k}$ for all k, with $e_{1k}'' < e_{1k}$ for at least some k. Let $w(m) = (s(m), b_1(m), b_2(m), b_3(m))$ be a contract that depends on subjective evaluations and let $w'' = (\bar{s}'', \bar{b}_1'', \bar{b}_2'', \bar{b}_3'')$ be a contract where $\bar{s}'', \bar{b}_1'', \bar{b}_2''$, and $\bar{b}_3''$ are all constant, with $q_1'' = a\mathbf{e}_1''{\cdot}\mathbf{f}$, $\bar{s}'' = s(q_1'')$, $\bar{b}_1'' = b_1(q_1'')$, $\bar{b}_2'' = b_2(q_1'')$, and $\bar{b}_3'' = b_3(q_1'')$. The evaluations can have a positive incentive effect only if there exists some $\mathbf{e}_1'' \leq \mathbf{e}_1$ such that the increase in effort from $\mathbf{e}_1''$ to $\mathbf{e}_1$ induces a larger increase in expected pay for the worker under contract $w(m)$ than under w'' :

$$\begin{aligned}
& E_a\left[[b_2(q_1) + b_3(q_1)a\mathbf{e}_1{\cdot}\mathbf{g}]\, a\mathbf{e}_2{\cdot}\mathbf{g} + s(q_1) + b_1(q_1)p_1\right] \\
& -E_a\left[[b_2(q_1'') + b_3(q_1'')a\mathbf{e}_1''{\cdot}\mathbf{g}]\, a\mathbf{e}_2''{\cdot}\mathbf{g} + s(q_1'') + b_1(q_1'')p_1''\right] \\
> \;& E_a\left[\left[\bar{b}_2 + \bar{b}_3a\mathbf{e}_1{\cdot}\mathbf{g}\right] a\mathbf{e}_2{\cdot}\mathbf{g} + \bar{s} + \bar{b}_1p_1\right] - E_a\left[\left[\bar{b}_2 + \bar{b}_3a\mathbf{e}_1''{\cdot}\mathbf{g}\right] a\mathbf{e}_2{\cdot}\mathbf{g} + \bar{s} + \bar{b}_1p_1''\right],
\end{aligned}$$

where $p_1'' = a\mathbf{e}_1''{\cdot}\mathbf{g}$ and $e_{k2}'' = [b_2(q_1'') + b_3(q_1'')a\mathbf{e}_1''{\cdot}\mathbf{g}]\, ag_k$. Rearranging, this condition yields

$$E_a\left[[b_2(q_1) + b_3(q_1)a\mathbf{e}_1{\cdot}\mathbf{g}]\, a\mathbf{e}_2{\cdot}\mathbf{g} + s(q_1) + b_1(q_1)p_1\right] > E_a\left[[b_2(q_1'') + b_3(q_1'')a\mathbf{e}_1{\cdot}\mathbf{g}]\, a\mathbf{e}_2''{\cdot}\mathbf{g} + s(q_1'') + b_1(q_1'')p_1\right],$$

which contradicts (A1). Hence, when $\cos\theta = 0$, subjective pay cannot induce effort.

(ii) This part will be proven by constructing a contract with subjective evaluations that are truthful and improve incentives whenever $\cos\theta > 0$. In particular, let $b_1 \geq 0$

and $b_2 \in (0, b^{SB})$ be independent of m, let $b_3 = 0$ (so that $\beta = b_2$), and let $s(m) = \frac{1}{2}m^2 \|\mathbf{g}\|^2 b_2 \left(b^{SB} - b_2\right) + D$, where D is a constant. Now, plug the expression for $s(m)$ into the principal's second period profit $\pi_2 = a\mathbf{e}_2 \cdot \mathbf{f} - \beta a \mathbf{e}_2 \cdot \mathbf{g} - s(m)$ and use $e_{k2} = \beta m g_k$ to get

$$\pi_2 = am \|\mathbf{g}\|^2 b_2 \left(b^{SB} - b_2\right) - \frac{1}{2}m^2 \|\mathbf{g}\|^2 b_2 \left(b^{SB} - b_2\right) - D.$$

The first order condition for maximization with respect to m yields

$$a \|\mathbf{g}\|^2 b_2 \left(b^{SB} - b_2\right) = m \|\mathbf{g}\|^2 b_2 \left(b^{SB} - b_2\right),$$

which demonstrates that this contract induces truthful evaluations.[18] To see that it improves incentives, observe that for any first period effort vector $\hat{\mathbf{e}}_1$, the worker expects evaluation $m(\hat{\mathbf{e}}_1) = \frac{a\hat{\mathbf{e}}_1 \cdot \mathbf{f}}{\mathbf{e}_1 \cdot \mathbf{f}}$. Hence, $E_a \frac{\partial s(m)}{\partial e_{1k}} = E_a[a \frac{f_k}{\mathbf{e}_1 \cdot \mathbf{f}} s'(m)] = b_2 \left(b^{SB} - b_2\right) \frac{f_k}{\mathbf{e}_1 \cdot \mathbf{f}} \|\mathbf{g}\|^2 E_a[am(\hat{\mathbf{e}}_1)] > 0$. ■

Proof of Proposition 2. From (ICW$_2$), the second period efforts are $e_{2k} = a\beta(m)g_k$, whereas the efficient efforts are $e_{2k}^{FB} = af_k$. Thus, $\mathbf{e}_2^* = \mathbf{e}_2^{FB}$ is possible only if $\beta(m)g_k = f_k$ for all k and m. This is precluded by the assumption that $\mathbf{f}$ and $\mathbf{g}$ are linearly independent.

Next consider $\mathbf{e}_1$. Because $\cos\theta < 1$, $\mathbf{e}_1 = \mathbf{e}_1^{FB}$ requires that $b_1(m) = 0$ almost everywhere. Suppose that a $\beta(m)$ and $s(m)$ that elicit $\mathbf{e}_1^{FB}$ exist (if not, then we are done) and denote them as $\hat{\beta}(m)$ and $\hat{s}(m)$. Assume first that $\hat{\beta}(m)$ maximizes the expected second period surplus $ETS_2 = E_a[a\mathbf{e}_2^* \cdot \mathbf{f} - \Psi(\mathbf{e}_2^*)]$ subject to (ICW$_2$), so that $\hat{\beta}(m) = b^{SB} = \frac{\mathbf{g} \cdot \mathbf{f}}{\|\mathbf{g}\|^2} = \frac{\|\mathbf{f}\|}{\|\mathbf{g}\|} \cos\theta$. The principal's truthtelling constraint (ICP) then becomes

$$\begin{aligned} a &\in \arg\max_m a\mathbf{e}_2 \cdot \mathbf{f} - \hat{\beta}(m) a\mathbf{e}_2 \cdot \mathbf{g} - b_1(m) a\mathbf{e}_1 \cdot \mathbf{g} - \hat{s}(m) \\ &= \arg\max_m a \|\mathbf{g}\|^2 m\hat{\beta}(m) \left[b^{SB} - \hat{\beta}(m)\right] - b_1(m) a\mathbf{e}_1 \cdot \mathbf{g} - \hat{s}(m) \\ &= \arg\max_m -\hat{s}(m). \end{aligned}$$

This can only hold if $\hat{s}(m) = \hat{s}$, where $\hat{s}$ is a constant. Hence, the whole contract is independent of m in this case, so that (ICW$_1$) reduces to

$$\mathbf{e}_1^* \in \arg\max_{\mathbf{e}_1} E_a[\hat{s} + b^{SB} a\mathbf{e}_2 \cdot \mathbf{g} - \Psi(\mathbf{e}_2^*)] - \Psi(\mathbf{e}_1).$$

This yields $e_{1k}^* = 0$ for all k, contrary to the assumption that $\mathbf{e}_1 = \mathbf{e}_1^{FB}$.

[18] The second order condition is satisfied because $\frac{\partial^2 \pi_2}{\partial m^2} = -\frac{\|\mathbf{g}\|^2}{(\mathbf{e}_1 \cdot \mathbf{f})^2} b_2 \left(\beta^{SB} - b_2\right) < 0$.

Thus, if $\mathbf{e}_1 = \mathbf{e}_1^{FB}$ then $\hat{\beta}(m) \neq b^{SB}$, which means that the first period surplus, ETS_1, is maximized with respect to β, but ETS_2 is not. The standard variational argument therefore implies that a small change in β can increase the total surplus, because its positive effect on ETS_2 is of first order magnitude, whereas its negative effect on ETS_1 is of second order magnitude. Consequently, $\hat{\beta}(m)$ cannot be optimal. Thus, it must be $\mathbf{e}_1 \neq \mathbf{e}_1^{FB}$. ■

Proof of Lemma 1. Plugging $e_{k2} = \beta(m)mg_k$ into the objective function and into (ICW$_1$) and (ICP), and using $\mathbf{g} \cdot \mathbf{f} = \|\mathbf{g}\| \|\mathbf{f}\| \cos\theta = \|\mathbf{g}\|^2 b^{SB}$, problem (P) can be written as

$$\max_{s(m),b_1(m),\beta(m),b_3(m)} E_a \left[a\mathbf{e}_1^* \cdot \mathbf{f} - \Psi(\mathbf{e}_1^*) + \beta(a)a^2 \|\mathbf{g}\|^2 \left[b^{SB} - \frac{\beta(a)}{2} \right] \right] \tag{A3}$$

subject to $\hat{m} = \frac{a\hat{\mathbf{e}}_1 \cdot \mathbf{f}}{\tilde{\mathbf{e}}_1 \cdot \mathbf{f}}$, $\beta(m) = b_2(m) + b_3(m)a\mathbf{e}_1 \cdot \mathbf{g}$, and

$$\begin{aligned} \mathbf{e}_1 &\in \arg\max_{\hat{\mathbf{e}}_1} E_a \left[s(\hat{m}) + b_1(\hat{m})a\hat{\mathbf{e}}_1 \cdot \mathbf{g} + \frac{1}{2}\beta^2(\hat{m})a^2 \|\mathbf{g}\|^2 \right] - \Psi(\hat{\mathbf{e}}_1); && \text{(A4)} \\ a &= \arg\max_{m} a \|\mathbf{g}\|^2 m\beta(m) \left[b^{SB} - \beta(m) \right] - b_1(m)a\mathbf{e}_1 \cdot \mathbf{g} - s(m). && \text{(A5)} \end{aligned}$$

Given piecewise differentiability in m, the first order condition for (A5) is

$$s'(m) + b_1'(m)a\mathbf{e}_1 \cdot \mathbf{g} = a \|\mathbf{g}\|^2 \left[\beta(m) \left[b^{SB} - \beta(m) \right] + m\beta'(m) \left[b^{SB} - 2\beta(m) \right] \right], \tag{A6}$$

except for the points of non-differentiability. Imposing truthtelling, $m = a$, and substituting (A6) into the first order condition for (A4) reduces the problem to

$$\max_{s(m),b_1(m),b_2(m),b_3(m)} E_a \left[a\mathbf{e}_1^* \cdot \mathbf{f} - \Psi(\mathbf{e}_1^*) + \beta(m)a^2 \|\mathbf{g}\|^2 \left[b^{SB} - \frac{\beta(m)}{2} \right] \right] \tag{A7}$$

subject to $\beta(m) = b_2(m) + b_3(m)a\mathbf{e}_1 \cdot \mathbf{g}$, $\beta'(m) = b_2'(m) + b_3'(m)a\mathbf{e}_1 \cdot \mathbf{g}$, and

$$e_{1k}^* = E_a \left[[b_1(a)a + \beta(a)b_3(a)a^3]g_k + \frac{\|\mathbf{g}\|^2 f_k}{\mathbf{e}_1 \cdot \mathbf{f}} a^2 \left[b^{SB} - \beta(a) \right] \left[\beta(a) + a\beta'(a) \right] \right]. \tag{A8}$$

Note that because neither (A7) nor (A8) depend on $b_1'(a)$, one can without loss of generality set $b_1(m) = b_1$, where b_1 is a constant such that $b_1 E(a) = E(b_1(a)a)$. Next observe that $b_3(m)$ enters only through the term $\beta(a)b_3(a)a^3 g_k$ in (A8). It is therefore again w.l.o.g. to replace $b_3(a)$ with $\hat{b}_3 = 0$, while replacing b_1 with $\hat{b}_1 \equiv b_1 + \frac{E_a[\beta(a)b_3(a)a^3]}{E(a)}$ and replacing $b_2(m)$ with $\hat{b}_2(m)$ such that $E_a \left[a \left[b^{SB} - \hat{b}_2(a) \right] \left[\hat{b}_2(a) + a\hat{b}_2'(a) \right] \right] = E_a \left[a \left[b^{SB} - \beta(a) \right] \left[\beta(a) + a\beta'(a) \right] \right]$.

This converts the problem (A7)-(A8) into problem (P') in the text.

Now, by construction, any solution to problem (A7)-(A8) also solves (A3)-(A5) if it induces truthtelling. One thus only needs to find a salary function $s(m)$ such that the truthtelling constraint (A5) holds. Let $\Phi(m) \equiv \frac{\partial\left[m\beta(m)\left[b^{SB}-\beta(m)\right]\right]}{\partial m}$ and let $s(m)$ be given by $s'(m) = m \left\|\mathbf{g}\right\|^2 \Phi(m)$. Then for $b_1 = const$ and $b_3 = 0$, the first order condition for (A5) is

$$a \left\|\mathbf{g}\right\|^2 \Phi(m) - s'(m) = \left\|\mathbf{g}\right\|^2 \Phi(m)(a-m) = 0,$$

which yields $m = a$. Moreover, $m = a$ is the maximum if $\Phi(m) \geq 0$ for all m, because then $\left\|\mathbf{g}\right\|^2 \Phi(m)(a-m) \leq 0$ for all $m < a$ and $\left\|\mathbf{g}\right\|^2 \Phi(m)(a-m) \geq 0$ for all $m > a$. On the other hand, if $\Phi(m') < 0$ for some m', then $\frac{\partial[\Phi(m)(a-m)]}{\partial m}|_{a=m'} = -\Phi(m') > 0$, which means that the local second order condition does not hold at m'. To sum up, one can find a salary function $s(m)$ such that (A5) holds if and only if $ab_2(a)\left[b^{SB} - b_2(a)\right]$ is non-decreasing in a. ■

Proof of Proposition 3. Part (i) is established in the analysis of Problem (2), and part (ii) in the analysis of Problem (1), in Appendix B. ■

Proof of Proposition 4. For $H(.)$ uniform, $\varepsilon(a) = 0$. Expression (3a) then follows immediately from (1). Now, to see that $s^*(m) = \frac{1}{2}m^2 \left\|\mathbf{g}\right\|^2 b_2^* \left(b^{SB} - b_2^*\right) + D$ induces truthtelling, rewrite the (ICP) constraint as in the proof of Lemma 1 to get

$$a \in \arg\max_m a \left\|\mathbf{g}\right\|^2 m b_2^* \left[b^{SB} - b_2^*\right] - b_1^* a \mathbf{e}_1 \cdot \mathbf{g} - s^*(m). \tag{A9}$$

Plugging $s^*(m)$ into (A9), the objective function in (A9) becomes

$$\left\|\mathbf{g}\right\|^2 b_2^* \left[b^{SB} - b_2^*\right] \left(am - \frac{m^2}{2}\right) - D - b_1^* a \mathbf{e}_1 \cdot \mathbf{g}. \tag{A10}$$

Because (by Proposition 3) $b_2^* < b^{SB}$ for all $a < \bar{a}$, (A10) is strictly maximized at $m = a$ when $a < \bar{a}$. When $a = \bar{a}$, then $b_2^* = b^{SB}$, so that $m = \bar{a}$ is weakly optimal.

Next, using $s^*(m)$ and that b_2^* is constant, the optimization problem can be written as

$$\max_{b_1, b_2} \left[E(a)\mathbf{e}_1 \cdot \mathbf{f} - \frac{1}{2}\sum_{k=1}^{K} e_{1k}^2 + E(a^2) \left\|\mathbf{g}\right\|^2 b_2 \left(b^{SB} - \frac{b_2}{2}\right)\right]$$

subject to

$$e_{1k} = E(a)b_1g_k + E(a^2)\frac{\|\mathbf{g}\|^2 f_k}{\mathbf{e}_1\cdot\mathbf{f}}b_2\left(b^{SB}-b_2\right), \quad k=1,2,...,K; \tag{A11}$$

$$\mathbf{e}_1\cdot\mathbf{f} = E(a)b_1\mathbf{g}\cdot\mathbf{f} + E(a^2)\frac{\|\mathbf{g}\|^2\|\mathbf{f}\|^2}{\mathbf{e}_1\cdot\mathbf{f}}b_2\left(b^{SB}-b_2\right), \tag{A12}$$

where (A12) was obtained by multiply (A11) by f_k and summing up. The FOC are then

$$b_2 \quad : \quad E(a)\frac{\partial\left(\mathbf{e}_1\cdot\mathbf{f}\right)}{\partial b_2} - \sum_{k=1}^{K}\frac{\partial e_{1k}}{\partial b_2}e_{1k} + E(a^2)\|\mathbf{g}\|^2\left(b^{SB}-b_2\right) = 0 \tag{A13}$$

$$b_1 \quad : \quad E(a)\frac{\partial\left(\mathbf{e}_1\cdot\mathbf{f}\right)}{\partial b_1} - \sum_{k=1}^{K}\frac{\partial e_{1k}}{\partial b_1}e_{1k} = 0. \tag{A14}$$

Substituting $\frac{\partial e_{1k}}{\partial b_2}$, $\frac{\partial e_{1k}}{\partial b_1}$,$\frac{\partial(\mathbf{e}_1\cdot\mathbf{f})}{\partial b_2}$, $\frac{\partial(\mathbf{e}_1\cdot\mathbf{f})}{\partial b_1}$ from (A11)-(A12), conditions (A13)-(A14) become $\|\mathbf{f}\|^2\left(b^{SB}-2b_2\right)R+b_2=0$ and $(\mathbf{g}\cdot\mathbf{f})\,(\mathbf{e}_1\cdot\mathbf{f})R-\mathbf{e}_1\cdot\mathbf{g}=0$, where $R\equiv\frac{E(a)+E(a^2)\frac{\|\mathbf{g}\|^2}{\mathbf{e}_1\cdot\mathbf{f}}b_2\left(b^{SB}-b_2\right)}{2\mathbf{e}_1\cdot\mathbf{f}-E(a)b_1\mathbf{g}\cdot\mathbf{f}}$. Solving for b_2 and using $\mathbf{g}\cdot\mathbf{f}=\|\mathbf{g}\|\,\|\mathbf{f}\|\cos\theta$ then yields

$$b_2^* = \frac{b^{SB}}{2-\frac{\|\mathbf{g}\|}{\|\mathbf{f}\|}\frac{\mathbf{e}_1\cdot\mathbf{f}}{\mathbf{e}_1\cdot\mathbf{g}}\cos\theta} > \frac{b^{SB}}{2}. \quad \blacksquare$$

Proof of Proposition 5. Suppose there is an interval $[a_1,a_2]\subseteq[0,\bar{a}]$ in which the principal sends the same message to all agents from $[a_1,a_2]$. (A proof for multiple pooling regions would follow similar steps.) The possible benefit of such a pooling contract is that it might improve the strength of the agent's incentives; the cost is that in the second period the agent cannot tailor his effort to his exact ability. Start with the second period cost. The total second period surplus from employing the agents in the interval $[a_1,a_2]$ is given by

$$TS_2(a_1,a_2) = \int_{a_1}^{a_2}\left(a\mathbf{e}_2.\mathbf{f}-\Psi(\mathbf{e}_2)\right)dH(a),$$

where $e_{k2}^*=\beta(m)x(m)g_k$, $k=1,2,...,K$. Under full separation, $x(m)=m\frac{\tilde{\mathbf{e}}_1\cdot\mathbf{f}}{\hat{\mathbf{e}}_1\cdot\mathbf{f}}$ and $\beta(m)=b_2^*(m)$ as given by 1. Under pooling, for the evaluation $\hat{m}$ that indicates that goes with $a\in[a_1,a_2]$, the worker's belief is $\hat{a}\equiv E(a|a_1\frac{\tilde{\mathbf{e}}_1\cdot\mathbf{f}}{\hat{\mathbf{e}}_1\cdot\mathbf{f}}\leq a\leq a_2\frac{\tilde{\mathbf{e}}_1\cdot\mathbf{f}}{\hat{\mathbf{e}}_1\cdot\mathbf{f}})$ and $\beta(\hat{m})=\hat{\beta}$, where $\hat{\beta}$ is a constant. The second period cost from pooling the agents in $[a_1,a_2]$ is therefore

$$\triangle TS_2(a_1,a_2) = \int_{a_1}^{a_2}\left[a^2b_2^*(m)\mathbf{g}.\mathbf{f}-\frac{1}{2}\left(b_2^*(m)\right)^2a^2\|\mathbf{g}\|^2\right]dH(a) - \int_{a_1}^{a_2}\left(a\hat{a}\hat{\beta}\mathbf{g}.\mathbf{f}-\frac{1}{2}\hat{\beta}^2\hat{a}^2\|\mathbf{g}\|^2\right)dH(a).$$

Let ρ be the measure of pooled agents under the optimal pooling contract, i.e., $\rho \equiv \int_{a_1}^{a_2} dH(a)$. Using $\lim_{\cos\theta\to 1} b_2^*(m) = b^{FB} = \frac{\|\mathbf{f}\|}{\|\mathbf{g}\|}$ and $\hat{a} = \frac{1}{\rho}\int_{a_1}^{a_2} adH(a)$ yields

$$\begin{aligned}
\lim_{\cos\theta\to 1} \triangle TS_2(a_1,a_2) &= \frac{1}{2}\|\mathbf{f}\|^2 \int_{a_1}^{a_2} a^2 dH(a) - \left[\hat{a}\hat{\beta}\,\|\mathbf{g}\|\,\|\mathbf{f}\| \int_{a_1}^{a_2} adH(a) - \frac{1}{2}\hat{\beta}^2\hat{a}^2\,\|\mathbf{g}\|^2\rho\right] \\
&= \frac{1}{2}\|\mathbf{f}\|^2 \int_{a_1}^{a_2} a^2 dH(a) - \hat{a}^2\rho\,\|\mathbf{g}\|^2\,\hat{\beta}\left(b^{FB} - \frac{\hat{\beta}}{2}\right) \\
&\geq \frac{1}{2}\rho\,\|\mathbf{f}\|^2\left[\frac{1}{\rho}\int_{a_1}^{a_2} a^2 dH(a) - \hat{a}^2\right] = \frac{1}{2}\rho\,\|\mathbf{f}\|^2\, Var(a|a_1 \leq a \leq a_2),
\end{aligned}$$

where the inequality follows from $\hat{\beta}\left(b^{FB} - \frac{\hat{\beta}}{2}\right) \leq \frac{1}{2}\left(b^{FB}\right)^2$.

Now, $\lim_{\cos\theta\to 1} b_2^* = b^{FB}$ implies that the total surplus from the separating contract converges to the first best surplus as $\cos\theta \to 1$, which means that the benefit from improving the agents' incentives through pooling converges to zero. Consequently, $\lim_{\cos\theta\to 1} \triangle TS_2(a_1,a_2)$ must also be zero. This in turn requires $\lim_{\cos\theta\to 1}\rho = 0$, which proves the claim. ■

Proof of Proposition 6. The only two claims that do not immediately follow from the analysis in the text are (i) that there exists a PBE with truthful evaluations and (ii) that the optimal contract with FDSE strictly dominates the optimal contract without FDSE. Claim (ii) follows from the fact, established in Proposition 2, that the optimal contract in the absence of FDSE entails $\mathbf{e}_1^* \neq \mathbf{e}_1^{FB}$ and $\mathbf{e}_2^* \neq \mathbf{e}_2^{FB}$.

To see that (i) holds, suppose the worker believes that the evaluations are truthful. Then $x(m) = m$ and, from (6), the firm's expected second period profit (again ignoring $s(m)$) is $E\pi_2 = \|\mathbf{g}\|^2 b_2\left(b^{SB} - b_2\right)\int_0^{\bar{a}} amh(a)da$. Truthtelling is a PBE if $m = a$ for all a maximizes $E\pi_2$ subject to the following constraint implied by the forced distribution:

$$\int_0^{\bar{a}} (m^2 - a^2)h(a)da = 0. \tag{A15}$$

This is an isoperimetric optimal control problem with Hamiltonian $H = amh + \omega(m^2 - a^2)h$, where ω is the multiplier (a constant) for the state variable m. By Pontryiagin's maximum principle, the optimum is given by $H_m = ah + 2\omega mh = 0$ and by (A15). Restricting attention to $m \in [0, \bar{a}]$, this yields $\omega = -\frac{1}{2}$ and $m = a$. ■

Proof of Proposition 9. Denote the two workers as A and B and consider first FDSE. With two workers, there are two possible evaluations, m_L and $m_H > m_L$. The expected ability of the worker who got the evaluation m_H (say, worker A) is $x_H \equiv x(m_H) = E(a_A|a_A > a_B) = 2\int_0^{\bar{a}} aH(a)h(a)da$. Similarly, the expected ability of the worker with evaluation m_L (worker

B) is $x_L \equiv x(m_L) = E(a_B|a_A > a_B) = 2\int_0^{\bar{a}} a[1-H(a)]h(a)da$. Worker i's second period effort is then $e_{2k}^i = b^{SB}x_i g_k$, $i = L, H$; $k = 1, 2, ..., K$. In $t = 1$, both workers provide efforts $e_{1k}^{FB} = E(a)f_k$. The total surplus under FDSE, TS^{FDSE}, is then

$$\begin{aligned} TS^{FDSE} &= 2E_a[a\mathbf{e}_1^{FB}\cdot\mathbf{f} - \Psi(\mathbf{e}_1^{FB})] + x_H\mathbf{e}_2^H\cdot\mathbf{f} + x_L\mathbf{e}_2^L\cdot\mathbf{f} - \Psi(\mathbf{e}_2^H) - \Psi(\mathbf{e}_2^L) \\ &= 2E_a\left[aE(a)\mathbf{f}\cdot\mathbf{f} - \frac{1}{2}\left[E(a)\right]^2\mathbf{f}\cdot\mathbf{f}\right] + b^{SB}\left(x_H^2 + x_L^2\right)\mathbf{g}\cdot\mathbf{f} - \left[b^{SB}\right]^2\frac{x_H^2 + x_L^2}{2}\mathbf{g}\cdot\mathbf{g} \\ &= \left[E(a)\right]^2\|\mathbf{f}\|^2 + b^{SB}\left(x_H^2 + x_L^2\right)\|\mathbf{f}\|\,\|\mathbf{g}\|\cos\theta - \left[b^{SB}\right]^2\frac{x_H^2 + x_L^2}{2}\|\mathbf{g}\|^2 \\ &= \left[E(a)\right]^2\|\mathbf{f}\|^2 + \left[b^{SB}\right]^2\frac{x_H^2 + x_L^2}{2}\|\mathbf{g}\|^2, \end{aligned}$$

so that $\lim_{\theta\to 0} TS^{FDSE} = \left[E(a)\right]^2\|\mathbf{f}\|^2$ (because $\lim_{\theta\to 0} b^{SB} = \lim_{\theta\to 0}\frac{\|\mathbf{f}\|}{\|\mathbf{g}\|}\cos\theta = 0$).

Without FDSE, $e_{2k}^* = ab^{SB}g_k$, and the total surplus, TS^0, is

$$\begin{aligned} TS^0 &= 2E_a[a\mathbf{e}_1^*\cdot\mathbf{f} - \Psi(\mathbf{e}_1^*) + a\mathbf{e}_2^*\cdot\mathbf{f} - \Psi(\mathbf{e}_2^*)] \\ &= 2E_a[a\mathbf{e}_1^*\cdot\mathbf{f} - \Psi(\mathbf{e}_1^*) + b^{SB}a^2\mathbf{g}\cdot\mathbf{f} - \frac{1}{2}\left[b^{SB}\right]^2 a^2\mathbf{g}\cdot\mathbf{g}] \\ &= 2E_a[a\mathbf{e}_1^*\cdot\mathbf{f} - \Psi(\mathbf{e}_1^*)] + \left[b^{SB}\right]^2 a^2\|\mathbf{g}\|^2. \end{aligned} \tag{A16}$$

Now, refer to the analysis in Appendix B: First, $\lim_{\theta\to 0} b^{SB} = 0$ implies that both $\delta^*(a)$ and $\delta^{*\prime}(a)$ converge to zero as $\theta \to 0$. Consequently, the L.H.S. of (B3) also converges to zero, and so does $C^{\max}$ in constraint (B20). This in turn implies $\lim_{\theta\to 0} C^* = 0$, where C^* solves Problem (2) in Appendix B. (B22) and (B24) then yield $\lim_{\theta\to 0} b_1^* = \lim_{\theta\to 0} b^{SB} = 0$. Hence, from (B19), it must be $\lim_{\theta\to 0} e_{1k}^* = E(a)g_k\lim_{\theta\to 0} b_1^* + \|\mathbf{g}\|^2 f_k\lim_{\theta\to 0}\frac{C^*}{\mathbf{e}_1^*\cdot\mathbf{f}} = 0$ for each k, so that $\lim_{\theta\to 0}\mathbf{e}_1^*\cdot\mathbf{f} = \lim_{\theta\to 0}\mathbf{e}_1^*\cdot\mathbf{e}_1^* = 0$. This shows that $\lim_{\theta\to 0} TS^0 = 0 < \lim_{\theta\to 0} TS^{FDSE}$. Therefore, there exists a $\theta^* > 0$ such that $TS^0 < TS^{FDSE}$ for all $\theta \leq \theta^*$. Setting $c^* \equiv \cos\theta^*$ concludes the proof of the first claim in the proposition.

To obtain the second claim, let $\theta \to 1$. Then $\lim_{\theta\to 1} b^{SB} = \frac{\|\mathbf{f}\|}{\|\mathbf{g}\|}$, so that $\lim_{\theta\to 1} TS^{FDSE} = \left[E(a)\right]^2\|\mathbf{f}\|^2 + \frac{x_H^2+x_L^2}{2}\|\mathbf{f}\|^2$. As for TS^0, set $C = 0$ and optimize over b_1 and $\mathbf{e}_1$. Denote the solution as b_1^0 and $\mathbf{e}_1^0$ and observe that, by definition, $b_1^0 = b^{SB}$, so that $\lim_{\theta\to 1}\mathbf{e}_1^0 = \mathbf{e}_1^{FB}$. Because $\mathbf{e}_1^*$ maximizes TS^0, (A16) then implies $\lim_{\theta\to 1} TS^0 \geq \lim_{\theta\to 1} 2E_a[a\mathbf{e}_1^0\cdot\mathbf{f} - \Psi(\mathbf{e}_1^0)] + \left[b^{SB}\right]^2 a^2\|\mathbf{g}\|^2 = \left[E(a)\right]^2\|\mathbf{f}\|^2 + E(a^2)\|\mathbf{f}\|^2$. Now, the true distribution of a is a mean-preserving spread of the beliefs $x(m_H)$ and $x(m_L)$; it therefore must be $E(a^2) > \frac{x_H^2+x_L^2}{2}$, which yields $\lim_{\theta\to 1} TS^0 > \lim_{\theta\to 1} TS^{FDSE}$. Consequently, there exists a $\theta^{**} < 1$ such that $TS^0 \geq TS^{FDSE}$ for all $\theta \geq \theta^{**}$. Setting $c^{**} \equiv \cos\theta^{**}$ concludes the proof. ■

B. Appendix B: Analysis of problem (P')

It will prove useful to restate the problem in terms of $\delta(a) \equiv ab_2(a)$:

$$\text{(P')} \qquad \max_{b_1,\delta(m),\mathbf{e}_1} \int_0^{\bar{a}} \left[a\mathbf{e}_1{\cdot}\mathbf{f} - \Psi(\mathbf{e}_1) + \|\mathbf{g}\|^2 \delta(a) \left[ab^{SB} - \frac{\delta(a)}{2}\right]\right] h(a)da$$

subject to $\delta(a) \geq 0$, $\delta(0) = 0$, and

$$e_{1k} = \int_0^{\bar{a}} \left[ab_1 g_k + \frac{\|\mathbf{g}\|^2 f_k}{\mathbf{e}_1{\cdot}\mathbf{f}} a\left[ab^{SB} - \delta(a)\right]\delta'(a)\right] h(a)da, \quad k = 1, 2, ..., K. \tag{B1}$$

Constraint (B1) can be rearranged as follows:

$$\int_0^{\bar{a}} a\left[ab^{SB} - \delta(a)\right]\delta'(a)h(a)da = \frac{\mathbf{e}_1{\cdot}\mathbf{f}}{\|\mathbf{g}\|^2}\frac{\left[e_{1k} - E(a)b_1 g_k\right]}{f_k}, \quad k = 1, 2, ..., K. \tag{B2}$$

Recalling that $b_3(m) = 0$, so that $\delta(a) = ab_2(a)$ is independent of $\mathbf{e}_1$, (B2) shows that (P') is separable into two self-contained problems: (1) Optimization over $\delta(m)$, taking b_1 and $\mathbf{e}_1$ as given, and (2) optimization over b_1 and $\mathbf{e}_1$, taking into account the effect on $\delta(m)$.

B.1. Problem (1): Optimization with respect to $\delta(m)$.

Step 1. Problem setup. Note that with respect to Problem (1), (B2) is just a single constraint: Because the L.H.S. of (B2) does not depend on k, the R.H.S. cannot depend on k either, i.e. it must be $\frac{\mathbf{e}_1{\cdot}\mathbf{f}}{\|\mathbf{g}\|^2}\frac{[e_{1k} - E(a)b_1 g_k]}{f_k} = C$ for all k, where C is a constant. The choice of C is analyzed in Problem (2) below; here, C is treated as exogenous. Thus, (ignoring the constant $\|\mathbf{g}\|^2$) the problem of optimizing with respect to $\delta(m)$ can be stated as

$$\text{(P1)} \qquad \max_{\delta(a)} \int_0^{\bar{a}} \delta(a)\left[ab^{SB} - \frac{\delta(a)}{2}\right] h(a)da$$

subject to

$$\int_0^{\bar{a}} a\left[ab^{SB} - \delta(a)\right]\delta'(a)h(a)da = C. \tag{B3}$$

It will be proven in Problem (2) and taken here as given that $C > 0$.

Program (P1) is an isoperimetric optimal control problem, i.e., an optimal control problem with an integral constraint. To formulate it as a proper optimal control problem, define a new control variable $u(a) = \delta'(a)$ and a new state variable $y(a) = \int_0^a t\left[tb^{SB} - \delta(t)\right]u(t)h(t)dt$.

This transforms (P1) to a problem with a control variable u and state variables y and δ:

$$\text{(P1')} \qquad \max_{\delta,y,u} \int_0^{\bar{a}} \delta(a)\left[ab^{SB} - \frac{\delta(a)}{2}\right] h(a)da$$

$$\begin{aligned} \text{subject to} \qquad y'(a) &= a\left[ab^{SB} - \delta(a)\right] u(a)h(a); && \text{(B4)} \\ \delta'(a) &= u(a); && \text{(B5)} \\ y(0) &= 0; \quad y(\bar{a}) = C; && \text{(B6)} \\ \delta(a) &\geq 0; \quad \delta(0) = 0. && \text{(B7)} \end{aligned}$$

Step 2. Necessary conditions. Let $\lambda(a)$ and $\mu(a)$ be the multiplier functions that go with y and δ respectively and $\eta(a)$ the multiplier that goes with the inequality in (B7). The Hamiltonian for this problem is then

$$H(a, y, \delta, u, \lambda, \mu) = \delta\left(ab^{SB} - \frac{\delta}{2}\right) h + \lambda a \left(ab^{SB} - \delta\right) uh + \mu u + \eta\delta.$$

Pontryiagin's maximum principle says that any solution to (P1'), denoted by $\delta^*(a)$, $y^*(a)$, $u^*(a)$, $\lambda^*(a)$, $\mu^*(a)$, must satisfy (B4)-B(7), plus

$$\begin{aligned} u &= \arg\max H = \arg\max \delta\left(ab^{SB} - \frac{\delta}{2}\right) h + \lambda a \left(ab^{SB} - \delta\right) uh + \mu u + \eta\delta && \text{(B8)} \\ \lambda' &= -H_y = 0 && \text{(B9)} \\ \mu' &= -H_\delta = -\left(ab^{SB} - \delta\right) h + \lambda auh - \eta && \text{(B10)} \\ \eta\delta &= 0, && \text{(B11)} \end{aligned}$$

and the transversality condition $\qquad \mu(\bar{a}) = 0. \qquad$ (B12)

Given that H is affine in u, the above is a singular control problem with an unbounded control. This suggests that the solution entails a singular arc on some interval $I \subset [0, \bar{a}]$. Along this arc, the solution must lie on the singular surface defined by $H_u = 0$, $\frac{d}{da}H_u = 0$, ..., and $\frac{d^r}{da^r}H_u = 0$, where r is the order of the singular arc, i.e., the smallest positive integer r such that $\frac{\partial}{\partial u}\left(\frac{d^r}{da^r}H_u\right) \neq 0$ (see, e.g. Chachuat, 2007, pp. 145-6). Noting that (B9) implies that λ is a constant, we have

$$H_u = \lambda a \left(ab^{SB} - \delta\right) h + \mu = 0. \tag{B13}$$

Using (B5) and (B10), this yields

$$\begin{aligned}
\frac{d}{da}(H_u) &= \lambda\left(2ab^{SB}-\delta-a\delta'\right)h+\lambda a\left(ab^{SB}-\delta\right)h'+\mu' \\
&= \lambda\left(2ab^{SB}-\delta-au\right)h+\lambda a\left(ab^{SB}-\delta\right)h'-\left(ab^{SB}-\delta\right)h+\lambda auh-\eta \\
&= \lambda\left(2ab^{SB}-\delta\right)h+\lambda a\left(ab^{SB}-\delta\right)h'-\left(ab^{SB}-\delta\right)h-\eta=0, \qquad \text{(B14)}
\end{aligned}$$

which does not depend on u. Next,

$$\begin{aligned}
\frac{d^2}{da^2}(H_u) &= \lambda\left(2b^{SB}-u\right)h+\lambda\left(4ab^{SB}-2\delta-au\right)h'+\lambda a\left(ab^{SB}-\delta\right)h'' \\
&\quad -\left(b^{SB}-u\right)h-\left(ab^{SB}-\delta\right)h', \qquad \text{(B15)}
\end{aligned}$$

so that $\frac{\partial}{\partial u}\left(\frac{d^2}{da^2}H_u\right)=(1-\lambda)(h+ah')\neq 0$ as long as $\lambda\neq 1$ (which will be verified shortly). Thus, the singular arc is given by (B13)-(B15).

If $1-\lambda(1+\varepsilon)\neq 0$ (again verified shortly), solving for δ from (B14) yields

$$\delta^*(a)=ab^{SB}\left[1-\frac{\lambda}{1-\lambda(1+\varepsilon)}\right]+\frac{\eta}{h(a)\left[1-\lambda(1+\varepsilon)\right]},$$

where $\varepsilon(a)\equiv\frac{ah'(a)}{h(a)}$ is the elasticity of the distribution function $h(.)$ at a. Combined with constraint (B11), this means that

$$\delta^*(a)=\max\left\{0,ab^{SB}\left[1-\frac{\lambda}{1-\lambda(1+\varepsilon)}\right]\right\}. \qquad \text{(B16)}$$

The optimal λ^* is then determined by plugging (B16) into constraint (B3) and solving for λ. Appendix C shows that a solution to (B3) exists if and only if $C\in[0,C^{\max}]$, where $C^{\max}>0$, and that $\lambda^*\in\left(0,\frac{1}{3}+\phi_2(M)\right)$, where M is as in Assumption 1 and $\phi_2(M)>0$, with $\lim_{M\to 0}\phi_2(M)=0$. Note that $\lambda^*<\frac{1}{3}+\phi_2(M)$ implies $1-\lambda(1+\varepsilon)>0$ for all ε when M is small, verifying that the denominator in (B16) is non-zero.

Step 3. End points. Equation (B16) satisfies the requirement that $\delta(0)=0$, but not the transversality condition (B12): Substituting (B12) to (B13) and using the fact that u is unbounded implies that $\delta(\bar{a})=\bar{a}b^{SB}$. This, however, is generically not compatible with (B16). Consequently, the optimal solution has an impulse at $a=\bar{a}$: The optimal δ is given by (B16) for $a\in I=[0,\bar{a})$ and is then transported via an impulse to $\delta(\bar{a})=\bar{a}b^{SB}$ at $a=\bar{a}$.[19]

[19] On impulses in singular optimal control problems see e.g. Bryson and Ho (1975).

Step 4. Sufficiency. It is straightforward to check that the Hessian of H is indefinite. Consequently, the Mangasarian sufficiency conditions for global maximum are not satisfied and an alternative way of proving that (B16) solves the problem is needed. This will be done by showing that (B16) cannot be a minimum. Since any solution must satisfy (B16), the above must be a maximum. Thus, suppose, as a way of contradiction, that the above singular arc is a global minimum. Then it is also a local minimum, which requires that the generalized Legendre-Clebsch condition for minimum holds, i.e., $-\frac{\partial}{\partial u}\left(\frac{d^2}{da^2}H_u\right) > 0$, or

$$-(1-\lambda)(1+\varepsilon) > 0. \tag{B17}$$

Given that $1+\varepsilon > 0$ for M small, (B17) holds only if $\lambda > 1$, which for M small contradicts $\lambda^* < \frac{1}{3} + \phi_2(M)$ established in Appendix C. Therefore, the above arc must be a maximum. (Observe that the generalized Legendre-Clebsch condition for local maximum is the reverse of (B17), which holds for M small.)

Step 5. The condition imposed by Lemma 1. Lemma 1 says that for the solution to problem (P1) to be a part of the solution to the original problem (P), $ab_2(a)\left[b^{SB} - b_2(a)\right]$ must be non-decreasing in a. Using (B16) and $\delta(a) = ab_2(a)$, we have

$$ab_2(a)\left[b^{SB} - b_2(a)\right] = ab^{SB}\frac{\lambda\left[1-\lambda\left(2+\varepsilon\right)\right]}{\left[1-\lambda\left(1+\varepsilon\right)\right]^2},$$

which is non-decreasing in a if and only if

$$b^{SB}\left[1-\lambda\left(2+\varepsilon\right)\right] + \lambda\varepsilon'\left[2\delta(a) - ab^{SB}\right] \geq 0. \tag{B18}$$

Given that $\lambda^* < \frac{1}{3} + \phi_2(M)$ and $\lim_{M\to 0}\phi_2(M) = \lim_{M\to 0}\varepsilon(a) = \lim_{M\to 0}\varepsilon'(a) = 0$, it must be that $\lim_{M\to 0} LHS$(B18)> 0. Thus, (B18) holds for M small.

B.2. Problem (2): Optimization with respect to e_1, b_1, and C.

Step 1. Problem setup. Let $V(\delta^*, \lambda^*, C)$ be the principal's optimal value function from problem (P1), i.e., $V(\delta^*, \lambda^*, C) \equiv \int_0^{\bar{a}} \delta^*(a)\left[ab^{SB} - \frac{\delta^*(a)}{2}\right]h(a)da$, and denote by π her total expected profit over the two periods. Problem (2) can then be written as

$$
\begin{aligned}
&\text{(P2)} & \max_{b_1,\mathbf{e}_1,C} \pi &= \max_{b_1,\mathbf{e}_1,C} E(a)\mathbf{e}_1\cdot\mathbf{f} - \Psi(\mathbf{e}_1) + \|\mathbf{g}\|^2 V(\delta^*,\lambda^*,C) \\
&\text{subject to} & e_{1k} &= E(a)b_1 g_k + \frac{C\|\mathbf{g}\|^2}{\mathbf{e}_1\cdot\mathbf{f}} f_k, \quad k=1,2,...,K; & \text{(B19)} \\
& & C &\geq 0;
\end{aligned}
$$

subject to $\delta(a)$ being determined by (B3) and B(6); and subject to C being feasible (i.e., such that the set of $\delta(a)$ that satisfy (B3) is non-empty). Appendix C shows that the feasibility constraint for C can be expressed, for some $C^{\max} > 0$, as

$$
C \leq C^{\max}. \tag{B20}
$$

Step 2. First order conditions. The dynamic Envelope Theorem for the fixed endpoint class of optimal control problems (e.g., Theorem 9.1 in Caputo, 2005, p. 232) implies that $\frac{\partial V(\delta^*,\lambda^*,C)}{\partial C} = -\lambda^*$. Let μ_C be the Lagrange multiplier associated with constraint (B20). The first order conditions for problem (P2) are then

$$
\begin{aligned}
\frac{\partial \pi}{\partial C} &= \sum_{k=1}^{K} \frac{\partial e_{1k}}{\partial C}\left[E(a)f_k - e_{1k}\right] - \lambda^* \|\mathbf{g}\|^2 + \mu_C = 0, & \text{(B21)} \\
\frac{\partial \pi}{\partial b_1} &= \sum_{k=1}^{K} \frac{\partial e_{1k}}{\partial b_1}\left[E(a)f_k - e_{1k}\right] = 0, \quad \text{and} & \text{(B22)} \\
\left(C^{\max} - C\right)\mu_C &= 0, \quad \mu_C \geq 0, & \text{(B23)}
\end{aligned}
$$

where, from (B19),

$$
\frac{\partial e_{1k}}{\partial C} = \frac{\|\mathbf{g}\|^2 f_k}{\mathbf{e}_1\cdot\mathbf{f} + \|\mathbf{g}\|^2 \frac{Cf_k^2}{\mathbf{e}_1\cdot\mathbf{f}}} > 0 \quad \text{and} \quad \frac{\partial e_{1k}}{\partial b_1} = \frac{E(a)g_k\mathbf{e}_1\cdot\mathbf{f}}{\mathbf{e}_1\cdot\mathbf{f} + \|\mathbf{g}\|^2 \frac{Cf_k^2}{\mathbf{e}_1\cdot\mathbf{f}}} > 0. \tag{B24}
$$

Step 3. $C^* > 0$. To see this, suppose $C = 0$. Then $\frac{\partial e_{1k}}{\partial C} = \frac{\|\mathbf{g}\|^2 f_k}{\mathbf{e}_1\cdot\mathbf{f}}$, $\frac{\partial e_{1k}}{\partial b_1} = E(a)g_k$, $e_{1k} = E(a)b_1 g_k$, and (from (B3) and (B16)) $\lambda^* = 0$, so that (B22) yields $b_1 = b^{SB} = \frac{\|\mathbf{f}\|}{\|\mathbf{g}\|}\cos\theta$

and $\frac{\partial \pi}{\partial C}$ becomes

$$\begin{aligned}
\frac{\partial \pi}{\partial C}|_{C=0} &= E(a)\frac{\|\mathbf{g}\|^2}{\mathbf{e}_1 \cdot \mathbf{f}} \sum_{k=1}^{K} \left[f_k^2 - b^{SB} f_k g_k \right] + \mu_C \\
&= E(a)\frac{\|\mathbf{g}\|^2}{\mathbf{e}_1 \cdot \mathbf{f}} \left[\|\mathbf{f}\|^2 - b^{SB} \|\mathbf{f}\| \|\mathbf{g}\| \cos\theta \right] + \mu_C \\
&= E(a)\frac{\|\mathbf{g}\|^2 \|\mathbf{f}\|^2}{\mathbf{e}_1 \cdot \mathbf{f}} \left(1 - \cos^2\theta \right) + \mu_C > 0.
\end{aligned}$$

Hence, it must be $C^* > 0$.

Step 4. $C^* < \frac{E(a)\mathbf{e}_1 \cdot \mathbf{f}}{\|\mathbf{g}\|^2}$. Suppose $C \geq \frac{E(a)\mathbf{e}_1 \cdot \mathbf{f}}{\|\mathbf{g}\|^2}$. Then (B19) implies $e_{1k} \geq E(a) b_1 g_k + E(a) f_k,\ k = 1, 2, ..., K$. Plugging this to $\frac{\partial \pi}{\partial C}$ in (21) yields

$$\frac{\partial \pi}{\partial C} \leq -\sum_{k=1}^{K} \frac{\partial e_{1k}}{\partial C} E(a) b_1 g_k - \lambda^* \|\mathbf{g}\|^2 + \mu_C.$$

Now suppose for the moment that (P2) is not constrained by (B20). Then $\mu_C = 0$ and the above implies $\frac{\partial \pi}{\partial C} < 0$ for all $C \geq \frac{E(a)\mathbf{e}_1 \cdot \mathbf{f}}{\|\mathbf{g}\|^2}$, where the inequality follows from $\lambda^* > 0$ for $C > 0$, established in Appendix C. Therefore, it must be $C^* < \frac{E(a)\mathbf{e}_1 \cdot \mathbf{f}}{\|\mathbf{g}\|^2}$ if C is unconstrained and hence also if C is constrained by (B20).

Step 5. $b_1^* > 0$. Suppose $b_1 = 0$. Because $C < \frac{E(a)\mathbf{e}_1 \cdot \mathbf{f}}{\|\mathbf{g}\|^2}$, (B19) then implies $e_{1k} < E(a) f_k,\ k = 1, 2, ..., K$, so that, from (B22), $\frac{\partial \pi}{\partial b_1}|_{b_1=0} > 0$. Hence, it must be $b_1^* > 0$.

C. Appendix C: Technical details regarding λ^* and constraint (B20)

Define

$$V(\lambda) \equiv \int_0^{\bar{a}} \delta^*(a) \left[a b^{SB} - \frac{\delta^*(a)}{2} \right] h(a) da, \text{ and} \tag{C1}$$

$$Z(\lambda) \equiv \int_0^{\bar{a}} a \left[a b^{SB} - \delta^*(a) \right] \delta^{*\prime}(a) h(a) da, \tag{C2}$$

where $\delta^*(a)$ is given by (B16). That is, $V(\lambda)$ is the principal's optimal value function from problem (P1) in Appendix B as a function of λ, and $Z(\lambda)$ is the L.H.S. of constraint (B3) evaluated at $\delta^*(a)$. Also, note that for M small,

$$\begin{aligned} \delta^*(a) &= 0 \text{ if } \lambda \in [\frac{1}{2+\varepsilon}, \frac{1}{1+\varepsilon}) && \text{(C3a)} \\ \delta^*(a) &> 0 \text{ otherwise.} && \text{(C3b)} \end{aligned}$$

Step 1. Shape of $V(\lambda)$. Let $\hat{V}(\lambda, \varepsilon) \equiv \delta^*(a)\left[ab^{SB} - \frac{\delta^*(a)}{2}\right]$. By (C3), $\hat{V}(\lambda, \varepsilon) = 0$ for $\lambda \in [\frac{1}{2+\varepsilon}, \frac{1}{1+\varepsilon})$. Let $\varepsilon^{\max} \equiv \max_a\{\varepsilon(a)\}$ and $\varepsilon^{\min} \equiv \min_a\{\varepsilon(a)\}$. Then $\hat{V}(\lambda, \varepsilon) = 0$ for all $\lambda \in [\frac{1}{2+\varepsilon^{\min}}, \frac{1}{1+\varepsilon^{\max}})$, so that $V(\lambda) = 0$ for $\lambda \in [\frac{1}{2+\varepsilon^{\min}}, \frac{1}{1+\varepsilon^{\max}})$.

Next, suppose $\lambda \notin [\frac{1}{2+\varepsilon}, \frac{1}{1+\varepsilon}]$. Then substituting (C3) into (C1) yields

$$\hat{V}(\lambda, \varepsilon) = \frac{1}{2}\left(b^{SB}\right)^2 a^2 \left[1 - \frac{\lambda^2}{[1-\lambda(1+\varepsilon)]^2}\right] = \frac{1}{2}\left(b^{SB}\right)^2 a^2 \left[1 - \frac{1}{\left[\frac{1}{\lambda} - (1+\varepsilon)\right]^2}\right],$$

so that $\frac{\partial \hat{V}(\lambda,\varepsilon)}{\partial \lambda} < 0$ if $0 < \lambda < \frac{1}{2+\varepsilon}$ and $\frac{\partial \hat{V}(\lambda,\varepsilon)}{\partial \lambda} > 0$ if $\lambda < 0$ or if $\lambda > \frac{1}{1+\varepsilon}$. Furthermore, we have $\hat{V}(\lambda, \varepsilon) = 0$ iff $\lambda^2 = [1-\lambda(1+\varepsilon)]^2$, i.e., iff either $\lambda = \lambda_1(\varepsilon) = \frac{1}{2+\varepsilon}$ or $\lambda = \lambda_2(\varepsilon) = \frac{1}{\varepsilon}$. Finally, note that $\hat{V}(\lambda, \varepsilon)$ is continuous in λ except for $\hat{\lambda}(\varepsilon) = \frac{1}{1+\varepsilon}$, and that $\lim_{\lambda \uparrow \hat{\lambda}(\varepsilon)} \hat{V}(\lambda, \varepsilon) = \lim_{\lambda \downarrow \hat{\lambda}(\varepsilon)} \hat{V}(\lambda, \varepsilon) = -\infty$ and $\hat{V}(0, \varepsilon) = \frac{1}{2}\left(b^{SB}\right)^2 a^2 > 0$ for all ε.[20] Hence,

$$\begin{aligned} V'(\lambda) &> 0 \text{ for } \lambda < 0 && \text{(C4a)} \\ V'(\lambda) &< 0 \text{ for } 0 < \lambda < \frac{1}{2+\varepsilon^{\max}} && \text{(C4b)} \\ V'(\lambda) &\leq 0 \text{ for } \frac{1}{2+\varepsilon^{\max}} \leq \lambda \leq \frac{1}{2+\varepsilon^{\min}} && \text{(C4c)} \\ V'(\lambda) &= 0 \text{ for } \frac{1}{2+\varepsilon^{\min}} \leq \lambda < \frac{1}{1+\varepsilon^{\max}} && \text{(C4d)} \\ V'(\lambda) &\geq 0 \text{ for } \lambda \geq \frac{1}{1+\varepsilon^{\max}}, && \text{(C4e)} \end{aligned}$$

[20] $\lambda \uparrow \hat{\lambda}(\varepsilon)$ indicates convergence of λ to $\hat{\lambda}(\varepsilon)$ from below; similarly, $\downarrow$ indicates convergence of from above.

and

$$
\begin{aligned}
V(\lambda) &> 0 \text{ for } \lambda_0 \leq \lambda \leq \frac{1}{2+\varepsilon^{\max}} \text{ where } \lambda_0 < 0 && \text{(C5a)}\\
V(\lambda) &\leq 0 \text{ for } \frac{1}{2+\varepsilon^{\max}} \leq \lambda \leq \frac{1}{2+\varepsilon^{\min}} && \text{(C5b)}\\
V(\lambda) &= 0 \text{ for } \frac{1}{2+\varepsilon^{\min}} \leq \lambda < \frac{1}{1+\varepsilon^{\max}} && \text{(C5c)}\\
V(\lambda) &< 0 \text{ for } \frac{1}{1+\varepsilon^{\min}} < \lambda < \frac{1}{\varepsilon^{\max}} && \text{(C5d)}\\
V(\lambda) &> 0 \text{ for } \lambda > \frac{1}{\varepsilon^{\min}}. && \text{(C5e)}
\end{aligned}
$$

Step 2. Shape of $Z(\lambda)$. Let $\hat{Z}(\lambda,\varepsilon) \equiv \left[ab^{SB} - \delta^*(a)\right]\delta^{*\prime}(a)$. Then (C3a) implies $\hat{Z}(\lambda,\varepsilon) = 0$ for $\lambda \in [\frac{1}{2+\varepsilon^{\min}}, \frac{1}{1+\varepsilon^{\max}})$, so that $Z(\lambda) = 0$ for all $\lambda \in [\frac{1}{2+\varepsilon^{\min}}, \frac{1}{1+\varepsilon^{\max}})$.

Next, suppose $\lambda \notin [\frac{1}{2+\varepsilon}, \frac{1}{1+\varepsilon}]$. Then substituting (C3) into (C2) yields

$$\hat{Z}(\lambda,\varepsilon) = \frac{\lambda\left[1-\lambda(2+\varepsilon)\right]}{\left[1-\lambda(1+\varepsilon)\right]^2} + \frac{a\lambda^3\varepsilon'}{\left[1-\lambda(1+\varepsilon)\right]^3}, \tag{C6}$$

so that $\frac{\partial \hat{Z}(\lambda,\varepsilon)}{\partial\lambda} = \frac{[1-\lambda(1+\varepsilon)][1-\lambda(3+\varepsilon)]+3a\lambda^2\varepsilon'}{[1-\lambda(1+\varepsilon)]^4}$. For M small, we thus have

$$
\begin{aligned}
Z'(\lambda) &> 0 \text{ if } \lambda < \frac{1}{3} - \phi_1(M) && \text{(C7a)}\\
Z'(\lambda) &\leq 0 \text{ if } \frac{1}{3} + \phi_2(M) < \lambda < \frac{1}{2+\varepsilon^{\min}} && \text{(C7b)}\\
Z'(\lambda) &= 0 \text{ if } \frac{1}{2+\varepsilon^{\min}} \leq \lambda < \frac{1}{1+\varepsilon^{\max}} && \text{(C7c)}\\
Z'(\lambda) &\gtrless 0 \text{ if } \frac{1}{1+\varepsilon^{\max}} \leq \lambda < \frac{1}{1+\varepsilon^{\min}} + \phi_3(M) && \text{(C7d)}\\
Z'(\lambda) &> 0 \text{ if } \lambda > \frac{1}{1+\varepsilon^{\min}} + \phi_3(M), && \text{(C7e)}
\end{aligned}
$$

where $\phi_i(M) > 0$ and $\lim_{M\to 0}\phi_i(M) = 0$, $i = 1,2,3$.

Now, from (C6), there exists a $\lambda^+ \in (0,\infty)$ such that $Z(\lambda) < 0$ for all $\lambda \geq \lambda^+$. Further, $Z(0) = 0$, $\lim_{M\to 0} Z(\frac{1}{3}) = \frac{1}{4}E(a) > 0$, and $\lim_{M\to 0} Z(\frac{1}{2}) = 0$. This, together with (C7), implies that, for M small,

$$\begin{aligned}
Z(\lambda) &< 0 \text{ for } \lambda < 0 && \text{(C8a)}\\
Z(\lambda) &> 0 \text{ for } 0 < \lambda < \frac{1}{2+\varepsilon^{\max}} && \text{(C8b)}\\
Z(\lambda) &\leq 0 \text{ for } \frac{1}{2+\varepsilon^{\min}} \leq \lambda < \frac{1}{1+\varepsilon^{\max}} && \text{(C8c)}\\
Z(\lambda) &\gtreqless 0 \text{ for } \frac{1}{1+\varepsilon^{\max}} \leq \lambda < \frac{1}{1+\varepsilon^{\min}} + \phi_3(M) && \text{(C8d)}\\
Z(\lambda) &< 0 \text{ for } \lambda > \frac{1}{1+\varepsilon^{\min}} + \phi_3(M). && \text{(C8e)}
\end{aligned}$$

Step 3. Solution to (B3) and the form of constraint (B20). Conditions (C8) imply that for $C > 0$, (C2) can hold only if either $\lambda \in \left(0, \frac{1}{2+\varepsilon^{\min}}\right)$, or, possibly, $\lambda \in \left(\frac{1}{1+\varepsilon^{\max}}, \frac{1}{1+\varepsilon^{\min}} + \phi_3(M)\right)$. But $\lim_{M\to 0} V(\lambda = \frac{1}{1+\varepsilon^{\min}}) = \lim_{M\to 0} V(\lambda = \frac{1}{1+\varepsilon^{\max}}) = -\infty$ implies that $\lambda \in \left(\frac{1}{1+\varepsilon^{\max}}, \frac{1}{1+\varepsilon^{\min}} + \phi_3(M)\right)$ cannot be an optimum. Therefore, it must be $\lambda^* \in \left(0, \frac{1}{2+\varepsilon^{\min}}\right)$. Furthermore, $Z(\lambda)$ is continuous on $\left[0, \frac{1}{2+\varepsilon^{\min}}\right]$, with $Z(0) = 0 \geq Z(\frac{1}{2+\varepsilon^{\min}})$ and with $Z(\frac{1}{3}) > 0$. Hence, $\max Z(\lambda)$ on $\left[0, \frac{1}{2+\varepsilon^{\min}}\right]$ exists and is positive. Denote this maximum as $C^{\max}$. Then by continuity, for every $C \in [0, C^{\max}]$ there exists a $\lambda \in \left[0, \frac{1}{2+\varepsilon^{\min}}\right]$ such that (C2) holds, whereas if there is a λ such that (C2) holds for $C > C^{\max}$, this λ cannot be a part of the solution to (P1). Consequently, the feasibility constraint on C can be expressed as $0 \leq C \leq C^{\max}$.

Finally, (C4) says that $V'(\lambda) \leq 0$ on $\left(0, \frac{1}{2+\varepsilon^{\min}}\right)$. Hence, if multiple λ solve (B3), then λ^* is the smallest of them. But, from (C7), $Z(0) = 0$ and $Z'(\lambda) \leq 0$ for all $\lambda \in \left(\frac{1}{3} + \phi_2(M), \frac{1}{2+\varepsilon^{\min}}\right)$. The smallest λ that solves (B3) therefore cannot exceed $\frac{1}{3} + \phi_2(M)$. That is, $\lambda^* \leq \frac{1}{3} + \phi_2(M)$.

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