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Three Stories about the Chance of Casting a Pivotal Vote

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Abstract: People vote from self-interest or from a sense of duty. Voting from self-interest requires there to be some chance, however small, that one's vote swings the outcome of the election from the political party one opposes to the political party one favours. This paper is a discussion of three models of how that chance might arise: the *common sense* model inferring the probability of a tied vote today from the distribution of outcomes in past elections, *person-to-person randomization* where each voter looks upon the political preferences of rest of the electorate as analogous to drawings from an urn with given proportions of red and blue balls, and *nation-wide randomization* where voters are lined up according to their valuations (positive or negative) of a win for one of the two competing parties, but where chance shifts the entire schedule of preferences up or down. Emphasis is on the third model about which this paper may have something new to say.

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A vote is pivotal if, all by itself, it changes the outcome of the election. A person's vote is pivotal if, by voting rather than abstaining, that person transforms outcome of the election from a loss to the party that the person votes for into a win. With no chance of one's vote being pivotal - if the party one votes for is destined either to win for sure or to lose for sure - then one might as well abstain to save oneself the cost of voting. The same is true when, as is almost always the case, the chance of one's vote being pivotal is too small for one's expected benefit from voting to cover the cost. If so, the preservation of democracy depends on a sufficiently large portion of the electorate choosing to vote rather than to abstain not from self-interest, as when people participate in a competitive market, but from a sense of duty.

Exemptions and qualifications to this line of argument will be mentioned below, but this paper is more narrowly focused upon how the chance of casting a pivotal vote may arise. There can be no such chance without uncertainty in the voter's mind about the outcome of the election. This paper tells three stories about how such uncertainty can arise, three models of the chance of casting a pivotal vote based upon what might be called "common sense", "person-to-person randomization" and "nation-wide randomization". The common sense and the person-by-person models have been analyzed in detail elsewhere and are discussed here as background for the model of nation-wide randomization which is the paper's main contribution and concern. In the "common sense" model, the probability of tie is estimated from the range of possible outcomes as inferred from the history of past elections, but with no explanation of why the range is what it is inferred to be. In "person-by-person randomization", all voters are assigned probabilities of voting for each of two competing political parties. In "nation-wide randomization", people are lined up according to their, positive or negative, valuations of a win by one of the two competing parties, and uncertainty in the outcome of an election is created by unpredictable shifts of opinion in the electorate as a whole. Nation-wide randomization is a useful framework for comparing a person's cost and benefit of voting, for determining whether and to what extent abstention might bias the outcome of an election and for identifying a duty to vote where self-interest alone is an insufficient incentive.

From here on, the paper contains a discussion of the mechanics of pivotal voting, descriptions of each of the three models in turn, a numerical example of nation-wide randomization and a critique of assumptions with emphasis on the consequences when assumptions are relaxed.

The Mechanics of Pivotal Voting

As formulated by by Riker and Ordeshook (1968), one's decision to vote or abstain depends upon whether or not

$$\pi B + D > C \qquad (1)$$

where B is one's personal benefit if one's preferred party wins the election

D is the value one places upon voting as a duty to the rest of the community.[1]
C is one's cost of voting
and π is the probability of casting a pivotal vote.

For a strictly self-interested person, the value of D is 0. Such a person votes if $\pi B > C$, and abstains otherwise. The three stories in this paper are about how the magnitude of π might be determined.

My vote is pivotal when the party I prefer wins if I vote it but loses if I abstain. Consider the outcome of an election between a left party and a right party[2] where a tied vote is broken in favour of one party or the other by the flip of a coin, and suppose that I prefer the left party so that I would either vote for the left party or abstain.[3]

For my vote to be pivotal, it must be the case that *either* the number of other votes cast is even, exactly half of these other votes are cast for the left party, and, if I abstain, the resulting tie

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[1]That voting may not be personally advantageous was recognized by Downs (1957) and by Tullock (1967). Tullock suggested people's willingness to vote rather than to abstain might be explained by the possibility that voting is not costly at all. Voting for a political party might be like supporting a local football team or voting for the American Idol or congregating in Times Square on New Year's Eve. Incorporating this possibility transforms equation (1) into

. $$\pi B + D + E > C$$

where E is the value one attaches to participating in voting as a public event. For some purposes, E might be incorporated into D, but duty to participate and enjoyment of participation are not the same. One is about the welfare of the community, while the other is about one's own pleasure.

[2]The probability of one's vote being pivotal becomes murky with more than two competing parties and with a possibility of coalition government when no party wins a clear majority of votes cast. Multiparty elections will not be discussed in this paper.

[3]That voting may not be personally advantageous was recognized by Downs (1957) and by Tullock (1967). Tullock suggested people's willingness to vote rather than to abstain might be explained by the possibility that voting is not costly at all. Voting for a political party might be like supporting a local football team or voting for the American Idol or congregating in Times Square on New Year's Eve. Incorporating this possibility transforms equation (1) into

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would be broken in favour of the right party which I oppose, *or* the number of other votes cast is odd, the number of other votes cast for the left party is one less than the number of other votes cast for the right party, and, if I vote, the resulting tie would be broken in favour of the left party which I prefer. Designate the number of other votes cast as N_v or $N_v + 1$ where N_v is an even number. In either case, since ties are broken by the flip of a coin, the probability, π, of my vote being pivotal is half the probability that the left party gets exactly $N_v/2$ votes. As long as N_v is not too small, the probabilities of the left party getting exactly $N_v/2$ votes are virtually the same. The left party's chance of getting exactly 200,000 votes is only infinitesimally different when the total number of votes cast is 400,001 from what it would be when the total number of votes cast is 400,000. From here on, it will be supposed for convenience that n, the number of other voters, is even.

The probability of my vote being pivotal is universal in one sense but not in another. It is universal in the sense that, whatever I believe my probability to be, I must believe that everybody else's probability is the same. By voting left rather than abstaining, everybody must have the same chance of swinging the outcome of the election. The effect of an extra vote is the same no matter whose vote it is. But estimates of the common probability of casting a pivotal vote may differ from one person to the next because people have different information about the political preferences of the community as a whole. I may believe that neither party can win by more than 1,000 votes, a gap of 2,000 votes with equal chances of anything in between, so that the probability, as I see it, of a tied vote is 1/2,000, and the probability, as I see it, of casting a pivotal vote is 1/4,000. You may believe the comparable numbers to be 2,000, 1/4,000 and 1/8,000. There is nothing illogical in that. However, in the models to follow, it will always be assumed that everybody's assessment of the uncertainty in the outcome of an election is the same, and that everybody's estimate of the chance of casting a pivotal vote is the same too. The purpose of these models is to explain the proportion of people eligible to vote who do vote rather than abstain and to construct a mechanism yielding the probability that a vote between left and right parties is tied.

A simple example shows what is at stake. A person who happens to favour the left party is considering whether to vote or abstain, where the cost of voting is $20 and where it is worth $10,000 to this person for the left party to win the election. The cost of voting is the dollar value of the time and trouble required to go to the ballot box and to cast one's vote. Compensation of $10,000 would be required to keep this person equally well off in the event of a win for the right party instead. If a sense of duty played no role in this person's decision whether to vote or to abstain and if strict self-interest were the only consideration, it would be disadvantageous for this person to vote unless the chance of his vote altering the outcome of the election - turning a tie into a win for the left party, or a loss by just one vote into a tie - exceeds 1 in 500 [The required $\pi = C/B = 20 \div 10{,}000 = 1/500$.] For most elections, this probability seems much too high. Suppose that this person is choosing to vote or abstain in a constituency where a million other people are expected to vote and where the left party is confidently expected to obtain somewhere between 48% and 52% of the votes, with equal chances of outcomes anywhere between these limits. If so, the chance of a tie between left and right parties becomes 1 in 40,000, and the probability of this person's vote being pivotal becomes 1 in 80,000.

In this example, the chance, π, of the person's vote altering the outcome of the election is no more than 1 in 80,000, but the minimal chance required to make voting advantageous to the strictly self-interested voter is 1 in 500. This person's chance of swinging the election is only a quarter of a percent of the chance required to make voting personally advantageous. Of course, these numbers are pulled out of thin air, but the gap between the estimated chance and the required chance is so large that it seems unlikely for any reasonable choice of numbers to induce more than a very small minority of eligible voters with enormous stakes in the outcome of the election to vote rather than to abstain. Hence the paradox of not voting.[4]

The Common Sense Model

The numerical example in the preceding section - with a million voters and a range of 80,000 equally likely outcomes - is an application of the common sense model, based upon the assumptions that observed pluralities will never be exceeded and that all outcomes within the observed range are equally likely. On these assumptions, the probability of a tie is 1 in 40,000, implying that the probability of one's vote being pivotal is 1 in 80,000, and, with a cost of voting of \$20, the strictly self-interested citizen would prefer to abstain unless his valuation of a win for his preferred party exceeds 1.6 million dollars. Not many citizens' valuations would be as high as that. It is hard to cook plausible numbers for which even a significant minority of eligible voters would choose to vote rather than to abstain if pure self-interest were the only consideration.

The great advantage of the common sense model is in supplying a rough and ready way to estimate the chance that a vote turns out to be pivotal. There are several disadvantages. The assumption that the distribution of votes is uniform between well-specified limits is convenient but completely arbitrary. It is hard to tell what the appropriate limits - assumed in the example to be 48% and 52% - might be. A more plausible distribution of outcomes might be inferred from information about the history of outcomes in past elections, a distribution with more weight in the middle and less at the tails. There is also some question as to whether outcomes in past elections are really predictive of the outcome in an election today. Regardless of the history of

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[4]The chance of casting a pivotal vote may be of interest in another context as well. Electoral rules may be designed or evaluated according to the distribution of "political power" among eligible voters or among legislators, where political power is interpreted as the probability of casting a pivotal vote. Shapley and Shubik (1954, page787) define the power of a voter as "the chance he has of being critical to the success of the winning coalition" where all arrangements of voters are assumed to be equally likely and where a voter in any particular arrangement is deemed to be pivotal when his appearance in the sequence of like-minded voters generates a majority for the party he favours. The Shapley-Shubik index of voting power can be thought of as a generalization of the probability of casing a pivotal vote in which, for example, a person who is part of a large majority is not deemed powerless despite the fact that he is never pivotal as defined here. For a review of the literature on voting power see, Felsenthal and Machover (1998).

elections, the probability of a voter being pivotal may be relatively high in some elections and relatively low in others, high when an election is expected to be close, and low when one party is almost certain to win. Most importantly, the probability of a voter being pivotal is observed rather than derived from assumptions about voters' behaviour. There is some question as to whether what is being called the common sense model is really a model at all, for it has nothing to say about why π is what it is inferred to be. It is a black box generating numbers. It is most useful as a check on other models to be discussed below, raising suspicion whenever outcomes are too far from what the common sense model would suggest.[5]

Person-by-Person Randomization

Person-by-person randomization is based upon an analogy between voting and sampling.[6] Think of all voters - everyone choosing to vote rather than to abstain - as lined up at the ballot box, and voting one by one. Before each person votes, the angel of chance assigns that person a preference, for the left party with a probability of p and for the right party with a probability (1 - p). With a total of N votes cast, the number of votes for the left party is analogous to the number of blue balls in N drawings from an urn containing blue and red balls in proportions p and (1 - p). The distribution of the number of blue balls is binomial with mean pN and standard deviation $[p(1 - p)N]^{1/2}$. Suppose for convenience that N is an even number.

The probability, T, of a tie among the N voters is precisely the probability that the number of blue balls is equal to N/2. It is the probability of picking exactly N/2 blue balls in N draws from this urn, where

$$T = \alpha\beta \qquad (2)$$

where α = the probability that the first N/2 balls are blue and the remaining N/2 balls are red

$$= (p)^{N/2}(1 - p)^{N/2} \qquad (3)$$

and β = the number of ways to place N/2 balls in N slots

$$= N!/[(N/2)!(N/2)!] \qquad (4)$$

The probability of a tie becomes

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[5]The common sense model is used in Edlin, Gelman and Kaplan(2008) to quantify a duty to vote. It is discussed at greater length in Gelman, Katz and Bafumi (2004) and Edlin, Gelman and Kaplan (2007).

[6]The first such models I know of are those of Beck(1975) and of Good and Mayer (1975). See also Myerson (1998).

$$T = \{N!/[(N/2)!(N/2)!]\}(p)^{N/2}(1 - p)^{N/2} \qquad (5)$$

which, when simplified by Stirling's formula [7], reduces to

$$T = \{2/\Pi N\}^{1/2} [(2p)^{N/2}(2(1- p))^{N/2}] \qquad (6)$$

where Π is the ratio of the circumference to the diameter of a circle.

Now imagine of a society with (N + 1) people, where the first N are assumed to vote in accordance with preferences as assigned by the angel of chance and the $(N + 1)^{st}$ person is deciding whether to vote of abstain. The probability, π, that the vote of the $(N + 1)^{st}$ person is pivotal is half the probability, T, of a tie among the other N people.

Suppose once again that there are a million other voters, i.e. N = 1,000,000. As the binomial distribution is bell shaped, the probability of any particular nnumber of votes for the left party, n_L, depends on its distance from the mean, highest when n_L is the mean, and steadily lower the farther from the mean n_L happens to be. Thus, the probability that $n_L = N/2$ is greatest when N/2 is itself the mean, that is, when p = ½. With p = ½, it follows from equation (6) that the probability of a tie between political parties - the probability $n_L = N/2 = 500{,}000$ - is just under one tenth of 1%. With a cost of voting of $20, the benefit from a win for one's preferred party sufficient to make voting advantageous must be at least $32,000, which is high but not impossible.

The probability of a tie diminishes sharply when p deviates from ½, causing the mean of the distribution of votes for the left party to deviate from N/2. Even a deviation of as little as 1% reduces the probability of a tie dramatically. Suppose the probability of a randomly-chosen person voting for the left party rises from .5 to .51, raising the mean, pN, of the distribution of n from 500,000 to 510,000, a difference of 10,000 votes. The effect of this difference upon the probability of a tied vote depends upon the standard deviation of n_L, [equal to $p(1 - p)N]^{1/2}$,

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[7]Stirling's formula is that

[N!] Is approximately equal to $(2\,\Pi N)^{½}(N/e)^N$

From Stirling's formula, it follows that

$$\alpha = N!/[(N/2)!(N/2)!] = \{(2\Pi N)^{½}(N/e)^N\}/\{(2\Pi N/2)^{½}(N/2e)^{N/2}\}\{(2\Pi N/2)^{½}(N/2e)^{N/2}\}$$

$$= [\{(2\Pi N)^{½}\}/\{(2\Pi N/2)^{½}\}\{(2\Pi N/2)^{½}\}][\{(N/e)^N\}/\{(N/2e)^{N/2}\}\{(N/2e)^{N/2}\}\}$$

$$= [(2/\Pi N)^{½}]\,[2^N]$$

from which equation (6) follows immediately.

which turns out in this example to be 500. Thus, the distance between the expected number of votes for the left party (550,000) and the number of votes for which the election is tied (500,000) amounts to 20 standard deviations [10,000/500 = 20].The probability of a variable being 20 standard deviations from its mean is infinitely tiny. Beck (1975, table 1) computes that probability to be about 10^{-90}. The entire national income of the world would not be sufficient to compensate for such a small chance of a pivotal vote. As there is no particular reason for supposing p to be exactly ½, something beyond mere self-interest would surely be required to explain why people vote at all.

Some features of person-to-person randomization should be noted. Hidden within person-to-person randomization is a special assumption about sampling. Imagine a population of exactly one million voters, with no abstentions and where exactly half a million people vote left and the other half vote right. Clearly, on these assumptions, the outcome is a tie, and any additional voter must be pivotal despite the fact that the probability of a randomly-chosen person voting for the left party remains equal to ½. With p = ½ , the probability of a tied election is 100% in this case, as opposed to only a tenth of 1% in Beck's calculation. The discrepancy arises from the distinction between sampling with and without replacement. A ball picked randomly from an urn may or may not be returned to the urn before the next ball is drawn. Beck's formula is for sampling with replacement. Voting may be more like sampling without replacement. Enormous in this example, the difference between estimated probabilities of a tie would be less pronounced when the proportion of population that votes is significantly less than 1.

As discussed so far, person-to-person randomization is of *interests*. Everybody knows what is best for himself, but some people would be made better off with a win by the left party while other people would be made better off with a win by the right. Alternatively, randomization may be of *opinions.* One of two parties may be better for everybody, but people may differ in their judgments about which party that is. Everybody gains the same fixed amount B when the better party wins, but a proportion, p, of eligible voters thinks the left party is better and a proportion (1 - p) thinks the right party is better. If people are correct on average - in the sense that p > ½ if and only if the left party is preferable - and as long as the value of p is not common knowledge, the probability of the better party winning the election is an increasing function of the number of people who vote rather than abstain. The optimal proportion of voters in the population becomes that for which a person's cost of voting is just equal to the expected gain to society as a whole from the chance of swinging the outcome of the election advantageously. It is also possible that some people know what is best, others do not, and everybody knows who is who. In such situations, public decision-making is best left to the experts rather than to the ballot box, or voting may be for people whose task it would be to identify experts.

There is also something unsatisfactory about supposing voters' preferences to be assigned in the first instances over political parties themselves rather than over policies that political parties may come to adopt. The angel of chance might more appropriately be seen as conveying tastes randomly on the understanding that people's votes would be determined accordingly. The implicit assumption that policies of political parties are fixed and inflexible is at variance with the median voter theorem according to which pressure to win elections drives all political parties to set platforms in accordance with the preferences of the median voter.

A discrepancy between assumptions is more apparent than real. The $(N+1)^{th}$ person has a genuine option to vote or to abstain. The other N people are given no such choice. They must all vote, left or right as the case may be, in accordance with preferences assigned by the angel of chance. The discrepancy disappears when it is recognized that the story about the angel of chance is no more than a specification of each person's uncertainty about what the rest of the electorate will do. The story is adequate if all that is required is an estimate of π for any given p and N, but it includes no explanation of either p or the proportion of eligible voters who vote rather than abstain. Presumably p would be a reflection of economic and political circumstances in the economy, but the model contains no list of relevant circumstances or mechanism connecting these to political behaviour.

Recognition of the huge gap between the estimates of the chance of casting a pivotal vote within the common sense model and within the person-to-person model when the probability, p, of voting for the left party differs, however slightly, from 50%, has provoked several attempts to close the gap by modifying the assumptions of the person-to-person model. Chamberlain and Rothschild (1981) assume instead that p itself is chosen from a probability distribution of possible values between 0 and 1. There are two angels of chance, the first choosing p for the entire population, and the second choosing each person's vote for the given value of p. Chamberlain and Rothschild show that, if the distribution of p is uniform (and in some other cases as well), the probability of a tie is proportional to 1/N. The probability of a tie is still very small, but not as small as in Beck's computation when p differs from ½ by as little as 1%. Chamberlain and Rothschild's model is in effect a cross between the common sense model and person-to-person randomization, replacing the black box in the common sense model with a mechanism that generates similar outcomes.

Other modifications in the assumptions raise the chance of casting a pivotal vote by systematically lowering the number of people who choose to vote. Models along these lines will be discussed later on in this paper.

Nation-wide Randomization

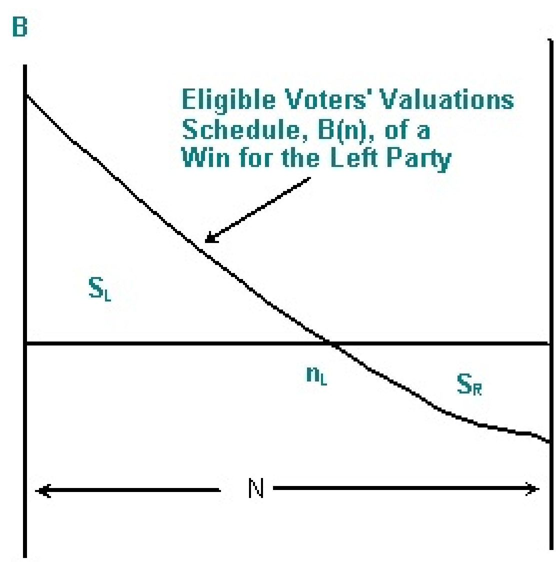
Nation-wide randomization begins by supposing not just that people favour the left party or the right party as the case may be, but that people place different monetary values on their preferences and that a schedule of voters' valuations may be identified. Each person knows for certain his own, positive or negative, value of a win for the left party, but nobody has more than a vague idea of the schedule of voters' valuations in the electorate as a whole. A person's value of a win for the left party is designated as B. If Joan places a value of $4,000 on a win for the left party, then, for Joan, B = 4,000. If Charles places a value of $10,000 on a win for the right party, then, for Charles, B = - 10,000. Along the true but unobservable schedule, people are ordered from left to right in accordance with their valuations, B, of a win for the left party. In an electorate with N people eligible to vote, B(1) is the dollar value of a win for the left party to the person with the highest such valuation, B(2) is the next highest valuation, and so on, until B(N) which is the lowest of all valuations and which must be negative if the right party is to capture any votes at all.

When the income of person n is $y^L(n)$ in the event of a win for the left party and $y^R(n)$ in the event of a win for the right party, then B(n) becomes

$$B(n) = y^L(n) - y^R(n) \quad (7)$$

A voters' valuations schedule is illustrated in figure 1 with benefits B(n) on the vertical axis and with the N eligible voters lined up appropriately on the horizontal axis. For convenience, the voters' valuations schedule is drawn continuously, but the schedule is really confined to integral values of n from 1 to N. If everybody votes and as long as all votes are cast selfishly, for the left party when $B(n) > 0$ and for the right party when $B(n) < 0$, then the left party wins whenever $n_L > N/2$ and the right party would win whenever $n_L < N/2$.

Figure 1: The Distribution of Voters' Valuations of a Win for the Left Party



The areas designated as S_L and S_R respectively are the total of the valuations by all left-supporters of a win for the left party and the total of the valuations by all right-supporters of a win for the right party. Specifically,

$$S_L = \text{sum of all } B(n) \text{ for which } B(n) > 0 \quad (8a)$$

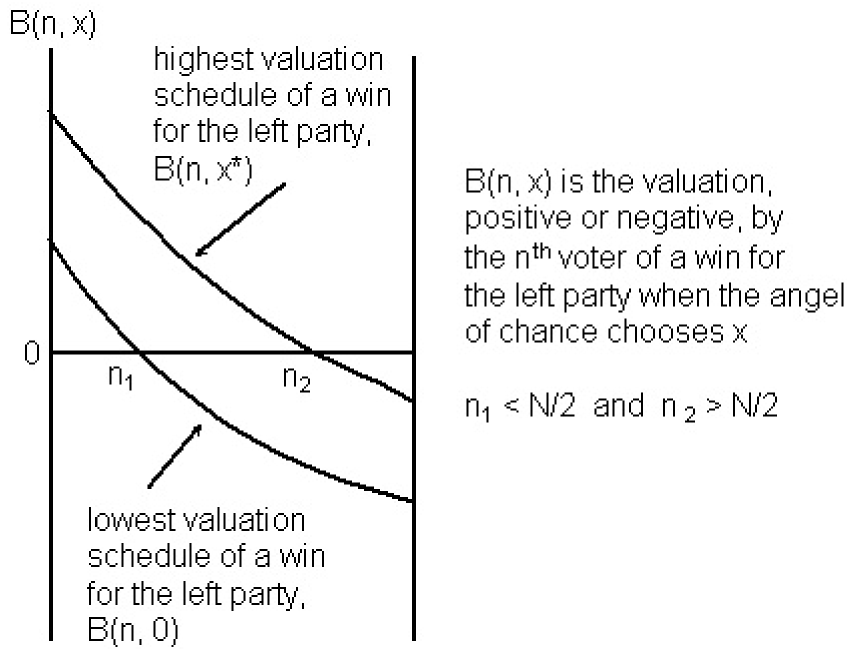
and

$$S_R = \text{sum of the absolute values of all } B(n) \text{ for which } B(n) < 0 \quad (8b)$$

As the figure is drawn, $n_L > N/2$ and $S_L > S_R$ signifying that the left party wins the election as long as everybody votes and that a win for the left party is best for the nation as a whole. A different postulated shape of the voters' valuation curve - with the same value of n_L but flatter to the left of n_L but steeper to the right - could create a discrepancy between number of votes and aggregate benefits.[8]

Uncertainty about whether or not one's vote is pivotal can only arise from each voter's uncertainty about the electorate as a whole. In person-to-person randomization, that uncertainty is created by the designation of each and every voter, one by one, as a left-supporter or a right-supporter, in accordance with the flip of a weighted coin. In nation-wide randomization, that uncertainty is about the exact location of a voters' valuations schedule. Each person looks upon the political preferences of the electorate as a whole as a voters' valuations schedule selected by the angel of chance from a set of feasible schedules within a range from highest to lowest as illustrated in figure 2.

Figure 2: Highest and Lowest Voters' Valuations Schedules



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[8]Note that $S_L - S_R = \sum_{n=1}^{N} y^L(n) - \sum_{n=1}^{N} y^R(n)$ which is the differ national income as it would be if the left party wins and the national income if the right party wins.

Selection by the angel of chance of one out of a range of possible voters' valuations schedules can be represented as the choice of a single parameter x. Imagine a "basic" schedule B(n) such as is illustrated in figure 1 that is shifted up or down in accordance with a random variable x selected by the angel of chance and converting the valuation schedule of a win for the left party from B(n) to B(n, x) where

$$B(n, x) = B(n) + x \qquad (9)$$

and where x varies from 0 to some maximal value, x*, and where an increase in x pushes the schedule up, raising all valuations of a win for the left party and lowering all valuations of a win for the right party accordingly. It is difficult to say *a priori* what the distribution of the random variable, x, might be, but, in the interest of simplicity, it is assumed here to be uniform, equally likely to take on any value from a minimum of 0 to a maximum of x*.

On these assumptions, the voters' valuations schedule is equally likely to lie anywhere between the highest schedule, B(n, x*), generating, for each person n, the largest possible valuation of a win for the left party (or the smallest possible valuation of a win for the right party), and the lowest schedule, B(n, 0), generating, for each person n, the largest possible valuation of a win for the right party.

It is important to emphasize what voters do and do not know. If they knew their rankings on the voters' valuations schedule, they could infer from their own dollar values of B which curve the angel of chance must have selected and would know for certain whether or not the votes of the rest of the electorate are tied. But voters are presumed not to know their rankings. Choice by the angel of chance of a voters' valuations schedule is no more than a rationalization of the general idea that voters have some imperfect perception about the preferences of the rest of the electorate. Thus, if B(2,172) = 4,000 on the lowest voters' valuations schedule, if B(10,956) = 4,000 on the highest voters' valuations schedule, and if Joan knows her own value of a win for the left party to be $4,000, Joan could infer that she is between the 2,172th and 10,956th left-leaning person in the entire electorate, but Joan would have no idea where between these limits her ranking lies.

To keep the model as simple as possible, a common stylized uncertainty is imposed. Everybody's perception of the shapes of the voters' valuations curves is the same. Perceptions about the locations of the maximal and minimal curves and about the distribution of x are the same too. The commonly-perceived distribution of x is uniform. A bell-shaped distribution of x would be more realistic but less tractable. As mentioned above, voters may in practice have different pictures in their minds about how the rest of the electorate behaves, but that possibility is being assumed away to ensure that everybody's estimate of π is the same.

On these assumptions, intervention by the angel of chance creates two distinct electoral probabilities, the probability of any person's vote being pivotal and the probability of the left party winning the election. With reference to figure 2 and assuming all numbers of left supporters between n_1 and n_2 to be equally likely, the probability, called π, of a person's vote becoming pivotal depends upon the *width* of the band between n_1 and n_2 and the probability, called P, of a win for the left party depends upon the *location* of the band, whether it is mainly to the right or mainly to the left of the center point, N/2.

Since all schedules within the band between n_2 and n_1 are equally likely, the chance of a tie must be $1/[n_2 - n_1]$, and a voter's chance of being pivotal must be

$$\pi = 1/[2\,(n_2 - n_1)] \qquad (10)$$

A slightly more complicated derivation of this formula is redundant in the present context where nobody abstains because voting is costless, but becomes useful for generalizing the formula once a cost of voting is introduced. Since there are votes N votes in total, the average number of votes for both parties together must be N/2, and the party with more than N/2 votes wins the election. It follows at once that the range of possible outcomes of the election is the sum of a) the largest possible number of votes for the left party in excess of the average and b) the largest possible number of votes for the right party in excess of the average. Since the largest possible number of votes for the left party is n_2 and the largest possible number of votes for the right party is $N - n_1$, the range of possible electoral outcomes must be

$$\{n_2 - N/2\} + \{(N - n_1) - N/2\} = n_2 - n_1 \qquad (11)$$

from which equation (10) follows at once.

Similarly, the probability, P, of a win for the left party becomes the ratio of "the largest number of votes for the left party in excess of the average" and " the sum of the largest numbers of votes for both parties in excess of the average". The probability, P, of the left party winning the election becomes

$$P = [n_2 - N/2]/[n_2 - n_1] \qquad (12)$$

which, as figure 2 is drawn, is somewhat less than ½.

Probabilities of a win for the left party and of casting a pivotal vote can be adjusted to take account of people's decisions to vote or abstain once a cost of voting is introduced. With costly voting and as long as people are strictly self-interested, the decision to vote or abstain is in accordance with equation (1) above with D set equal to 0. With a common cost of voting, C, a common probability, π, of casting a pivotal vote, a person with a valuation B votes for the left party when

$$B > C/\pi \qquad (13a)$$

for the right party when

$$- B > C/\pi \qquad (13b)$$

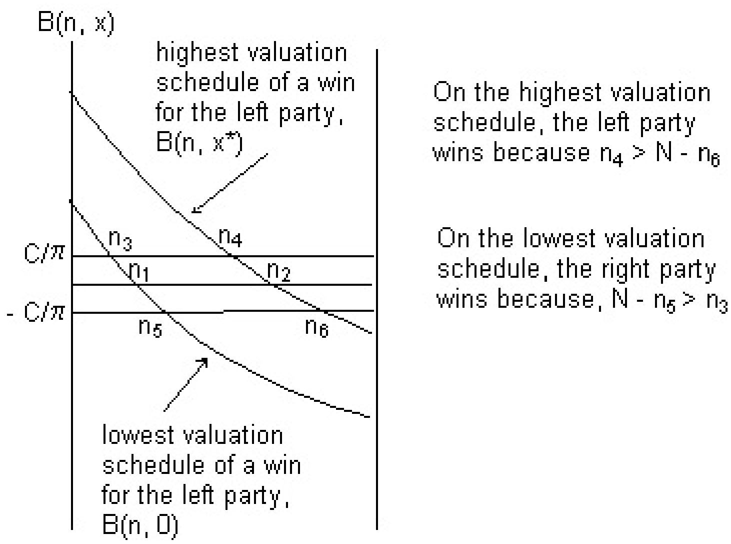
and abstains otherwise, i.e. when

$$C/\pi > |B| \qquad (13c)$$

Like market prices, π is at once a signpost for each and every voter and a characteristic of the community of voters as a whole. A person votes or abstains in accordance with the electorate's value of π, but the electorate's value of π depends on what voters choose to do.

Numbers of voters and abstainers are illustrated in figure 3, a reproduction of figure 2 with the addition of two horizontal lines at distances C/π above and below the horizontal axis. In accordance with equation (13), one votes for the left party of one's value of B is above the higher line, one votes for the right party if one's value of B is below the lower line, and one abstains in between.

Figure 3: Uncertainty about Other's Preferences, the Cost of Voting and the Chance of Casting a Pivotal Vote



For any given π and for any given maximal and minimal voters' valuation schedules, the number of votes for each party and the number of abstentions can be inferred from equation (13). If the highest valuation schedule (with x = x*) is chosen, then n_4 people vote for the left party, N - n_6 people vote for the right party, n_6 - n_4 people abstain and the left party wins the election. If the lowest valuation schedule (with x = 0) is chosen, then n_3 people vote for the left party, N - n_5 people vote for the right party, n_5 - n_3 people abstain and the right party wins the election. A situation could arise where one party is so much preferred to the other that it wins the election regardless of which valuation schedule is chosen. Were that so, no voter could ever be pivotal and the outcome of the election would depend upon the mobilization of each party's supporters, a matter not discussed in this paper. Assume for the present that that is not so. Assume the left party wins when the highest valuation schedule is chosen, and the right party wins when the lowest valuation schedule is chosen. The assumption is that

$$n_4 > N - n_6 \quad (14a)$$

and $$n_3 < N - n_5 \quad (14b)$$

yielding a unique value of the probability of casting a pivotal vote.

The derivation of the probability of casting a pivotal vote is similar to what it was with costless voting, but with one important exception. It remains true that the range of possible outcomes of the election is the sum of a) the largest possible number of votes for the left party in excess of the average and b) the largest possible number of votes for the right party in excess of the average, but the sum of the votes for both parties is reduced by the number of abstentions and the average vote is reduced accordingly.

On the highest voters' valuations schedule where the left party is destined to win the election, the number of votes cast is $n_4 + (N - n_6)$, the average is half that, and the number of votes for the left party in excess of the average is $n_4 - \{n_4 + (N - n_6)\}/2 = ½(n_4 + n_6 - N)$. On the lowest voters' valuations schedule where the right party is destined to win the election, the number of votes cast is $n_3 + (N - n_5)$, the average is half that, and the number of votes for the right party in excess of the average is $(N - n_5) - \{n_3 + (N - n_5)\}/2 = ½(N - n_5 - n_3)$. Thus, "the sum of a) the largest possible number of votes for the left party in excess of the average and b) the largest possible number of votes for the right party in excess of the average" becomes

$$½(n_4 + n_6 - N) + ½(N - n_5 - n_3) = ½(n_4 + n_6 - n_5 - n_3) \quad (15)$$

so that the probability of a tied vote becomes $2/(n_4 + n_6 - n_5 - n_3)$ and the probability of casting a pivotal vote becomes

$$\pi = 1/(n_4 + n_6 - n_5 - n_3) \quad (16)$$

which boils down to equation (10) when $C = 0$, in which case $n_4 = n_6 = n_2$ and $n_5 = n_3 = n_1$.

The probability, P, that the left party wins the election must be the ratio of the maximal win for the left party to the full range between the maximal win for the left and the maximal win for the right, i.e.

$$P = [n_4 + n_6 - N] / [n_4 + n_6 - n_3 - n_5] \quad (17)$$

The information on votes, abstentions and the chance of casting pivotal vote is summarized in table 1.

Table 1: Numbers of Votes and Abstentions
(On the assumption that each party has some chance of winning the election)

	highest valuation schedule	lowest valuation schedule
abstentions	$n_6 - n_4$	$n_5 - n_3$
votes for the left party	n_4	n_3
votes for the right party	$N - n_6$	$N - n_5$
total votes	$N - n_6 + n_4$	$N - n_5 + n_3$
votes for the winning party in excess of the average	$\frac{1}{2}(n_4 + n_6 - N) > 0$	$\frac{1}{2}(N - n_5 - n_3) > 0$

Equation (16) completes the equilibrium in figure 3. The critical numbers in the table - n_3, n_4, n_5 and n_6 - are dependent upon the value of π, but π itself is dependent on these numbers. Note the family resemblance between the what was called "common sense model" and "nation-wide randomization" in the determination of the probability of casting a pivotal vote. The common assumption in these models is that if one party might win by up to Q votes and the other party might win by up to R votes and if all outcomes in between are equally likely, the probability of a person's vote being pivotal must be $1/2(Q + R)$. Nation-wide randomization can be thought of as an explanation of the outcome in the common sense model. As long as the highest and lowest voters' valuations schedules are parallel, one's chance of casting a pivotal vote is just the inverse of the horizontal distance between the two schedules regardless of the size of C/π.

Note finally the independence of P and π. The highest and lowest voters' valuations schedules may both shift upward increasing the left party's chance of winning the election without at the same time changing anybody's chance of casting a pivotal vote. That is because P depends on the average height of the two curves while π depends on the distance between them. Person-by-person randomization has a very different implication. There, a change in the left party's chance of winning the election would normally be accompanied by a massive change in the probability of casting a decisive vote.

A Numerical Example

Consider a society with a population, N, of 10,000 eligible voters where a person's (positive or negative) valuation, B, of a win for the left party is

$$B(n, x) = 90{,}000 - 20n + x \qquad (18)$$

where n is an ordering of people from less to more right wing and where the angel of chance picks x from a minimum of 0 to a maximum of 30,000. Along the lowest voters' valuations schedule where x = 0, people's valuation of a win for the left party varies from a maximum of 90,000 when n = 0 to a minimum of - 110,000 (meaning that the most right wing person values a win for th right party at 110,000) when n = 10,000. The outcome when voting is costless is shown in table 2.

Equation (17) is a linear representation of the voters' valuations schedules in figure 2. On the highest schedule, the number of votes for the left party is n_2 and the number of votes for the right party is 10,000 - n_2 because, by assumption, everybody for whom $n < n_2$ votes left and everybody for whom $n > n_2$ votes right. On the lowest schedule, the number of votes for the left party in n_1 and the number of votes for the right party is 10,000 - n_1. Values of n_1 and n_2 can be read off equation (18) with the appropriate values of x. Specifically, when voting is costless,

$$B(n_1, 0) = 90{,}000 - 20n_1 = 0 \qquad (19a)$$

and

$$B(n_2, 30{,}000) = 90{,}000 - 20n_2 + 30{,}000 = 0 \qquad (19b)$$

from which it follows immediately that n_1 = 4,500 and n_2 = 6,000. Depending upon the angel of chance, the outcome for the left party may with equal probabilities lie anywhere between a win by 2,000 votes [6,000 - 4,000] and a loss by 1,000 votes [5,500 - 4,500]. Equivalently, the outcome of the election lies anywhere between a win fro the left party with 1,000 votes above the average and a win for the right party with 500 votes above the average, a range of 1,500 possible outcomes. The probability of a tied vote becomes 1/ 1,500, and, in accordance with equation (10), the probability of a person's vote turning out to be pivotal is half that,

$$\pi = 1/[2(1{,}000 + 500)] = 1/3{,}000 \qquad (20)$$

Table 2: Electoral Outcomes Dependant on the Angel of Chance when Voting is Costless

	highest valuation schedule x = 30,000	lowest valuation schedule x = 0
votes for the left party	6000	4500
votes for the right party	4000	5500
votes for the winning party in excess of the average	1,000 (for the left party)	500 (for the right party)
probability of a pivotal vote π = ½ (1,000 + 500) = 1/3,000		
probability of win for left party (P) = [maximal plurality for left party]/[sum for both pluralities] = 2/3		

Of no consequence when voting is costless, the probability, π, of one's vote being pivotal becomes important when voting is costly. The decision whether to vote or abstain becomes a trade off between the benefit of a win for one's preferred party and the expected cost of procuring it. It becomes necessary to compute the number of votes, n_3 and n_4, for the left party corresponding to the lowest and highest voters' valuations schedule, as well as the corresponding numbers, $N - n_4$ and $N - n_6$, of votes for the right party as shown in figure 3. With the linear valuation curves in equation (18), this becomes unconscionably simple, for the value of π in table 2 carries over to the more complex case where voting is costly. Each of n_1, n_2, n_3 and n_4 is derived in a separate equation, equating a variant of B(n,x) in equation (18) to the voter's expected cost, C/π, of turning the election from a loss to a win for the party he favours.

With an assumed cost of voting, C, of 20 and with a value of π of 1/3,000 from equation (20), the value of C/π becomes 60,000, the values of n_3 and n_4 can be estimated by setting the appropriate B equal to C/π, and the values of n_5 and n_6 can be estimated by setting the appropriate B equal to $- C/\pi$. Numbers of votes for the left party in accordance with the lowest and highest valuations schedules are derived from the equations

$$B(n_3, 0) = 90{,}000 - 20n_3 = C/\pi = 60{,}000 \qquad (21a)$$

and $$B(n_4, 30{,}000) = 90{,}000 - 20n_4 + 30{,}000 = C/\pi = 60{,}000 \qquad (21b)$$

Similarly, numbers of votes for the right party are derived from the equations

$$B(n_5, 0) = 90{,}000 - 20n_5 = - C/\pi = - 60{,}000 \qquad (21c)$$

and $$B(n_6, 30{,}000) = 90{,}000 - 20n_6 + 30{,}000 = - C/\pi = - 60{,}000 \qquad (21b)$$

Thus, $$n_3 = 1{,}500,\ n_4 = 3{,}000,\ n_5 = 7{,}500 \text{ and } n_6 = 9{,}000 \qquad (22)$$

Votes for the left party along the lower and higher schedules are 1,500 and 3,000. Votes for the left party along the lower and higher schedules are $N - n_5$ and $N - n_6$ equal to 2,500 and 1,000 respectively.[9] The numbers are brought together in table 3, and the valuation curves themselves are shown as figure 4.

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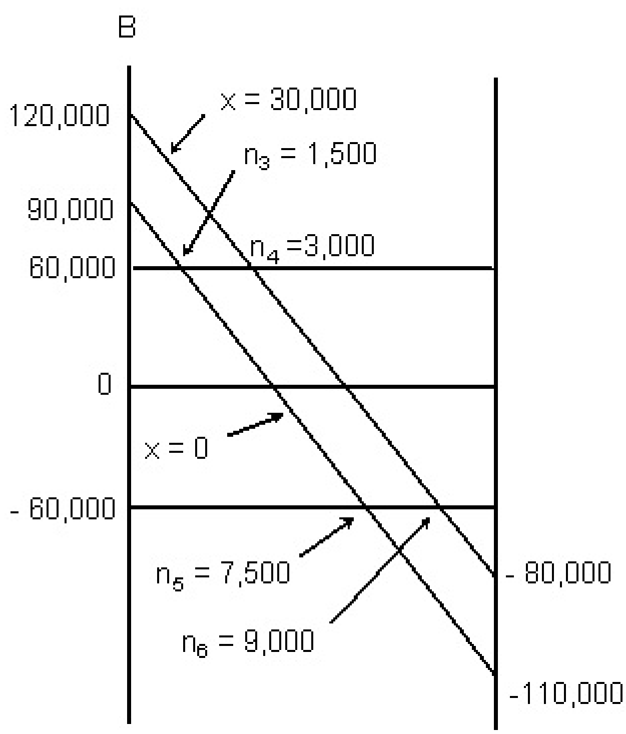
[9]Values of n_3, n_4, n_5, n_6 and π could be computed simultaneously from the four parts of equation (21) together with equation (15). Instead, use was made of the fact that the highest and lowest voters' valuations schedules are assumed to be parallel so that π is the horizontal distance between them.

Table 3: Electoral Outcomes Dependant on the Angel of Chance when Voting is Costly
(Cost of voting equals 20)

	highest valuation schedule x = 30,000	lowest valuation schedule x = 0
votes for the left party	3000	1500
votes for the right party	1000	2500
abstentions	6000	6000
votes for the winning party in excess of the average	1,000 (for the left party)	500 (for the right party
probability of a pivotal vote (π) = 1/(1,000 + 500) = 1/3,000		

The information in table 3 and figure 4 can be looked upon from two points of view. It is, on the one hand, a picture of how the collectivity behaves, showing everybody's probability of being pivotal, the range of possible pluralities for each party, the number of abstentions, and the valuations of a win for one's preferred party required to induce a person to vote. It is, on the other hand, a guide for deciding whether to vote or to abstain, supplying the critical value of π in equation (1). These points of view are consistent as long as nobody can infer from his own value of B what valuation schedule the angel of chance has selected.

Figure 4: Votes and Abstentions when Voting is Costly



Artificial as it is, this example suggests an important principle about who votes and who abstains. In so far as voting is from self-interest rather than from a sense of duty, it is the extremists who vote and the moderates who abstain. One votes if and only if the value of a win for one's preferred party exceeds 60,000. Everybody else abstains. Symmetries in the example prevent this consideration from influencing the outcome of the election, but, as will be discussed below, there is no guarantee of such symmetry in other, more realistic situations.

Qualifications and Exceptions:

The numerical example serves to establish that there *may* be an equilibrium – with some people eligible voters voting for the left party, others voting for the right party and still others choosing to abstain - in circumstances where each voter knows his own preference but has only a rough idea about the distribution of preferences in the rest of the population, and, above all, where those who vote do so out of pure self interest-rather than from a sense of duty. The parameters of the example were chosen to generate such an outcome. The example does not establish that there *must* be an equilibrium or that the outcome is necessarily a reflection of the will of the electorate.

1) <u>A Sure Winner</u>: The locations of maximal and minimal valuation schedules in figures 2, 3 and 4 were chosen to ensure that both parties have some chance of winning the election. The maximal schedule delivers a win the left party, the minimal valuation schedule delivers a win for the right party and some schedule in between delivers a tie. That need not always be so. Regardless of what value of x is chosen by the angel of chance, significantly higher schedules than those in equation (18) above would deliver a sure win to the left party, and significantly lower schedules would deliver a sure win to the right.

Suppose, for example, the constant term in equation (18) were reduced from 90,000 to 60,000, converting the schedule to

$$B(n, x) = 60{,}000 - 20n + x \qquad (23)$$

while everything else remains the same. The right party would then be sure to win the election regardless of the value of x selected by the angel of chance. The most favourable outcome for the left is when x is as large as possible, that is when $x = x^* = 30{,}000$. Even so, there would be only 1,500 out of the 10,000 eligible voters for whom $B \geq 60{,}000$, as against 2,500 eligible voters for whom $B \leq - 60{,}000$. The best the left party could hope for is a three-to-five loss. The right party's proportion of the votes can only increase when the angel of chance picks a lower value of x.

Recall the derivation of π in equation (10). The formula required that the set of options open to the angel of chance be such that some lead to a win by the left party and others lead to a win by the right. Figure 4 demonstrates how this may occur, but not that it must always do so. If not, the formula for π becomes inapplicable. Despite the randomness introduced by the angel of chance, the situation may remain essentially as illustrated in figure 1 where n_L is substantially larger than N/2 so that nobody's vote can be pivotal. Though it is probably rare for a political party to have absolutely no chance of winning in a two-party race, the possibility cannot be ruled out altogether.

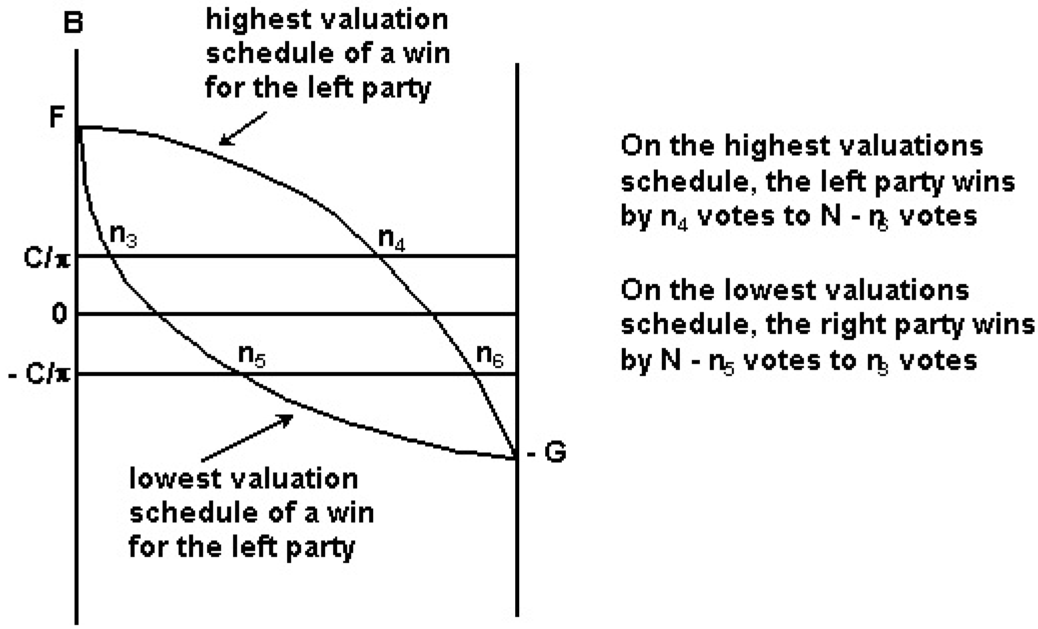
2) Evidence about the Angel of Chance: Nation-wide randomization requires people to know their own values of B without at the same time having more than a general idea of the preferences of the electorate as a whole. Nobody must know which voters' valuations schedule the angel of choice has selected. But if the highest and lowest schedules are linear as in figure 4, such information cannot be completely concealed. In particular, a person who observes his own B to be 120,000 (where the highest schedule cuts the left-hand axis) cannot help knowing the highest schedule has been chosen, and a person who observes his own B to be - 110,000 cannot help but knowing that the lowest schedule has been chosen. In both cases, the person must know for certain whether the left party wins, or the right party wins, or there is a tie. Also, anybody who observes his value of B to lie between 120,000 and 90,000 - the intersection of the highest and lowest schedules with the left-hand vertical axis - must know that the angel of chance has selected a voters' valuations schedule within a restricted portion of the full range of feasible schedules bordering on the highest possible schedule, and anybody who observes his value of B to lie between - 80,000 and -110,000 must know that the scheduled is confined in the opposite direction.

This difficulty is more apparent than real. A way out is to change the postulated shape of the voters' valuations schedule as shown in figure 5 with all possible schedules intersecting at both horizontal axes. The pattern of schedules in figure 5 serves to suppress everybody's knowledge about which schedule the angel of chance has chosen by bunching all schedules together at the maximal and minimal values of B (shown in the figure as F and - G), so that neither the person with the highest B nor the person with the lowest B nor anybody in between can have any idea whatsoever which among the set of feasible voters' valuations schedules the angel of chance has chosen. As in figure 3, everybody's probability of casting a pivotal is determined in accordance with equation (16) above, reflecting the largest and smallest possible numbers of votes for the left party among people whose votes are just marginally pivotal. As in the numerical example, values of the five relevant variables, n_3, n_4, n_5, n_6 and π, are determined simultaneously from equations (16) and (21), once equation (21) is altered to take account of the curvature of the voters' valuations curve.

The important consideration here is that imposition of common upper and lower limits to the voters' valuations schedule is a device imposed to make uncertainty precise. In practice, voters would have no more than a vague idea of the location of the true schedule on election day, with no firm boundary between what may and what may not happen. One's location on the voters' valuations curve may shift up or down over time, so that if you favour the right party and if your valuation of a win for the right party has increased since the last election, you cannot be sure how much of this increase is due to a nation-wide shift in preferences and how much is due to a change in the ranking within the electorate as a whole. Similarly, opinion polls showing an increase in the strength of, say, the left party may change estimates of the probability, P, of a win for the left party, causing parallel shifts in the highest and lowest voters' valuations schedule

without changing the probability, π, of casting a pivotal vote and without affecting your decision to vote or abstain.[10] The variable x in measure,
B(n, x), of the value to person n of a win for the left party might vary in accordance with a normal distribution, with no absolute maximum or minimum, rather than in accordance with a uniform distribution as postulated in equation (18). The uniform distribution is introduced for simplicity. In addition,

Figure 5: Valuation Schedules with Common Maximal and Minimal Values



Even the common value of π is an abstraction. In the numerical example, one's chance of casting a pivotal vote was a property of the electorate as a whole, analogous to the price of an ordinary good or service in a competitive market. As mentioned above, there is a sense in which that must be so. If I believe that my chance of casting a pivotal vote is, say, one-in-a-million, then,

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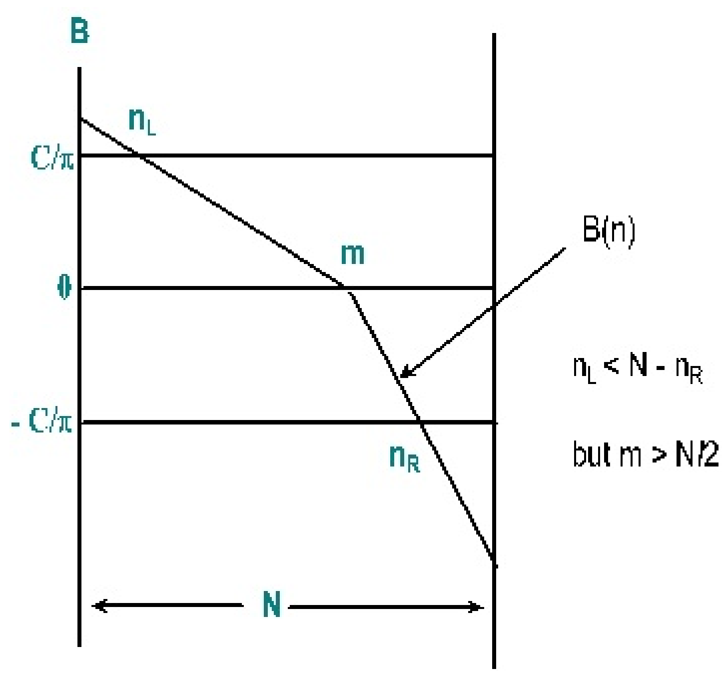
[10]Goeree and Grosser (2007) get a different result from a different model of randomization. In their model, opinion polls are self-defeating. If a poll predicts a win for the left party, then fewer left-leaning people are inclined to vote, more right-leaning people are inclined to vote, and the probability of a win for the left party may decrease. The famous headline "Dewey Defeats Truman" could have turned out to be correct if only the defeat had not been predicted in the polls. Goeree and Grosser's result is obtained within a model where everybody has the same C and the same absolute value of B so that everybody's value of π must be the same as well, forcing voters to randomize - voting with a certain probability and otherwise abstaining - if an equilibrium is to be attained. Evidence from opinion polls can influence the outcomes of an election by causing abstentions of left-leaning and of right-leaning people to change in different proportions.

as a matter of plain logic, I must believe that your chance is one-in-a-million too. But if you and I differ in our knowledge of politics and our assessments of the prospects of competing political parties, it is entirely possible for us to differ in our judgments about the locations of the highest and lowest voters' valuations schedules, or, more generally, in the range of possible voters' preferences and in the expected outcome of the election. This complicates the story of nation-wide randomization, but makes it even more difficult for anybody to draw inferences from his own B about the location of the true voters' valuations schedule as selected by the angel of chance.

On the other hand, the schedules of voters' valuations in figure 5 have one distinct advantage over the linear schedules in equation (18). With linear schedules, the probability of casting pivotal vote is independent of cost of voting, C, and is therefore unaffected by the increase in the number of abstentions as the cost of voting is increased. With curved schedules as illustrated in figure 5, the horizontal distance between the highest and the lowest schedules diminishes steadily as the distance above or below the horizontal axis is increased, automatically increasing every person's probability of casting a pivotal vote. Suppose C is increased. With linear schedules, the number of abstentions increases but the chance of casting a pivotal vote remains the same. With curved schedules as shown in figure 5, the number of abstentions increases but so too does the chance of casting a pivotal vote.

Figure 6: The More Popular Party Loses the Election



<u>3) Bias in Favour of Small Groups with Strong Preferences</u>: A situation can easily arise where a majority of the population prefers one party, but where the other party wins the election because its supporters place higher values on a win for their preferred party and are therefore less likely to abstain. The electoral triumph of a small group with strong preferences is illustrated in figure

6 where the voters' valuations schedule is flatter to the left than to the right. Think of B(n) in figure 6 as the true schedule selected by the angel of chance, and suppose the highest and lowest schedules are sufficiently above and below the true schedule that both left and right parties are seen as having some chance of winning the election, allowing everybody some probability, π, of casting a pivotal vote. If voting were costless, m people would vote for the left party where, as the figure is drawn, $m > N/2$, signifying that more people prefer the left party to the right. But when voting is costly, n_L people vote for the left party, $N - n_R$ people vote for the right party, but $(N - n_R) > n_L$, signifying that the right party wins the election. This is not an implausible situation. Rich people may well place the greater dollar value on a win for their preferred political party and have a greater incentive than poor people to vote rather than abstain. With a voters' valuations schedule as in figure 6, a majority of the population prefers the left party to the right, but a "decisive minority" with a sufficiently strong preference for the right party enables the right party to win the election.[11]

4) Social Welfare: Parallel to the possible discrepancy between numbers of supporters of each party and numbers of votes cast is a second possible discrepancy between number of voters and social welfare. The party with the most supporters need not convey the greatest social welfare. This is illustrated in figure 7 where the voters' valuation schedule is such that more people favour the left party than the right despite the fact that the right party conveys the greater social welfare; $n_L > N - n_R$ despite the fact that $S_R > S_L$. A relatively indifferent majority of the population may prefer that shopping on Sunday be allowed, while a passionate minority prefers that shopping on Sunday be forbidden. The dollar value of the potential loss to the minority may well exceed the dollar value of the potential gain to the majority.

Implicitly, surplus has so far been graduated in dollars to ensure that B and C/π are commensurate. Nevertheless, if one party tends to favour the rich while the other tends to favour the poor and if the disputed public policy is to narrow the distribution of income, then it may be of some interest to measure surplus in utils as well. For any given voters' valuations curve, the surpluses, S_L and S_R, measured in dollars as illustrated in figure 7, are

$$S_L = \int_0^m B(n)dn \qquad\qquad S_R = \int_m^N - B(n)dn \tag{24}$$

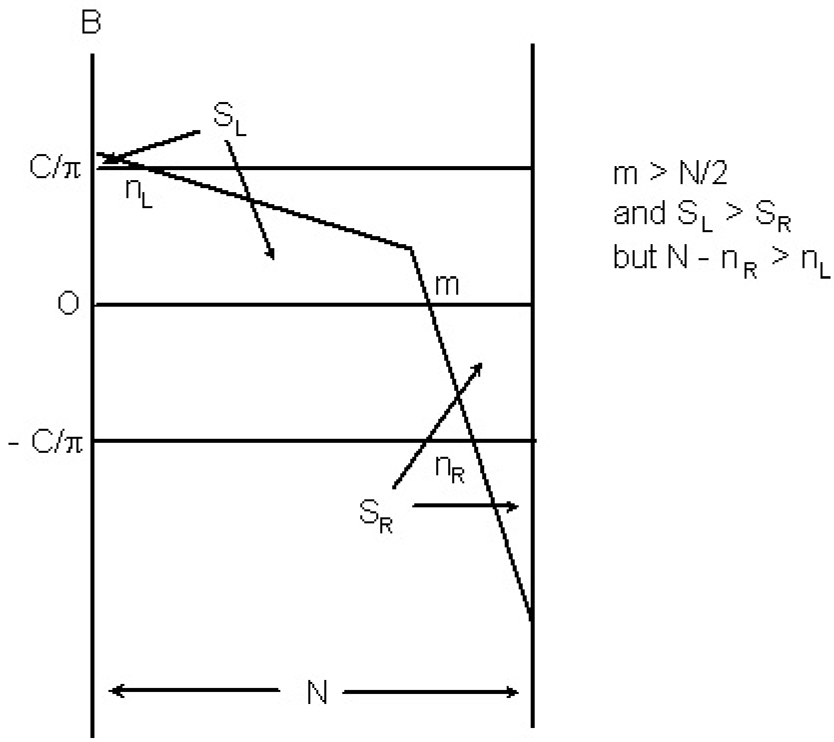
With a common utility of income function, the surpluses, S_L and S_R, can be transformed from dollars to utils. Expressed in utils, the surpluses become

[11] "..for large electorates, no matter how great the proportion of the electorate that prefers a particular outcome, that outcome is likely to lose the election if the opposing outcome is preferred by an expected majority of voters with a sufficiently large incentive to vote", Campbell (1999, 1203).

$$U_L = \int_0^m [u(y^L(n) - u^R(y^R(n)]dn \qquad U_R = \int_m^N [u(y^R(n) - u(y^L(n)]dn \qquad (25)$$

where $y^L(n)$ and $y^R(n)$ are the incomes of person n depending on which party wins the election. Equation (25) is constructed on the assumption that everybody`s utility of income function is the same. Otherwise, each person would have his own version of equation (25).

Figure 7: A Majority of the Population Prefers the Left Party to the Right Party and the Left Party Supplies the Larger Aggregate Surplus, but the Right Party Wins the Election



5) <u>The Political Environment</u>: To focus upon paradox of not voting, much of the context of democratic politics has been assumed away:

i) Party platforms are fixed. There is no discussion or explanation of how political parties go about choosing their platforms or of why the voters' valuations schedule is what it is assumed to be. No account is taken of political parties' incentives to adopt platforms appealing to the largest possible share of the electorate. Between the lines of this paper one may detect that a right party is favoured by the rich and a left party is favoured by the poor, but left and right may have many dimensions, each contributing to the schedule B(n).[12]

ii) There are only two parties, one of which will form the government. A paradox of not voting could be discussed in the framework of proportional representation, but that lead is not followed here.

iii) Nothing is said about the motives, interests or competence of politicians. In particular, no distinction is drawn between support of a political party because of its platform and support because its leaders, once in office, can be expected to serve nation-wide common interests effectively,

iv) Voters are never mistaken in their valuations, B, of their benefits or costs from the election of one party rather than the other. There is no campaign financing or electioneering because there is nothing about which the voter can be persuaded.

v) Eligible voters choose to vote or to abstain individually and self-interestedly, with no sense of duty and no coordination among them. Duty will be discussed presently.

6) Conflicting Interests vs Conflicting Opinions about Common Interests

The distinction between interests and opinions, discussed already in connection with person-by-person randomization, has its counterpart in nation-wide randomization. When valuations reflect *interests* as has been assumed so far, $B(n_\alpha) > B(n_\beta)$ means that person n_α has more to gain than person n_β from a win by the left party. If $B(n_\alpha) = 10$ and $B(n_\beta) = -7$, then everybody expects a win for the left party to make person n_α better off by 10 and to make person n_β worse off by 7 than if the right party had won instead. By contrast, when valuations reflect *opinions* about common interests, everybody expects a win for the left party to convey the same benefit, or cost as the case may be, to everybody, but people disagree about what that common benefit or cost would be. If $B(n_\alpha) = 10$ and $B(n_\beta) = -7$, then person n_α would expect a win for the left party to make everybody better off by 10 and person n_β would expect a win for the left party to make everybody worse off by 7.

When valuations represent opinions rather than interests and when each person is absolutely sure of his judgment, voting becomes an enormous act of charity to one's fellow citizens, not all observant or clever enough to appreciate what the true consequence of the

[12] Parties need brand names, but it is often hard to see why the range of products in each brand is what it happens to be, why particular bundles of policies get lumped together as left or as right. Some policies might be identified as pro-rich or pro-poor, but not all policies can be classified that way. Why, for example, would the party opposed to public provision of health care be opposed to gay marriage as well. Why might one party favour killing of people before they are born, while the other favours killing of people afterward.

election might be.[13]

If people differ in their opinions about common benefits or costs of a win for the left party, it would be odd, though not impossible, for everybody to hold their opinions with certainty. One may be certain about benefits to oneself. It is difficult to be certain about benefits or costs to others when one knows that others, with the same interests as oneself, disagree. People might more appropriately be thought to have probability distributions of outcomes with different means or variances. There may also be a tendency to believe that what is best for oneself is best for society as well.[14]

Surplus becomes ambiguous. With B(n) is interpreted as person n's opinion about a common benefit, the surpluses, S_L and S_R, become meaningless unless it is supposed that each person is equally likely to be correct, converting S_L and S_R into expected benefits as seen by different groups of people, those for whom $B > 0$ and those for whom $B < 0$.

With one exception below, this paper adheres to the assumption that B(n) refers to person n's assessment of his own benefit from a win for the left party rather than to person n's opinion about the benefit to everybody.

7) Constants and Variables: A key assumption in the model of nation-wide randomization as so far described is that the (positive or negative) value, B(n), of a win for left party varies from one person to the next but that the cost of voting. C, is the same for everybody. In support of this assumption, it may be argued that B can reasonably be expected to vary a great deal more than C. A win for the left party may be worth millions of dollars to one person, may be worth virtually nothing to another, and may create a loss of millions of dollars to somebody else, but everybody's cost of voting, the hour or two required going to and coming from the polling booth, is more or less the same. There, nevertheless, is a body of literature that reverses these assumptions, treating the benefit of a win for one's preferred constant and the cost of voting as variable from one person to the next.

The assumption that C is a random variable chosen from a uniform distribution is sometimes coupled with the a special restriction on the voters' valuation curve in figure 1 above. The downward-sloping curve would be replaced by two parallel lines, one above the horizontal

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[13]See Edlin, Gelman, and Kaplan, (2008). If I am one of 10 million an eligible voters, if I am absolutely sure that a win for the right party conveys a benefit of $1,000 to each of my fellow citizens, and, if, by chance, just enough of my fellow citizens mistakenly vote for the left party to create a tie or a win by one vote for the left party, then my vote conveys a benefit of $10 billion to my fellow citizens. If I am at all altruistic, the required probability of my vote being pivotal may be very, very small.

[14]Upton Sinclair said that, "It is difficult to get a man to understand something when his salary depends on his not understanding it."

axis and the other below it, the lower line beginning where the upper one stops Specifically, the assumption is that

$$B(n) = B^L > 0 \text{ for all n from 0 to n*} \qquad (26a)$$

and

$$B(n) = - B^R > 0 \text{ for all n from n* to N} \qquad (26b)$$

where N is the total population of eligible voters. On this assumption, the angel of chance may operate through the choice of n* which then becomes a random variable so that the chance of one's vote becoming pivotal comes to depend on the value of n* that the angel of chance selects.

Other modifications in the assumptions raise the chance of casting a pivotal vote by systematically lowering the number of people who choose to vote. Palfrey and Rosenthal (1985) assume all eligible voters to adopt a mixed strategy, choosing to vote or abstain by the flip of a weighted coin. Left-party weights would differ from right-party weights, but both weights, and the corresponding numbers of people voting for both parties, would be small enough to make voting advantageous in accordance with equation (1) with D = 0. Typically, though not invariably, estimated numbers of voters are very, very small.[15]

Alternatively, equilibrium numbers of voters can be reduced by allowing the cost of voting to vary from zero to some maximal value on the understanding that the distribution of voting cost is the same for supporters of both political parties. Ledyard (1984) develops such a model to explain the convergence of party platforms at the first preference of the median voter when politics is about the selection of some point, z, on a left-right continuum over which the preferences of all voters are single-peaked. The outcome is that a) both parties choose the same value of z, and b) all eligible voters abstain (so that the election is decided by the flip of a coin), but c) the parties' common choice of z (typically the first preference of the median voter) is such that any party deviating from the common value of z would be sure to lose the election because some voters would be roused to vote against it.

Variations in people's cost of voting can be incorporated into the model of nation-wide randomization by changing the interpretation of the vertical axis of figure 3 from B to B/C,

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[15] Multiple equilibria are rare but not impossible. With random voting and with equal numbers of eligible voters to the left and to the right, one's chance of casting a pivotal vote is relatively high when almost everybody votes and when almost everybody abstains, for the chance of an even split is the same among a given number of abstainers as among a given number of voters. In the former case, the number of abstentions is small enough that the number of voters for each party approaches the number of eligible voters who favour it. In the latter case, the number of abstentions is large enough to contract the total number of votes cast to the point where each voter has a significant impact on the outcome of the election. The former case has no counterpart when numbers of eligible voters supporting each party are not the same. See Palfrey and Rosenthal (1985) Figure 1.

automatically converting the heights, above and below the horizontal axis, of the two horizontal lines in the figure from C/π and - C/π to $1/\pi$ and - $1/\pi$ respectively, but leaving the rest of the figure unchanged. Now voting in the hope that one's vote will be pivotal is like investing in a very risky project where B/C is a person's benefit-cost ratio in the event that the project is successful and π is the probability of success dependent on characteristics of the voting market as a whole. An advantage of this version of nation-wide randomization is that any distribution among people of the cost of voting and the benefit of a win by one's preferred party can be accommodated. Among its disadvantages are that areas under the voters' valuations schedule can no longer be interpreted as surpluses and that it becomes difficult even to guess the characteristics of people occupying different parts of the schedule.

8) Partial vs General Equilibrium: All of the voting models discussed in this paper are partial, with nothing said about the content of the platforms of political parties or about how platforms of political parties are chosen. The restriction is comparable to the study of demand curves without studying supply curves as well. It is a legitimate way to proceed, but its limitations should be recognized.

Throughout this paper, as in much of the literature on the paradox of not voting, the benefit of voting to the self-interested voter is looked upon as nothing more than the voter's gain from a win by one party or the other on the assumption that the platforms of the two parties are fixed. With a given value of B, the sole concern of the self-interested voter is whether π is large enough for the absolute value of πB to exceed C in the current election. A pivotal vote for the winning party is beneficial; otherwise, one's vote is useless. That picture is too narrow because, even if not pivotal, one's vote may be beneficial to the voter himself in at least three distinct ways. First, a vote for the losing party in today's election might pave the way for a win in some subsequent election, helping to communicate to the rest of the electorate that the party one votes for is supported by a significant share of the population. The British Labour party hung on for generations before finally attaining office.

Second, dropping the assumption that party platforms are fixed allows for an indirect influence of a person's vote upon the platforms and policies of competing political parties. Even if not pivotal, one's vote can affect the margin of victory of the winning political party, and that in turn can affect the winning party's choice of public policy today, its platform in the next election and the platform of its rival. One may vote for the right party not just in the hope that the right party wins now, but to shift policies of both parties tomorrow. Over time, public opinion may swing one way or another, or voters may become disillusioned with the personnel of the party in office. Facing such risks, politicians seeking to maximize their chances of being elected and re-elected would be inclined to adjust policy and platforms in whatever direction the electorate appears to be moving. Suppose politics is about the choice of x on a left-right scale where a low value of x is left and a high value of x is right. If the right party wins the election, it must choose a policy x today and a platform x for the next election. The right party's choice of x is likely to be larger if its win is by a large margin than if its win is by a narrow margin, and the left party's choice of platform x for the next election would be larger too. (Austin-Smith, 1987). Recognizing this, the voter acquires an incentive to vote over and above the incentive from the chance of casting a pivotal vote. In Canada, the Cooperative Commonwealth Federation never won a federal election but is said to have provoked the ruling Liberal party to adopt many social programs.

Third, one's vote can affect the quality of democracy, increasing the spontaneous turnout and thereby reducing the influence of money, organization and small fanatical groups upon the outcome of elections.

Like the chance of casting a pivotal vote, the magnitude of the influence of one's vote upon outcomes in future elections, the choice of public policy and the quality of democracy are minute. One's personal benefit from these effects of one's decision to vote rather than to abstain are very unlikely to cover any significant portion of the cost of voting. It is worth stating that B in equation (1) comprises more than one's immediate benefit from a win by the party one favours, but the paradox of not voting remains. On the other hand, the corresponding externalities magnify personal benefits by a factor equal to the population as a whole. Voting supplies a public good to like-minded voters or even to the population as a whole. A duty to vote arises from the fact that, unlike roads and guns, these externalities may not be supplied in any other way.

Conclusion

The model of nation-wide randomization differs from the more commonly employed model of person-by-person randomization in several respects.

- Political preferences are monetized. People are not simply partitioned as left-supporters or right supporters, but are assigned dollar values of a win for one party over the other.

- Every eligible voter is assigned a preference on a left-right scale. The further left you are, the larger is your value of a win by the left party.

- Uncertainty in the outcome of elections - without which nobody's vote could be pivotal - is generated by shifts in the entire scale of left-right preferences rather than, as in person-by-person randomization, by the random assignment of people as left or right.

- As long as both parties have some chance of winning the election, the probability of casting a pivotal vote is independent of the parties' probabilities of winning the election.

- People differ in their (positive or negative) benefits of a win for the left party rather than in their cost of voting. Strictly speaking, it would make no difference if the cost of voting varied instead as long as the average cost of voting is allowed to differ between political parties, so that, for example, the cost of voting might be systematically higher for supporters of the left party than for supporters of the right. But this qualification is implausible. In models where cost of voting is variable, it is usually determined in the same random process for supporters of both political parties.

The paradox of not voting becomes less troublesome in one respect but more so in another. It becomes less troublesome than with person-by-person randomization in that microscopically low probabilities of casting a pivotal vote are eliminated within the range of outcomes where both competing parties have some chance, however small, of winning the election. In a sense, the model of nation-wide randomization inherits the variability of the common sense model. In an electorate of a million people where the left party is confidently expected to win between 47% and 51% of the votes, each voter's chance of becoming pivotal is 1-in-8,000 which may still be too small to justify voting on purely selfish grounds, but is astronomically larger than the chance of becoming pivotal under person-by-person randomization when each voter has a 49%, [(.47 + .51)/2], chance of voting for the left party.

The paradox of not voting becomes more troublesome in that nation-wide randomization may give rise to systematic bias - as illustrated in figure 7 - where the party most likely to win the election when everybody votes becomes likely to lose when a a significant portion of the electorate abstains because the expected benefit of voting, πB, falls short of the cost of voting, C. Such bias is particularly likely to emerge when one party is relatively favourable to the rich and the other is relatively favourable to the poor. Portrayal of such bias is, alas, a strength of rather than a weakness of nation-wide randomization because the bias can in practice arise.

The paradox of not voting is by no means eliminated. Neither model of voting explains why more than a tiny and typically unrepresentative minority of eligible voters would choose to vote rather than to abstain if pure self-interest were the citizens' only concern, and there is no guarantee whatsoever that the outcome of the election is as satisfactory for the great majority of citizens as it would be if everybody chose to vote rather than to abstain.

The central proposition in economics is that the outcome in markets with well-established property rights and where everybody does what is best for himself alone is not just determinate, but best for the community as a whole. The proposition is subject to many well-known qualifications and "best for the community" must be understood in a very special sense. Even so, the proposition is at first sight implausible. Only familiarity dulls our surprise that it is in fact true. The central question in the economics of voting is whether there is an analogous proposition when property rights in markets are replaced by voting rights in elections. A strong case can be made that there is not. The outcome when everybody chooses to vote or to abstain from self-interest alone is not the best for the community as a whole. To sustain majority rule voting there must be a willingness of politicians to compromise[16] and a recognition among a sufficient proportion of citizens of a duty to vote.[17]

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[16]See, for example, Usher (2010) and Usher (2011a)

[17] Discussed in Usher (2011b)

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