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"Reverse Bayesianism": A Choice-Based Theory of Growing Awareness

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Abstract

This paper introduces a new approach to modeling the expanding universe of decision makers in the wake of growing awareness, and invokes the axiomatic approach to model the evolution of decision makers’ beliefs as awareness grows. The expanding universe is accompanied by extension of the set of acts, the preference relations over which are linked by a new axiom, invariant risk preferences, asserting that the ranking of lotteries is independent of the set of acts under consideration. The main results are representation theorems and rules for updating beliefs over expanding state spaces and null events that have the flavor of “reverse Bayesianism.”

Keywords: Awareness, unawareness, reverse Bayesianism, null events

JEL classification: D8, D81, D83

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1 Introduction

According to the Bayesian paradigm, as new discoveries are made and new information becomes available, the universe shrinks: With the arrival of new information, events replace the prior universal state space to become the posterior state space, or universe of discourse. This process of "destruction" reflects the impossibility, in the Bayesian framework, of expanding the state space and of updating the probabilities of null events, coupled with the fact that conditioning on new information renders null events that, a-priori, were nonnull. Yet, experience and intuition alike contradict this view of the world. Becoming accustomed to possibilities that were once inconceivable is part of history and our own life experience. There is a sense, therefore, in which our universe expands as we become aware of new opportunities.

In this paper we take a step toward modeling the process of growing awareness and expansion of the universe, or state space, in its wake.[1] To model the evolution of beliefs in response to growing awareness, we invoke the theory of choice under uncertainty; borrowing its language and structure while modifying it to fit our purpose. In particular, we allow for new consequences and feasible acts to be discovered and for new evidence to establish, in the mind of decision makers, new links between acts and consequences. The interpretation of the updating is somewhat different for the discovery of new feasible acts and consequences on the one hand and the discovery of new links between feasible acts and consequences on the other. The discovery of new feasible acts and consequences represents growing awareness and leads to genuine expansion of the decision maker's universe. By contrast, new evidence suggesting the existence of links between feasible acts and consequences that were previously considered conceivable but unfeasible, results in rendering nonnull events that prior to the discovery of the new links were null. This updating of zero probability events is part of the reverse Bayesianism nature of our model.

In this paper, a decision maker's initial perception of the state-space is determined by primitive sets of what he considers to be feasible acts and feasible consequences. The *conceivable state space* consists of all the mappings from the set of feasible acts to that of the consequences.[2] Taking only those mappings from the set of feasible acts to the set

[1] Dekel, Lipman, and Rustichini (1998) argue that standard state spaces preclude unawareness. A choice theoretic approach therefore needs a more general point of departure than Savage (1954) and Anscombe and Aumann (1963).

[2] Here we follow the approach to defining a state space described in Schmeidler and Wakker (1987) and

of consequences that the decision maker actually considers possible, defines a (subjective) *feasible state space*. The discovery of new consequences and/or new feasible acts expands both the conceivable and feasible state spaces, capturing the decision maker's growing awareness. The discovery of new feasible states (that is, new links between feasible acts and consequences) that the decision maker previously believed to be impossible expands the feasible state space but not the conceivable state space. Within this framework, we model the evolution of beliefs in a way that can be described as "reverse Bayesianism."

We assume throughout that, within a given conceivable state space, decision makers' choice behavior is governed by the axioms of subjective expected utility theory. Our axioms linking preferences under different levels of awareness imply that as the state space expands, probability mass is shifted proportionally away from the nonnull events in the prior state space to events created as a result of the expansion of the state space. When new links between feasible acts and consequences are discovered, null events become nonnull, requiring the shifting of probability mass, proportionally, away from the prior nonnull events to the prior null events that have now become nonnull. We note that the same process applies in the inverse direction. The discovery that certain hypotheses about the connections between feasible acts and consequences are invalid render some events null. This requires redistributing the probability mass assigned to prior nonnull events, proportionally, among the remaining nonnull events. This process amounts to Bayesian updating.

Preference relations corresponding to different levels of awareness are defined over different domains. To link the preference relations across their corresponding domains, we introduce a new axiom, dubbed invariant risk preferences, asserting that the ranking of lotteries is independent of the set of acts under consideration.

The main novelties of this paper are the analytical framework within which growing awareness may be formalized and its consequences analyzed, and an axiomatic depiction of the evolution of preferences as the decision maker's awareness grows. The reverse Bayesianism aspect of our approach is driven by axioms that have the flavor of Savage's (1954) sure thing principle.

The systematic evolution of beliefs depicted by our approach, makes it possible to predict, at least partially, the decision maker's behavior when something unforeseen occurs. With the discovery of a contingency that he was unaware of, the decision maker's prior

Karni and Schmeidler (1991).

conception – or "model" – of the universe is falsified. When this happens, the decision maker's prior model need not be discarded; it can still provide some guidance for behavior in the "new" expanded universe. In other words, decision makers can use their experiences and understanding of the prior state space to guide their choices when their growing awareness enables them to construct an expanded state space.

The exploration of the issue of unawareness in the literature has invoked at least three different approaches. (a) the epistemic approach (see Fagin and Halpern [1988], Modica and Rustichini [1999], Halpern [2001], Li [2009], Hill [2010], and Heifetz, Meier, and Schipper [2006]); (b) the game-theoretic, or interactive decision making, approach (see Halpern and Rego [2008], Heifetz, Meier, and Schipper [2011], Grant and Quiggin [2011]); and (c) the choice-theoretic approach (see Kochov [2010], Schipper [2011], Li [2008], Lehrer and Teper [2011], Ahn and Ergin [2010]).

Our approach falls within the third category. However, unlike other studies that take the choice-theoretic approach, we do not take the state space as given.[3] Instead, we construct the relevant state space from the sets of feasible acts and consequences and the perceived links between them. In so doing, we abstract from concrete interpretations of the states and treat them as abstract resolutions of uncertainty. Consequently, decision makers' unawareness concerns feasible acts, feasible consequences, and/or their links.

Kochov (2010) considers a decision maker who knows that his perception of the universe may be incomplete. He characterizes the collection of foreseen events and shows that the result of the decision maker being aware of his incomplete perception of the environment is that his beliefs are represented by a non-singleton set of priors, which he updates as his perception of the environment becomes more precise.

Schipper (2011) focuses on detecting unawareness. Taking as primitive a lattice of disjoint state spaces in the Anscombe and Aumann (1963) model,[4] he defines acts as

[3] In the epistemic approach it is possible to construct, canonically, the state space from syntax (e.g., Halpern and Rego [2008], Heifetz, Meier, and Schipper [2008]). This allows a more direct comparison between the epistemic approach and the decision-theoretic approach pursued in this paper. In particular, the two approaches attribute unawareness to the limitation of language rather than to the power of reasoning. The choice theoretic approach is a special case of the epistemic approach in that its depiction of the state space does not contain anything related to the epistemic status of the decision maker.

[4] A lattice in Schipper's framework could be constructed from primitives of the present model as follows: Fix finite sets of acts and consequences and consider the power sets corresponding to these sets. For each subset of acts and consequences, define a conceivable state space, as we do below. The set of all conceivable

mappings from the union of these state spaces to the set of consequences. Thus an act in Schipper's model corresponds to an equivalence class of acts in our model. Another difference between Schipper's model and the approach taken here is that he defines the preference relation on the set of all acts while we define the preference relation on the set of conceivable acts given a state space. Consequently, unlike in this work, in Schipper (2011) the decision maker may not understand how an act assigns consequences to states (because he may be unaware of some event). These differences reflect diverse motivations. While our main interest is modeling growing awareness, Schipper's main interest is the behavioral implications of unawareness of some events.

Li (2008) takes as primitives a fixed set of consequences and two, exogenously given, state spaces that correspond to a decision maker being less than fully aware and fully aware. Decision makers are characterized by preference relations, conditional on the level of awareness, over Anscombe-Aumann acts on the corresponding state spaces. Li considers two types of unawareness: "pure unawareness," depicting situations in which the decision maker's perception of the environment is coarse, and "partial unawareness," depicting situations in which the decision maker's perception of the universe is a subset of the full state space. Partial unawareness has a flavor of unawareness of consequences or links between acts and consequences. However, since Li uses a traditional approach in taking her two state spaces as exogenously given, unawareness is directly about states, and thus her model is silent on how the discovery of new consequences or new scientific links would translate into the evolution of the state space and into behavior. Also, her model is unable to accommodate the discovery of new acts.

Ahn and Ergin (2010) introduce a theory of decision making under uncertainty designed to capture the idea that the evaluation of acts may depend on the manner in which the underlying events, or contingencies, are described. Formally speaking, these descriptions, or frames, are finite partitions of the state space, and the subjective likelihoods of events are quantified by a partition-dependent probability measure which is the normalized nonadditive set function over events. Acts are functions measurable with respect to the algebras generated by the partitions. Decision makers in this model are characterized by a family of partition-dependent preference relations representable by expected utility functionals. This analytical framework accommodates what Ahn and Ergin refer to as over-

state spaces thus defined, constitute a complete lattice of spaces with partial order defined by set inclusion on acts and consequences.

looked events. Growing alertness in their model takes the form of refining the partitions so that overlooked events become cells of the new partitions.

As will become clear below, the approach of Ahn and Ergin is different from the approach in this paper in several important respects. First, their model is not designed to, and therefore, cannot accommodate, the discovery of new consequences and associated expansion of the state space following such discoveries. Second, their model is neither designed to nor can it accommodate the discovery of new links between acts and consequences and the updating of the probabilities of null events. Where the two approaches display similar features is in the case in which awareness grows in the wake of discovery of new acts. In our model this implies that what was considered a state at a lower level of awareness becomes an event at the higher level of awareness. Consequently, the higher level of awareness requires the partition of such events. In the work of Ahn and Ergin the refined partition of the state space expands the set of acts. In their model, however, the nonaditivity of the support function implies that the sum of the probabilities of the subevents may exceed (fall short of) that of the original event. By contrast, in the model presented here, the probabilities of the subevents of the finer partition sum up to the probability of the original event. Since the relative probabilities of the cells of the original partition change, and as a result the preferences among acts that are measurable with respects to these partitions change, this difference implies distinct behavioral implications of the two models.

Lehrer and Teper (2011) model growing awareness due to the decision maker's improved ability, in the wake of information acquisition, to distinguish among events. As in Ahn and Ergin (2010), the state space in the model of Lehrer and Teper takes the form of finer partitions of the existing state space. Thus, most of the discussion in the previous paragraph applies to Lehrer and Teper as well, with the modification that their decision maker has Knightian preferences on the expanded set of acts.

2 The Meanings of Growing Awareness

The examples below illustrate the sense in which a decision maker's universe expands in the wake of his growing awareness.

2.1 Discovery of new consequences

The discovery of the New World. Columbus set out to discover a new sea route to India, presumably taking into account consequences such as ending the trip at the bottom of the ocean, having to turn back, losing some ships and crew members, reaching India, etc. He could not have included, among the set of consequences, the discovery of a new continent. This discovery expanded the universe for mankind.

The discovery of syphilis. The discovery of the New World ushered in its wake a new consequence of sexual intercourse. Presumably, sexual intercourse and the risk of contracting venereal diseases were well known in the Old World. Syphilis, however, was a new consequence whose discovery expanded the universe of the Europeans.

Discovery of a "new" consequence expands the state space and may affect the decision maker's ordinal preferences over acts. In other words, two acts that agree on the "old" state space may become distinct when associated with new consequences; as a result, one of the newly defined acts may be strictly preferred over the other.

2.2 Discovery of new feasible acts

Artificial self-sustaining nuclear chain reaction. After the discovery of nuclear fission, Szilárd and Fermi discovered neutron multiplication in uranium, proving that a nuclear chain reaction by this mechanism was possible. On December 2, 1942, Fermi created the first artificial self-sustaining nuclear chain reaction, thus making it feasible to use nuclear energy, for peaceful and military purposes. The consequences, such as the use of energy to produce electricity or explosions, were known long before the scientists became aware of the new possibilities of producing these consequences by means of nuclear fission.

The invention of sound recordings. By making it possible to preserve sounds, the invention of sound recording devices expanded the state space to include future replays of currently produced sounds.

The invention of new financial instruments. The invention of option trading opened up new possibilities of creating portfolios and diversifying risks. Again, the consequences, monetary gains and losses, were there before the invention. The new financial instruments represent new processes of attaining the same consequences.

2.3 Discovery of new links and changing beliefs

Yellow fever. To prevent ants from crawling into hospitals' beds, French doctors working in Panama during the French attempt to build the Panama Canal, placed the legs of the beds in bowls of water. These pools of water provided breeding grounds for the mosquitoes carrying yellow fever. Not being aware of the way the yellow fever was transmitted, the French did not conceive that their actions contributed to the propagation of the disease. Later, when the connection between stagnant water, mosquitoes, and yellow fever was understood, the Americans were able to eradicate yellow fever, eliminating a major stumbling point to the construction of the Panama Canal.

The velocity of light. According to Newton's mechanics the speed of light emitted by a flashlight moving in the forward direction should exceed the known speed of light by the speed of the flashlight. The discovery that, despite the expected boost from being emitted by a very fast source, the light is going forward at the usual speed of 186,300 miles per second, ushered in a revision of our understanding of the physical world. If we interpret the emission of light by a flashlight moving in the forward direction as an act, then the consequence, reaching a target in the direction of the movement at the speed of 186,300 miles per second, was not considered possible according to the Newtonian view of the universe. Establishing that this is not only possible but a necessary outcome, led to a revision of our beliefs about the feasible state space.

3 The Analytical Framework

We introduce a unifying framework within which the different sources of growing awareness and changing beliefs may be described and analyzed. We also illustrate how the different notions of growing awareness can be formalized in this framework.

3.1 Conceivable state spaces

States of nature, or *states* for short, are abstract representations of resolutions of uncertainty. To define the state space, we invoke the approach of Schmeidler and Wakker (1987) and Karni and Schmeidler (1991).[5] According to this approach, there is a finite, nonempty

[5] See also Gilboa (2009, Chapter 11) for a detailed discussion and an ingenious use of this approach to formulating the state space as means of resolving Newcomb's paradox.

set, F, of *feasible acts*, and a finite, nonempty set, C, of *feasible consequences.* Together these sets determine a *conceivable state space*, C^F, whose elements depict the resolutions of uncertainty. In other words, being a function on the set of feasible acts to the set of consequences, a state specifies the unique consequence that is associated with every act, thereby resolving all uncertainty.

Once the set of conceivable states is fixed, the set of acts is expanded to include what we refer to as *conceivable acts.* The notion of conceivable acts captures the idea of acts that are imaginable given the conceivable state space. In particular, we assume that the decision maker can imagine acts whose outcomes are lotteries with consequences in C as prizes. Let $\Delta(C)$ be the set of all such lotteries. Formally, $p \in \Delta(C)$ is a function $p : C \to [0,1]$ satisfying $\Sigma_{c \in C} p(c) = 1$. Then the set of conceivable acts consists of the functions in the set

$$\hat{F} := \{f : C^F \to \Delta(C)\}. \tag{1}$$

Notice that with the definition of $\Delta(C)$ above, for any $C \subset C'$, any $p \in \Delta(C)$ is also an element of $\Delta(C')$ with $p(c) = 0$ for all $c \in C' - C$. Likewise, $q \in \Delta(C')$ is an element of $\Delta(C)$ if $q(c) = 0$ for all $c \in C' - C$. We identify $c \in C$ with the degenerate lottery $\delta_c \in \Delta(C)$ that assigns c the unit probability mass. Hence, $F \subset \hat{F}$. As is usually done, we abuse notation and use p to also denote the constant act that returns the lottery p in each state.

To illustrate these concepts we introduce the following simple example. Let $C = \{x, y\}$, $F = \{f_1, f_2\}$, then the conceivable state space consists of four states as described below:

$F \backslash C^F$	s_1	s_2	s_3	s_4
f_1	x	x	y	y
f_2	x	y	x	y

Once the set of conceivable states is fixed, the set of acts may be expanded. For example, by adding the constant acts, f_3 and f_4, whose respective payoffs are x and y in every state. More generally, conceivable acts may be regarded as bets on the outcomes of the feasible acts whose payoffs are lotteries on the set of feasible consequences. Karni and Schmeidler (1991) observe that "In practice, the distinction between feasible and conceivable acts is not always crucial, and in many applications the sets of states and consequences are taken as primitives." (p. 1766). In the present context, however, the distinction between feasible and conceivable acts is crucial. It is the set of feasible acts, together with the feasible consequences that shape the decision maker's image of the state space.

Discovery of new consequences expand the conceivable state space. For instance, let C denote the initial set of consequences and suppose that a new consequence, $\bar{c}$, is discovered. The set of consequences of which the decision maker is aware then expands to $C' = C \cup \{\bar{c}\}$. The discovery of $\bar{c}$ requires a reformulation of the initial model, incorporating the new consequence into the range of the feasible acts. We denote the set of feasible acts with extended range by F^*. Using these notations, the expanded conceivable state space is $(C')^{F^*}$. The event $(C')^{F^*} - C^F$ represents the expansion of the decision maker's conceivable state space. The corresponding expanded set of conceivable acts is given by,

$$\hat{F}^* := \{f : (C')^{F^*} \to \Delta(C')\}. \tag{2}$$

As an illustration, let there be two feasible acts, $F = \{f_1, f_2\}$, and two consequences, $C = \{c_1, c_2\}$. The resulting conceivable state space is given by C^F and thus consists of four states as depicted in the following matrix:

$$\begin{array}{ccccc} F \backslash C^F & s_1 & s_2 & s_3 & s_4 \\ f_1 & c_1 & c_2 & c_1 & c_2 \\ f_2 & c_1 & c_1 & c_2 & c_2 \end{array} \tag{3}$$

Suppose that a new consequence, c_3, is discovered. The new conceivable state space consists of the 9 states in the set $(C')^{F^*}$:

$$\begin{array}{cccccccccc} F^* \backslash (C')^{F^*} & s_1 & s_2 & s_3 & s_4 & s_5 & s_6 & s_7 & s_8 & s_9 \\ f_1 & c_1 & c_2 & c_1 & c_2 & c_3 & c_3 & c_1 & c_2 & c_3 \\ f_2 & c_1 & c_1 & c_2 & c_2 & c_1 & c_2 & c_3 & c_3 & c_3 \end{array}$$

Discovery of new feasible acts also expands the conceivable state space, albeit in a different way. To grasp the difference, assume again that $F = \{f_1, f_2\}$ and $C = \{c_1, c_2\}$. The resulting conceivable state space is depicted in the matrix labeled (3). Suppose that a new feasible act, say f_3, is discovered. The set of feasible acts is now $F' = \{f_1, f_2, f_3\}$. The discovery of the new feasible act changes the decision maker's conceivable state space.[6] The new conceivable state space, $C^{F'}$, consists of the eight states depicted as follows

[6] Unlike in this paper, in which the state space expands and is partitioned more finely as a result of the discovery of new acts, in Ahn and Ergin (2010) new acts are defined as a result of a finer partition of an

$F'\backslash C^{F'}$	s_1	s_2	s_3	s_4	s_5	s_6	s_7	s_8
f_1	c_1	c_2	c_1	c_2	c_1	c_2	c_1	c_2
f_2	c_1	c_1	c_2	c_2	c_1	c_1	c_2	c_2
f_3	c_1	c_1	c_1	c_1	c_2	c_2	c_2	c_2

The elements of the expanded state space $C^{F'}$ constitute a finer partition of the original state space C^F. In other words, each state in C^F is a non-degenerate event in the expanded state space $C^{F'}$. For example, the state $s_1 := (c_1, c_1) \in C^F$ is the event $E = \{s_1, s_5\}$ in the state space $C^{F'}$. The new set of conceivable acts is

$$\hat{F}' := \{f : C^{F'} \to \Delta(C)\}.$$

Note that, unlike the discovery of new consequences, the discovery of new acts requires that the length of the vector of consequences defining each state increases. As we show later, this aspect of the evolving state space requires special treatment.

3.2 The feasible state space

The decision maker's perception of the state space is bounded by his awareness of the sets of feasible acts and consequences. However, he also entertains beliefs about the possible links between feasible acts and their potential consequences. These beliefs manifest themselves in, and may be inferred from, the decision makers' choice behavior.

To formalize this idea we consider a decision maker whose choice behavior is characterized by a preference relation $\succcurlyeq_{\hat{F}}$ on $\hat{F}$.[7] We denote by $\succ_{\hat{F}}$ and $\sim_{\hat{F}}$ the asymmetric and symmetric parts of $\succcurlyeq_{\hat{F}}$, with the interpretations of strict preference and indifference, respectively. For any $f \in \hat{F}$, $p \in \Delta(C)$, and $s \in C^F$, let $f_{-s}p$ be the act in $\hat{F}$ obtained from f by replacing its $s-th$ coordinate with p. A state $s \in C^F$ is said to be *null* if $f_{-s}p \sim_{\hat{F}} f_{-s}q$ for all $p, q \in \Delta(C)$. A state is said to be *nonnull* if it is not null. Denote by E^N the set of null states and let $S(F, C) = C^F - E^N$ be the set of all nonnull states. Henceforth we refer to $S(F, C)$ as the *feasible state space*. Consistently with the revealed-preference approach,

existing state space. These acts represent growing alertness to possibilities that were always present and were overlooked. The uniqueness result (Theorem 2) of Ahn and Ergin depends on the notion of gradual filtration. This rules out the discovery of new acts which, as we shall show below, implies a simultaneous partition of all events.

[7] More formally, the preference relation is a binary relation on $\hat{F}$.

we make no distinction between feasible states and nonnull states. "Unawareness" of links between feasible acts and consequences, according to our interpretation, means awareness that such links are conceivable but believed to be impossible.

New information (e.g., scientific evidence, an observation) may change the decision maker's beliefs concerning the links between feasible acts and consequences and his perception of the feasible state space. Unlike the discovery of new feasible consequences and/or new feasible acts, which expands both the set of conceivable and the set of feasible states, changes of the decision maker's beliefs concerning the links between them changes the set of feasible states but leaves the set of conceivable states intact. Consequently, the discovery of new feasible consequences and/or new feasible acts represent a genuine expansion of the decision maker's universe (that is, his perception of the state space), while new information concerning the links between feasible acts and consequences entails expansion or contraction of the feasible state space. This may require the updating of zero probability events in the existing conceivable state space that, under the new information, might occur, or nullifying positive probability events that are considered no longer possible. When new links become possible, the decision maker includes the consequences $f(s)$, for all $f \in F$ and some $s \in C^F - S(F,C)$, in the ranges he considers possible of the feasible acts. Vice versa when old links are eliminated. We denote the newly defined feasible state space incorporating the new links by $S'(F,C)$, the corresponding set of conceivable acts by $\hat{F}_{S'}$, and the decision maker's posterior preference relation by $\succcurlyeq_{\hat{F}_{S'}}$.[8]

To see how changes in the perceived links change the feasible state space, consider the case in which there are two feasible acts, $F = \{f_1, f_2\}$ and two consequences, $C = \{c_1, c_2\}$. The conceivable state space C^F consists of four states as in matrix (3). If the decision maker does not believe that the act f_2 may result in the consequence c_2, (that is, whether f_1 results in c_1 or c_2, the consequence c_2 is considered impossible if f_2 is chosen) then the states (c_2, c_2) and (c_1, c_2) are null. In other words, the feasible state space is $S(F,C) = \{(c_1, c_1), (c_2, c_1)\}$. Suppose that new evidence establishes that f_2 may result in c_2, (independently of the consequence that is associated with the choice of f_1) and, as a result, the decision maker realizes that his belief that certain states cannot possibly obtain is untenable. Then, following the discovery of the new (and final) link, the feasible and conceivable state spaces coincide (that is, $S'(F,C) = C^F$). By the same logic, the discovery that a link that the

[8] In fact, the set of conceivable acts is unchanged, but the notation is used to index the preference relation.

decision maker believed possible is, in fact, impossible, results in rendering null an event that was considered to be nonnull before the discovery.

What is a reasonable updating rule for probabilities of events that were considered impossible (null) and, as a result of scientific progress and growing understanding of the structure of the universe, become possible (nonnull)? Clearly, the Bayesian approach is useless for this purpose. Here we explore an alternative approach.

To discuss the various types of unawareness with which we are concerned, we use the following notational convention throughout. We denote by F and C, respectively, the initial sets of feasible acts and consequences, and we let $S(F, C)$ denote the corresponding feasible state space. When new elements are introduced into each of these sets we denote the corresponding new sets by F' and C'. When new acts are discovered, the new feasible state space is denoted by $S(F', C)$. When new consequences are discovered, the new feasible state space is denoted by $S(F^*, C')$, where the asterisk indicates that the ranges of the feasible acts now include the new consequence.

4 Growing Awareness and Choice Behavior

The discovery of new feasible consequences and acts expands the decision maker's perception of the state space and its structure. The discovery of new links between acts and consequences expands what he considers to be the feasible state space. How does the decision maker's growing awareness manifest itself in his choice behavior? In this section we address this question.

4.1 Basic preference structure

Decision makers in our model are supposed to be able to express preferences among conceivable acts. Formally, let $\mathcal{F}$ be a family of sets of conceivable acts corresponding to increasing levels of awareness from all sources (that is, from the discovery of new feasible acts, consequences, and links between them). Because the set of conceivable acts is a variable in our model, we denote the preference relation on $\hat{F}$ by $\succcurlyeq_{\hat{F}}$. When the state space expands, so does the set of conceivable acts, which means that the preference relations must be redefined on the extended domain. For instance, if $\hat{F}^*$ is the expanded set of conceivable acts in the wake of discoveries of new feasible consequences, then the corresponding preference relation is denoted by $\succcurlyeq_{\hat{F}^*}$. If the state space is expanded in the wake of the discovery of new feasible acts, then the new set of conceivable acts is denoted by $\hat{F}'$ and the expanded preference relation by $\succcurlyeq_{\hat{F}'}$.

For each $\hat{F} \in \mathcal{F}$, $f, g \in \hat{F}$, and $\alpha \in [0,1]$ define the convex combination $\alpha f + (1-\alpha) g \in \hat{F}$ by: $(\alpha f + (1-\alpha) g)(s) = \alpha f(s) + (1-\alpha) g(s)$, for all $s \in C^F$. Then, $\hat{F}$ is a convex subset in a linear space.[9]

We assume that, for each $\hat{F} \in \mathcal{F}$, $\succcurlyeq_{\hat{F}}$ abides by the axioms of expected utility theory. Formally,

(A.1) (**Weak order**) For all $\hat{F} \in \mathcal{F}$, the preference relation $\succcurlyeq_{\hat{F}}$ is transitive and complete.

(A.2) (**Archimedean**) For all $\hat{F} \in \mathcal{F}$ and $f, g, h \in \hat{F}$, if $f \succ_{\hat{F}} g$ and $g \succ_{\hat{F}} h$ then $\alpha f + (1-\alpha) h \succ_{\hat{F}} g$ and $g \succ_{\hat{F}} \beta f + (1-\beta) h$, for some $\alpha, \beta \in (0,1)$.

(A.3) (**Independence**) For all $\hat{F} \in \mathcal{F}$, $f, g, h \in \hat{F}$, and $\alpha \in (0,1]$, $f \succcurlyeq_{\hat{F}} g$ if and only if $\alpha f + (1-\alpha) h \succcurlyeq_{\hat{F}} \alpha g + (1-\alpha) h$.

In addition we suppose that, for each $\hat{F} \in \mathcal{F}$, $\succcurlyeq_{\hat{F}}$ abides by the following axioms. The first axiom, monotonicity, is analogous to Savage's (1954) postulate P3.[10]

(A.4) (**Monotonicity**) For all $\hat{F} \in \mathcal{F}$, $f \in \hat{F}$, $p, q \in \Delta(C)$ and nonnull event $E \subseteq C^F$, $f_{-E}p \succcurlyeq_{\hat{F}} f_{-E}q$ if and only if $p \succsim_{\hat{F}} q$.

(A.5) (**Nontriviality**) For all $\hat{F} \in \mathcal{F}$, $\succ_{\hat{F}} \neq \varnothing$.

4.2 Invariant Risk Preferences

To link the preference relations across expanding sets of conceivable acts, we introduce a new axiom, which we refer to as invariant risk preferences. The essence of this axiom is that the ranking of constant acts, which capture the decision maker's risk preferences, is independent of the set of acts on which the preference relation is defined. The axiom delivers the commonality of risk attitudes regardless of whether the conceivable and/or

[9] Throughout this paper we use Fishburn's (1970) formulation of Anscombe and Aumann (1963). According to this formulation, mixed acts, (that is, $\alpha f + (1-\alpha) g$) are, by definition, conceivable acts.

[10] It is well known that, under the axioms (A.1) -(A.3) and (A.5), monotonicity is equivalent to Anscombe and Aumann's (1963) state-independence axiom.

feasible state space expands as a result of growing awareness of consequences, acts, or links between them.

(A.6) (**Invariant risk preferences**) For every given $\hat{F}, \hat{F}' \in \mathcal{F}$, if C and C' are the sets of consequences associated with $\hat{F}$ and $\hat{F}'$, respectively, then $p \succcurlyeq_{\hat{F}} q$ if and only if $p \succcurlyeq_{\hat{F}'} q$ for all $p, q \in \Delta(C \cap C')$.

When new consequences are discovered, $C \subset C'$, then $C \cap C' = C$. When new feasible acts are discovered, the invariant risk preferences axiom may be stated as follows: For all F, F' and $p, q \in \Delta(C)$, $p \succcurlyeq_{\hat{F}} q$ if and only if $p \succcurlyeq_{\hat{F}'} q$. When new links are discovered (or old links eliminated) between the original sets of acts, F, and consequences, C, the invariant risk preferences axiom asserts that, for all $p, q \in \Delta(C)$, $p \succcurlyeq_{\hat{F}} q$ if and only if $p \succcurlyeq_{\hat{F}_{S'}} q$.

5 The Main Results

The analysis of the effects of growing awareness on choice behavior and the evolution of decision makers' beliefs is divided into three parts. In the first part, we explore the implications of the discovery of new consequences. In the second part we explore the implications of the discovery of new feasible acts. In the third part, we explore the implications of the discovery of new acts-consequences links. The discovery of new consequences increases the number of conceivable and, in general, also of feasible states, but the "dimension" of each state is unchanged. By contrast, the discovery of new feasible acts increases the number of both conceivable and feasible states and, at the same time, changes the characterization of each state in such a way that what used to be a state before the discovery of the new act is an event in the reconstructed state space following the discovery. The discovery of new acts-consequences links increases the set of feasible states without affecting the conceivable state space.

5.1 Discovery of new consequences and its representation

The following axiom requires that, as the decision maker's awareness of consequences grows and his state space expands, his preference relation conditional on the prior state space remains unchanged. In other words, the discovery of new consequences does not alter the preference relation conditional on the original set of feasible states.[11] To formalize this idea, let $C' \supset C$, F^*, and $S(F^*, C')$ denote, respectively, the new set of consequences, the new set of feasible acts redefined to accommodate the new consequences, and the resulting new feasible state space.[12]

(A.7) (**Awareness consistency**) For every given F, for all C, C' with $C \subset C'$, $S(F,C) \subseteq S(F^*, C')$, $f, g \in \hat{F}$, and $f', g' \in \hat{F}^*$, such that $f' = f$ and $g' = g$ on $S(F,C)$ and $f' = g'$ on $S(F^*, C') - S(F,C)$ it holds that $f \succcurlyeq_{\hat{F}} g$ if and only if $f' \succcurlyeq_{\hat{F}^*} g'$.

Our first result describes the evolution of a decision maker's beliefs in the wake of discoveries of new consequences. Specifically, a decision maker whose preferences have the structure depicted by the axioms above is a subjective expected utility maximizer. Moreover, as he becomes aware of new consequences, the decision maker updates his beliefs in such a way that likelihood ratios of events in the original state space remain intact. That is to say, probability mass is shifted away from states in the prior state space to the posterior state space, proportionally. We refer to this property as "reverse Bayesianism."

Theorem 1 For each $\hat{F} \in \mathcal{F}$, let $\succcurlyeq_{\hat{F}}$ be a binary relation on $\hat{F}$ then, for all $\hat{F}, \hat{F}^* \in \mathcal{F}$, the following two conditions are equivalent:

(i) Each $\succcurlyeq_{\hat{F}}$ satisfies (A.1) - (A.5) and, jointly, $\succcurlyeq_{\hat{F}}$ and $\succcurlyeq_{\hat{F}^*}$ satisfy (A.6) and (A.7).

(ii) There exist real-valued, non-constant, affine functions, U on $\Delta(C)$ and U^* on $\Delta(C')$, and for any two $\hat{F}, \hat{F}^* \in \mathcal{F}$, there are probability measures, $\pi_{\hat{F}}$ on C^F and $\pi_{\hat{F}^*}$ on $(C')^{F^*}$, such that for all $f, g \in \hat{F}$,

$$f \succcurlyeq_{\hat{F}} g \Leftrightarrow \sum_{s \in C^F} U(f(s))\, \pi_{\hat{F}}(s) \geq \sum_{s \in C^F} U(g(s))\, \pi_{\hat{F}}(s). \tag{4}$$

and, for all $f', g' \in \hat{F}^*$,

$$f' \succcurlyeq_{\hat{F}^*} g' \Leftrightarrow \sum_{s \in (C')^{F^*}} U^*\left(f'(s)\right) \pi_{\hat{F}^*}(s) \geq \sum_{s \in (C')^{F^*}} U^*\left(g'(s)\right) \pi_{\hat{F}^*}(s). \tag{5}$$

[11]This axiom is reminiscent of Savage's (1954) sure thing principle in that it requires that preference between acts be independent of the aspects on which they agree.

[12]Below, $f' = f$ on an event E means that $f'(s) = f(s)$ for all $s \in E$ (i.e. it is defined pointwise for the states in E). Also, recall that for any $C \subset C'$, any $p \in \Delta(C)$ is also an element of $\Delta(C')$.

Moreover, U and U^* are unique up to positive linear transformations, $U(p) = U^*(p)$ for all $p \in \Delta(C)$, $\pi_{\hat{F}}$ and $\pi_{\hat{F}^*}$ are unique, $\pi_{\hat{F}}(S(F,C)) = \pi_{\hat{F}^*}(S(F^*,C')) = 1$, and

$$\frac{\pi_{\hat{F}}(s)}{\pi_{\hat{F}}(s')} = \frac{\pi_{\hat{F}^*}(s)}{\pi_{\hat{F}^*}(s')}, \tag{6}$$

for all $s, s' \in S(F,C)$.

The proof is in the Appendix.

It follows from the result in Theorem 1 that U^*, the von Neumann-Morgenstern utility function associated with the larger set of consequences C', is an extension of U, the von Neumann-Morgenstern utility function associated with the smaller set of consequences C. This is an implication of the invariant risk preferences Axiom (A.6).

5.2 Discovery of new feasible acts and its representation

Recall that the introduction of new feasible acts increases the number of conceivable states as well as the number of coordinates defining a state. Hence, the newly defined states constitute a finer partition of the original state space. To state the next axiom, which is analogous to axiom (A.7), we introduce the following additional notations: If $F \subset F'$ then $C^F \cap C^{F'} = \varnothing$, and for each $s \in C^F$ there corresponds an event $E(s) \subset C^{F'}$ defined by $E(s) = \{s' \in C^{F'} \mid \boldsymbol{P}_{C^F}(s') = s\}$, where $\boldsymbol{P}_{C^F}(\cdot)$ is the projection of $C^{F'}$ on C^F.[13] For $s \in C^F$, we refer to the set $E(s)$ as the inverse image of s on $C^{F'}$. For all $f \in \hat{F}'$, $p \in \Delta(C)$ and $s \in C^F$, define the act $f_{-E(s)}p$ by $\left(f_{-E(s)}p\right)(s') = p$ for all $s' \in E(s)$ and $\left(f_{-E(s)}p\right)(s') = f(s')$ for all $s' \in C^{F'} - E(s)$.

Projection consistency requires that if two acts on the original state space disagree on two states, then the preference ranking of these acts is the same as that of two acts that disagree, in the same way, on the corresponding events in the expanded state space.

(A.8) (**Projection consistency**) For every given C, for all F, F' such that $F \subset F'$, $p, q, \bar{p}, \bar{q} \in \Delta(C)$, $h \in \hat{F}$, $h' \in \hat{F}'$, and $s, s' \in S(F,C)$, it holds that $\left((h_{-s}p)_{-s'}\bar{p}\right) \succcurlyeq_{\hat{F}} \left((h_{-s}q)_{-s'}\bar{q}\right)$ if and only if $\left(\left(h'_{-E(s)}p\right)_{-E(s')}\bar{p}\right) \succcurlyeq_{\hat{F}'} \left(\left(h'_{-E(s)}q\right)_{-E(s')}\bar{q}\right)$.

The representation theorem below describes how a decision maker's beliefs evolve as he becomes aware of new feasible acts. As before, the decision maker is a subjective expected

[13] Suppose that $\mid F \mid = r$ and $\mid F' \mid = k > r$. Let $s = (c_1, ..., c_k) \in C^{F'}$, then $\boldsymbol{P}_{C^F}(s) = (c_1, ..., c_r) \in C^F$. Analogous projections across spaces appear in Modica (2008) and Heifetz, Meier and Schipper (2012).

utility maximizer. When he becomes aware of new feasible acts, the decision maker updates his beliefs in a way that the likelihood ratios of events in the original state space remain intact. Because of the difference in the evolution of the state space, probability mass is shifted from states in the prior state space to the corresponding events the posterior state space, in such a way as to preserve the likelihood ratios of the events in the posterior state space and their corresponding projected states in the prior state space.[14]

Theorem 2 For each $\hat{F} \in \mathcal{F}$, let $\succcurlyeq_{\hat{F}}$ be a binary relation on $\hat{F}$. Then for all $\hat{F}, \hat{F}' \in \mathcal{F}$, the following two conditions are equivalent:

(i) Each $\succcurlyeq_{\hat{F}}$ satisfies (A.1) - (A.5) and, jointly, $\succcurlyeq_{\hat{F}}$ and $\succcurlyeq_{\hat{F}'}$ satisfy (A.6) and (A.8).

(ii) There exists a real-valued, non-constant, affine function, U on $\Delta(C)$ and, for any two $\hat{F}, \hat{F}' \in \mathcal{F}$, there are probability measures, $\pi_{\hat{F}}$ on C^F and $\pi_{\hat{F}'}$ on $C^{F'}$, such that for all $f, g \in \hat{F}$,

$$f \succcurlyeq_{\hat{F}} g \Leftrightarrow \sum_{s \in C^F} U\left(f\left(s\right)\right) \pi_{\hat{F}}\left(s\right) \geq \sum_{s \in C^F} U\left(f\left(s\right)\right) \pi_{\hat{F}}\left(s\right), \tag{7}$$

and, for all $f', g' \in \hat{F}'$,

$$f' \succcurlyeq_{\hat{F}'} g' \Leftrightarrow \sum_{s \in C^{F'}} U\left(f'\left(s\right)\right) \pi_{\hat{F}'}\left(s\right) \geq \sum_{s \in C^{F'}} U\left(g'\left(s\right)\right) \pi_{\hat{F}'}\left(s\right). \tag{8}$$

Moreover, U is unique up to positive linear transformations, $\pi_{\hat{F}}$ and $\pi_{\hat{F}'}$ are unique, $\pi_{\hat{F}}\left(S\left(F, C\right)\right) = \pi_{\hat{F}'}\left(S(F', C)\right) = 1$, and

$$\frac{\pi_{\hat{F}}\left(s\right)}{\pi_{\hat{F}}\left(s'\right)} = \frac{\pi_{\hat{F}'}\left(E\left(s\right)\right)}{\pi_{\hat{F}'}\left(E\left(s'\right)\right)}, \tag{9}$$

for all $s, s' \in S\left(F, C\right)$ and $E\left(s\right), E\left(s'\right) \subset S\left(F', C\right)$, where $E\left(s\right)$ and $E\left(s'\right)$, are the inverse images of s and s' on $S\left(F', C\right)$.

The proof is in the Appendix.

5.3 Discovery of new feasible states and their representation

The discovery of new acts-consequences links, or the discovery that some links that were believed to exist are, in fact, nonexistent, does not affect the conceivable state space.

[14] This is "reverse Bayesianism" applied to the present context. Li (2008) conjectures an axiomatization of the link between preferences under full awareness and those under pure unawareness and states a proposition linking the evolution of beliefs. This is in the spirit of Theorem 2.

Rather such discoveries expand or contract only the feasible state space. To model this, fix C and F, and suppose that a new link is established. Then, $S(F,C) \subset S'(F,C)$, and $\succcurlyeq_{\hat{F}_{S'}}$ denotes the posterior preference relation as discussed in section 3.2. Using these notations we restate axiom (A.7) as follows:

(A.7a) (**Updating consistency**) For all $\hat{F}, \hat{F}_{S'} \in \mathcal{F}$, if $S'(F,C) \supset S(F,C)$ and $f', g' \in \hat{F}_{S'}$, $f' = f$ and $g' = g$ on $S(F,C)$ and $f' = g'$ on $S'(F,C) - S(F,C)$, $f \succcurlyeq_{\hat{F}} g$ if and only if $f' \succcurlyeq_{\hat{F}_{S'}} g'$.

Similarly, if the feasible state space is contracted due to the nullification of a link that was supposed to exist, (that is, $S'(F,C) \subset S(F,C)$), then Axiom (A.7a) can be restated as:

(A.7b) (**Bayesian updating**) For all $\hat{F}, \hat{F}_{S'} \in \mathcal{F}$, if $S'(F,C) \subset S(F,C)$ and $f, g \in \hat{F}$, $f = f'$ and $g = g'$ on $S'(F,C)$ and $f = g$ on $S(F,C) - S'(F,C)$, $f' \succcurlyeq_{\hat{F}_{S'}} g'$ if and only if $f \succcurlyeq_{\hat{F}} g$.

Nullification of a link that was believed to hold corresponds to the shrinking of the feasible state space as new information arrives within the Bayesian paradigm.

We show next that the process of updating the zero probability events in the wake of discovery of new links between acts and consequences is the exact counterpart of Bayesian updating in the wake of discovery that some links that were presumed to exist are, in fact, non-existent.

Theorem 3 For each $\hat{F} \in \mathcal{F}$, let $\succcurlyeq_{\hat{F}}$ be a binary relation on $\hat{F}$ then, for all $\hat{F}, \hat{F}_{S'} \in \mathcal{F}$, the following two conditions are equivalent:

(i) Each $\succcurlyeq_{\hat{F}}$ satisfies (A.1) - (A.5) and, jointly, $\succcurlyeq_{\hat{F}}$ and $\succcurlyeq_{\hat{F}_{S'}}$ satisfy (A.6), (A.7a) and (A.7b).

(ii) There exists a real-valued, non-constant, affine function, U, on $\Delta(C)$ and, for any two $\hat{F}, \hat{F}_{S'} \in \mathcal{F}$, there are probability measures, $\pi_{\hat{F}}$ and $\pi_{\hat{F}_{S'}}$ on C^F, such that, for all $f, g \in \hat{F}$,

$$f \succcurlyeq_{\hat{F}} g \Leftrightarrow \sum_{s \in C^F} U(f(s)) \pi_{\hat{F}}(s) \geq \sum_{s \in C^F} U(g(s)) \pi_{\hat{F}}(s), \tag{10}$$

and, for all $f', g' \in \hat{F}_{S'}$,

$$f' \succcurlyeq_{\hat{F}_{S'}} g' \Leftrightarrow \sum_{s \in C^F} U\left(f'(s)\right) \pi_{\hat{F}_{S'}}(s) \geq \sum_{s \in C^F} U\left(g'(s)\right) \pi_{\hat{F}_{S'}}(s). \tag{11}$$

Moreover, U is unique up to positive linear transformations, $\pi_{\hat{F}}$ and $\pi_{\hat{F}_{S'}}$ are unique, $\pi_{\hat{F}}(S(F,C)) = \pi_{\hat{F}_{S'}}(S'(F,C)) = 1$, and

$$\frac{\pi_{\hat{F}}(s)}{\pi_{\hat{F}}(s')} = \frac{\pi_{\hat{F}_{S'}}(s)}{\pi_{\hat{F}_{S'}}(s')}, \tag{12}$$

for all $s, s' \in S(F,C) \cap S'(F,C)$.[15]

The proof is in the Appendix.

6 Concluding Remarks

The model presented in this paper predicts that, as awareness grows and the state space expands, the relative likelihoods of events in the original state space remain unchanged. The model is silent about the absolute levels of these probabilities. In other words, our theory does not predict the probability of the new events in the expanded state space. This may appear as a serious limitation of our approach. However, this appearance is misleading. In fact, the relation between the prior and posterior probabilities in our model is not essentially different from the Bayesian model.

To grasp this claim, consider the Bayesian model. In that model, new information shrinks the state space by rendering null events that were assigned positive prior probabilities. Furthermore, given the prior probability of an event that has been rendered null, the Bayesian model predicts the absolute levels and, consequently, the likelihood ratios, of the posterior probabilities of all the events in the original algebra. These predictions, however, are predicated on the prior, about which the Bayesian model is silent. In Savage's (1954) model, the prior is derived from a primitive preference relation over acts.

Our approach is analogous. Rather than being silent on the prior, it is silent on the posterior probabilities of the newly discovered events. If we proceed analogously to Savage (1954), the posterior is derived from a primitive preference relation on the acts defined over the expanded state space. Given the posterior, our model predicts the absolute probabilities and, consequently, the likelihood ratios, of all the events in the original algebra, including those between newly discovered and previously known events.

Finally, we observe that, in reality, one often becomes aware of multiple new consequences and acts (and links among them) at the same time. Changes of this kind may be

[15] Notice that $S(F,C) \cap S'(F,C) = S(F,C)$ or $S(F,C) \cap S'(F,C) = S'(F,C)$.

handled by defining new conceivable and feasible state spaces and applying Theorems 1 through 3, respectively. Our theory then predicts the likelihood ratios on the intersection of the feasible state spaces.[16] If subjective probabilities are elicited at any point during these successive discoveries, the entire posterior will be known for the corresponding awareness level, and likelihood ratios can be predicted on the intersection of the feasible state spaces associated with that and the successive levels of awareness.

7 Proofs

7.1 Proof of theorem 1

(Sufficiency) Fix F and C, then, by (A.1) - (A.5), the theorem of Anscombe and Aumann (1963) and the von Neumann-Morgenstern expected utility theorem, there exists a real-valued, non-constant, function $u_{\hat{F}}$ on C such that for all $p, q \in \Delta(C)$

$$p \succcurlyeq_{\hat{F}} q \Leftrightarrow \sum_{c \in Supp(p)} u_{\hat{F}}(c)\, p(c) \geq \sum_{c \in Supp(q)} u_{\hat{F}}(c)\, q(c)\,. \tag{13}$$

Let $C' \supset C$, $\hat{F}^* \in \mathcal{F}$, with corresponding feasible state space $S(F^*, C') \supseteq S(F, C)$. Then, by the same argument as above, there exists a real-valued function $u_{\hat{F}^*}$ on C' such that for all $p', q' \in \Delta(C')$

$$p' \succcurlyeq_{\hat{F}^*} q' \Leftrightarrow \sum_{c \in Supp(p')} u_{\hat{F}^*}(c)\, p'(c) \geq \sum_{c \in Supp(q')} u_{\hat{F}^*}(c)\, q'(c)\,. \tag{14}$$

But, by (A.6), for all $p, q \in \Delta(C \cap C') = \Delta(C)$,

$$p \succcurlyeq_{\hat{F}} q \Leftrightarrow p \succcurlyeq_{\hat{F}^*} q. \tag{15}$$

The uniqueness of the von Neumann-Morgenstern utility function implies that for all $\hat{F}, \hat{F}^* \in \mathcal{F}$, $u_{\hat{F}^*}(c) = b u_{\hat{F}}(c) + a$, $b > 0$, for all $c \in C$. Hence, $u_{\hat{F}^*}$ is an extension of $u_{\hat{F}}$.

Let $u = u_{\hat{F}^*}$ and define $U(f(s)) := \sum_{c \in Supp(f(s))} u(c)\, f(s)(c)$, for all $f \in \hat{F}^*$ and $s \in S\left(\hat{F}^*, C'\right)$. Then, by Anscombe and Aumann (1963), for all $\hat{F} \in \mathcal{F}$, and $f, g \in \hat{F}$,

$$f \succcurlyeq_{\hat{F}} g \Leftrightarrow \sum_{s \in C^F} U(f(s))\, \pi_{\hat{F}}(s) \geq \sum_{s \in C^F} U(g(s))\, \pi_{\hat{F}}(s)\,, \tag{16}$$

[16] When the growing awareness involves new acts, the likelihood ratios are predicted on the intersection of the lower dimensional state spaces and the inverse image of the states in this intersection.

and, for all $f', g' \in \hat{F}^*$,

$$f' \succcurlyeq_{\hat{F}^*} g' \Leftrightarrow \sum_{s \in (C')^{F^*}} U\left(f'(s)\right)\pi_{\hat{F}^*}(s) \geq \sum_{s \in (C')^{F^*}} U\left(g'(s)\right)\pi_{\hat{F}^*}(s), \tag{17}$$

where $\pi_{\hat{F}}(S(F,C)) = \pi_{\hat{F}^*}(S(F^*,C')) = 1$.

Let $f, g \in \hat{F}$ and $f', g' \in \hat{F}^*$ be as in Axiom (A.7) (that is, $f' = f$ and $g' = g$ on $S(F,C)$ and $f' = g'$ on $S(F^*,C') - S(F,C)$) then

$$f' \succcurlyeq_{\hat{F}^*} g' \Leftrightarrow \sum_{s \in S(F,C)} U(f(s))\pi_{\hat{F}^*}(s) \geq \sum_{s \in S(F,C)} U(g(s))\pi_{\hat{F}^*}(s). \tag{18}$$

Since $\pi_{\hat{F}}(S(F,C)) = 1$, the representation (16) implies

$$f \succcurlyeq_{\hat{F}} g \Leftrightarrow \sum_{s \in S(F,C)} U(f(s))\pi_{\hat{F}}(s) \geq \sum_{s \in S(F,C)} U(g(s))\pi_{\hat{F}}(s). \tag{19}$$

But Axiom (A.7) implies

$$f \succcurlyeq_{\hat{F}} g \Leftrightarrow f' \succcurlyeq_{\hat{F}^*} g'. \tag{20}$$

Thus the expressions in (18) and (19) are equivalent. Hence, by the uniqueness of the probabilities in Anscombe and Aumann (1963),

$$\frac{\pi_{\hat{F}^*}(s)}{\sum_{s \in S(F,C)} \pi_{\hat{F}^*}(s)} = \pi_{\hat{F}}(s), \text{ for all } s \in S(F,C). \tag{21}$$

(Necessity) The necessity of (A.1)-(A.5) is an implication of the Anscombe and Aumann (1963) theorem. The necessity of (A.6) and (A.7) is immediate.

The uniqueness part is an implication of the uniqueness of the utility and probability in Anscombe and Aumann (1963). ■

7.2 Proof of theorem 2

(Sufficiency) By (A.1) - (A.6), and the argument in the proof of Theorem 1, $u_{\hat{F}'}(c) = bu_{\hat{F}}(c) + a$, $b > 0$, for all $c \in C$ and $\hat{F}, \hat{F}' \in \mathcal{F}$. Let $u_{\hat{F}} = u$ and define $U(f(s)) := \sum_{c \in Supp(f(s))} u(c) f(s)(c)$, for all $f \in \hat{F}$ and $s \in S(F,C)$. Then, since $C^F - S(F,C)$ is a null event, by Anscombe and Aumann (1963), for all $\hat{F} \in \mathcal{F}$, and $f, g \in \hat{F}$,

$$f \succcurlyeq_{\hat{F}} g \Leftrightarrow \sum_{s \in S(F,C)} U(f(s))\pi_{\hat{F}}(s) \geq \sum_{s \in S(F,C)} U(g(s))\pi_{\hat{F}}(s). \tag{22}$$

Let $\hat{F}, \hat{F}' \in \mathcal{F}$ and, without loss of generality, suppose that $S(F', C)$ is a refinement of the states in $S(F, C)$.[17] Take $s, s' \in S(F, C)$ and $\left(\left(h'_{-E(s)}p\right)_{-E(s')}\bar{p}\right)$ and $\left(\left(h'_{-E(s)}q\right)_{-E(s')}\bar{q}\right)$ in $\hat{F}'$ as defined in Axiom (A.8). For these acts, (22) is equivalent to

$$\left(\left(h'_{-E(s)}p\right)_{-E(s')}\bar{p}\right) \succcurlyeq_{\hat{F}'} \left(\left(h'_{-E(s)}q\right)_{-E(s')}\bar{q}\right) \tag{23}$$

if and only if

$$U(p)\,\pi_{\hat{F}'}(E(s)) + U(\bar{p})\,\pi_{\hat{F}'}\left(E\left(s'\right)\right) \geq U(q)\,\pi_{\hat{F}'}(E(s)) + U(\bar{q})\,\pi_{\hat{F}'}\left(E\left(s'\right)\right), \tag{24}$$

since common terms cancel out. By Axiom (A.8),

$$\left(\left(h'_{-E(s)}p\right)_{-E(s')}\bar{p}\right) \succcurlyeq_{\hat{F}'} \left(\left(h'_{-E(s)}q\right)_{-E(s')}\bar{q}\right) \Leftrightarrow \left((h_{-s}p)_{-s'}\bar{p}\right) \succcurlyeq_{\hat{F}} \left((h_{-s}q)_{-s'}\bar{q}\right). \tag{25}$$

By (22),

$$\left((h_{-s}p)_{-s'}\bar{p}\right) \succcurlyeq_{\hat{F}} \left((h_{-s}q)_{-s'}\bar{q}\right)$$

if and only if

$$\sum_{s \in S(F,C)} U\left(\left((h_{-s}p)_{-s'}\bar{p}\right)(s)\right)\pi_{\hat{F}}(s) \geq \sum_{s \in S(F,C)} U\left(\left((h_{-s}q)_{-s'}\bar{q}\right)(s)\right)\pi_{\hat{F}}(s),$$

which, since common terms cancel out, is equivalent to

$$U(p)\,\pi_{\hat{F}}(s) + U(\bar{p})\,\pi_{\hat{F}}\left(s'\right) \geq U(q)\,\pi_{\hat{F}}(s) + U(\bar{q})\,\pi_{\hat{F}}\left(s'\right). \tag{26}$$

By (25), the expressions (24) and (26) are equivalent, which holds for all $p, \bar{p}, q, \bar{q} \in \Delta(C)$, if and only if

$$\frac{\pi_{\hat{F}}(s)}{\pi_{\hat{F}}(s')} = \frac{\pi_{\hat{F}'}(E(s))}{\pi_{\hat{F}'}(E(s'))}, \tag{27}$$

for all $s, s' \in S(F, C)$ and $E(s), E(s') \subset S(F', C)$, where $E(s)$ and $E(s')$ are the projections of s and s' on $S(F', C)$.

(Necessity) The necessity of (A.1)-(A.5) is an implication of the Anscombe and Aumann (1963) theorem. The necessity of (A.6) and (A.8) is immediate.

The uniqueness part is an implication of the uniqueness of the utility and probability in Anscombe and Aumann (1963). ■

[17] Hence, $F \subset F'$.

7.3 Proof of theorem 3

(Sufficiency) By (A.1) - (A.6) and the argument in the proof of Theorem 1, $u_{\hat{F}_{S'}}(c) = bu_{\hat{F}}(c) + a$, $b > 0$, for all $c \in C$.

Let $u_{\hat{F}} = u$ and define $U(f(s)) := \sum_{c \in Supp(f(s))} u(c) f(s)(c)$. Consider the case in which $S(F,C) \subset S'(F,C)$ (that is, a new link has been discovered). By Anscombe and Aumann (1963), for all $f, g \in \hat{F}$

$$f \succcurlyeq_{\hat{F}} g \Leftrightarrow \sum_{s \in S(F,C)} U(f(s)) \pi_{\hat{F}}(s) \geq \sum_{s \in S(F,C)} U(g(s)) \pi_{\hat{F}}(s), \tag{28}$$

and, for all $f', g' \in \hat{F}_{S'}$,

$$f' \succcurlyeq_{\hat{F}_{S'}} g' \Leftrightarrow \sum_{s \in S'(F,C)} U\left(f'(s)\right) \pi_{\hat{F}_{S'}}(s) \geq \sum_{s \in S'(F,C)} U\left(g'(s)\right) \pi_{\hat{F}_{S'}}(s). \tag{29}$$

Let $f', g' \in F_{S'}$ be as in axiom (A.7a), then (A.7a) implies that

$$\sum_{s \in S(F,C)} U(f(s)) \pi_{\hat{F}}(s) \geq \sum_{s \in S(F,C)} U(g(s)) \pi_{\hat{F}}(s), \tag{30}$$

if and only if

$$\sum_{s \in S(F,C)} U(f(s)) \pi_{\hat{F}_{S'}}(s) \geq \sum_{s \in S(F,C)} U(g(s)) \pi_{\hat{F}_{S'}}(s). \tag{31}$$

Hence,

$$\pi_{\hat{F}}(s) = \frac{\pi_{\hat{F}_{S'}}(s)}{\sum_{s \in S(F,C)} \pi_{\hat{F}_{S'}}(s')} \tag{32}$$

for all $s \in S(F,C)$. Thus, for all $s, s' \in S(F,C)$,

$$\frac{\pi_{\hat{F}}(s)}{\pi_{\hat{F}}(s')} = \frac{\pi_{\hat{F}_{S'}}(s)}{\pi_{\hat{F}_{S'}}(s')}. \tag{33}$$

The case in which new evidence entails the severance of existing links and contraction of the feasible state space is treated analogously, with axiom (A.7b) in place of (A.7a).

(Necessity) The necessity of (A.1)-(A.6) is an implication of Theorem 1. The necessity of (A.7a) and (A.7b) is immediate.

The uniqueness part is an implication of the uniqueness of the utility and probability in Anscombe and Aumann (1963). ■

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