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Endogenizing Growth via a Lag for Apprenticing

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Abstract

We take up a growth model with both skilled and unskilled labor, and a steady migration of some unskilled workers, who undertake apprenticing, to the skilled group of workers. Apprenticing involves a period of observing and thus labor output foregone. The time-out for observing represents a cost to the economy and this results in the rate of balanced growth being endogenous. We examine the balanced growth path and report on the stability of our dynamic system.

- key words: skilled and unskilled labor, apprenticing, balanced growth, endogenous growth
- JEL article classification: O110, O150, J000

1. Introduction

An uncomplicated extension of the Solow [1956] growth model has two types of labor, skilled and unskilled, and two perpetual flows of labor, one "up" from unskilled to skilled in volume proportionate with current net investment in durable capital and the other "down" and proportionate with current numbers of skilled workers. This Solow variant becomes interesting when the transition of an unskilled worker to skilled is costly. We model this cost as a simple time-out (one period in discrete time) during which the unskilled worker is observing or apprenticing. Apprenticing represents two costs: one is simply that some labor is foregone while the trainee is observing or apprenticing and the other is that trainees end up reproducing with a lag, the time-out for apprenticing. These two costs result in the the growth rate of the economy becoming endogenous and somewhat less than the rate of population growth along a balanced growth path. Thus the introduction of a simple time-out for apprenticing becomes a parsimonious route to endogenizing the growth rate in a Solow-type model. We work in discrete time and establish conditions for the existence of a balanced growth path and remark on stability in an Appendix.

We have in mind the emergence of a middle class of families headed by a skilled person. For Britain, Mokyr [2009] remarks: "The period under discussion here, 1700-1850, saw the rise of many other "white collar professions" that would be classified today in the service industries. In 1700, very few Britons were engaged in such occupations as land agents, dentists, architects, surveyors, apothecaries, or even attorneys. Apart from the very top, most of these specialists were trained through an apprenticeship system rather than through the universities." (p. 250) Earlier, he observes "The great English engineers of the Industrial Revolution learned their skills by being apprenticed to able masters, and otherwise were largely self-taught. James Brindley, the canal engineer, was taught by his mother and, like many other pivotal figures in the Industrial Revolution, never went to a formal school. Many of the others were educated in Scotland." (p. 232) In addition to the Industrial Revolution being led by skilled, but often not formally schooled individuals, the apprenticeship system itself was evolving over this crucial interval. Mokyr remarks: "The regulations that set up the strict requirements for craftsmen to undergo many years of appreticeship were enforced less and less during the eighteenth century and more and more exceptions to them could be found. As was noted earlier, the mandated length of the period of apprenticeship changed, and outdoor apprenticeship became more prevalent. Domestic industry expanded a great deal into low-skill full-time occupations, and permitted young couples to set up shop in a small cottage." (p. 288) In our closing remarks we note the significant increase in the investment rate during the middle years of the

Industrial Revolution.

2. The Model

Time is decrete, $t = 0, 1, 2, ...$ At the beginning of period t, there are M_t potentially skilled persons, and N_t potentially unskilled persons. In the middle of period t, the number of skilled workers is $M_t - \delta M_t + L_{t-1} \equiv S_t$ where L_{t-1} is the number of apprentices in period $t-1$, and δ is the fraction of nominally skilled persons who fail to realize their potential (δM_t workers are sliding back from the skilled to the unskilled cohort). Also, in the middle of period t, the numbers of unskilled workers is $N_t + \delta M_t - L_t \equiv U_t$, where L_t are the apprentices (drawn from N_t).

The volume of unskilled workers currently transitting into apprenticing and then, with a one period lag, into the skilled worker cohort is proportional to the current volume of investment in durable capital (there is no decay in durable capital K_t in the model). That is,

$$L_t = \gamma I_t,$$

where by definition

$$K_{t+1} - K_t = I_t.$$

We assume that the $M_t - \delta M_t + L_{t-1}$ skilled workers give birth to η nominally skilled workers in the next period, so that

$$M_{t+1} = (1 + \eta)(M_t - \delta M_t + L_{t-1}).$$

It is as if skilled parents are able to raise children who costlessly enter the cohort of skilled workers. Note that it is apprentices who "departed" from the unskilled cohort one period back that now reproduce as skilled people. These apprentices thus reproduce with a one period lag compared with other people. This reproduction lag is then a central cost to the economy of having apprenticing workers on the sidelines for a period.

Similarly, we assume

$$N_{t+1} = (1+\eta)(N_t + \delta M_t - L_t).$$

Unskilled workers reproduce at the same rate as the skilled workers.

Output, denoted by Q_t, is produced using K_t, S_t and U_t :

$$Q_t = K_t^a \left(AS_t\right)^{\beta} \left(U_t\right)^{1-\alpha-\beta}.$$

A constant fraction σ of output is saved, and is invested:

$$I_t = \sigma Q_t.$$

Consider the system of 4 difference equations

$$M_{t+1} = (1+\eta)\left(M_t - \delta M_t + \gamma I_{t-1}\right) \tag{1}$$

$$N_{t+1} = (1+\eta)(N_t + \delta M_t - \gamma I_t) \tag{2}$$

$$I_t = \sigma K_t^a A^{\beta} \left(M_t - \delta M_t + \gamma I_{t-1}\right)^{\beta} \left(N_t + \delta M_t - \gamma I_t\right)^{1-\alpha-\beta} \tag{3}$$

$$K_{t+1} - K_t = I_t. \tag{4}$$

We proceed to define

$$x_t \equiv \frac{I_t}{K_t}, m_t \equiv \frac{M_t}{K_t} \text{ and } n_t \equiv \frac{N_t}{K_t},$$

and also we have $x_{t-1} = I_{t-1}/K_{t-1}$ etc., and $K_{t+1} = K_t + x_t K_t = (1+x_t)K_t$, and $K_t = (1+x_{t-1})K_{t-1}$. Divide each of the three equations (1)-(3) by K_t to get

$$(1+x_t)m_{t+1} = (1+\eta)\left(m_t(1-\delta) + \gamma \frac{x_{t-1}}{1+x_{t-1}}\right) \tag{5}$$

$$(1+x_t)n_{t+1} = (1+\eta)\left(n_t + \delta m_t - \gamma x_t\right) \tag{6}$$

$$x_t = \sigma A^{\beta} \left(\frac{K_t^{\alpha}}{K_t^{\alpha}}\right) \frac{\left(M_t - \delta M_t + \gamma I_{t-1}\right)^{\beta}}{K_t^{\beta}} \left[\frac{\left(N_t + \delta M_t - \gamma I_t\right)^{1-\alpha-\beta}}{K_t^{1-\alpha-\beta}}\right]$$

or

$$x_t = \sigma A^\beta \left[(1-\delta)m_t + \frac{\gamma x_{t-1}}{(1+x_{t-1})}\right]^\beta [n_t + \delta m_t - \gamma x_t]^{1-\alpha-\beta} \tag{7}$$

Is there a steady-state triple (m^*, n^*, x^*) where $m^* > 0, n^* > 0$ and $x^* > 0$? We turn to this question.

Given x^*, equation (5) gives

$$m^* = \frac{(1+\eta)\gamma x^*}{(1+x^*)(\delta(1+\eta) - \eta + x^*)}. \tag{8}$$

Then eq (6) gives

$$(x^* - \eta)n^* = (1+\eta)(\delta m^* - \gamma x^*) = \gamma x^*(1+\eta)\left(\frac{\delta m^*}{\gamma x^*} - 1\right)$$

$$n^* = \frac{(1+\eta)\delta m^*}{(x^*-\eta)} - \frac{(1+\eta)\gamma x^*}{(x^*-\eta)}$$

$$= \frac{(1+\eta)\gamma x^*}{(x^*-\eta)}\left[\frac{(1+\eta)\delta}{(1+x^*)(\delta(1+\eta)-\eta+x^*)} - 1\right]$$

$$= \frac{(1+\eta)\gamma x^*}{(x^*-\eta)}\left[\frac{(1+\eta)\delta}{(1+x^*)(\delta(1+\eta)-\eta+x^*)} - 1\right];$$

$$\text{which implies } n^* = \frac{(1+\eta)\gamma x^*}{(x^*-\eta)}\left[\frac{-(1+x^*)(x^*-\eta) - \delta(1+\eta)x^*}{(1+x^*)(\delta(1+\eta)-\eta+x^*)}\right].$$

Assume that $(\delta(1+\eta)-\eta+x^*) > 0$. Then, for $n^* > 0$ and $x^* > 0$, the term $(\eta - x^*)$ must have the same sign as the term $x^*\delta(1+\eta) + (1+x^*)(x^*-\eta)$, ie,$(\eta - x^*)$ cannot be negative. Hence, for $(\eta - x^*) > 0$, we have

$$n^* = \frac{\gamma x^*(1+\eta)}{(\eta - x^*)}\left(\frac{x^*\delta(1+\eta) + (1+x^*)(x^*-\eta)}{(1+x^*)(\delta(1+\eta)-\eta+x^*)}\right) > 0. \tag{9}$$

Consider

$$x_t = \sigma A^\beta \left[(1-\delta)m_t + \gamma\frac{x_{t-1}}{1+x_{t-1}}\right]^\beta [n_t + \delta m_t - \gamma x_t]^{1-\alpha-\beta} \tag{10}$$

Suppose that we are in the hypothetical steady state version of the model. Then routine substitution into (10) yields the following equation in x^* :

$$\frac{1}{\sigma} = (x^*)^{-\alpha} A^{\beta} \left(\frac{(1-\delta)(1+\eta)\gamma}{(1+x^*)(\delta(1+\eta)-\eta+x^*)} + \frac{\gamma}{(1+x^*)} \right)^{\beta} \times$$

$$\left(\frac{(1+x^*)}{(x^*-\eta)} \left(\frac{\delta(1+\eta)\gamma}{(1+x^*)(\delta(1+\eta)-\eta+x^*)} \right) - \frac{(1+\eta)\gamma}{(x^*-\eta)} - \gamma \right)^{1-\alpha-\beta} . \qquad (11)$$

We turn to simplifying $\left(\frac{(1+x^*)}{(x^*-\eta)} \left(\frac{\delta(1+\eta)\gamma}{(1+x^*)(\delta(1+\eta)-\eta+x^*)} \right) - \frac{(1+\eta)\gamma}{(x^*-\eta)} - \gamma \right) - \frac{(1+\eta)\gamma}{(x^*-\eta)} - \gamma = -\gamma \left[\frac{1+x^*}{(x^*-\eta)} \right]$.

We have $\frac{(1+x^*)}{(x^*-\eta)} \left(\frac{\delta(1+\eta)\gamma}{(1+x^*)(\delta(1+\eta)-\eta+x^*)} \right) - \frac{(1+\eta)\gamma}{(x^*-\eta)} - \gamma =$

$$\frac{(1+x^*)\gamma}{(x^*-\eta)} \left(\frac{\delta(1+\eta)}{(1+x^*)(\delta(1+\eta)-\eta+x^*)} - 1 \right) =$$

$$\frac{(1+x^*)\gamma}{(\eta-x^*)} \left(\frac{x^*\delta(1+\eta)+(1+x^*)(x^*-\eta)}{(1+x^*)(\delta(1+\eta)-\eta+x^*)} \right) =$$

$$\frac{\gamma}{(\eta-x^*)} \left(\frac{x^*\delta(1+\eta)+(1+x^*)(x^*-\eta)}{\delta(1+\eta)-\eta+x^*} \right).$$

If $(x^*-\eta) < 0$ (the balanced growth rate less than the common population growth rate) then we need

$$\frac{\delta(1+\eta)}{(1+x^*)(\delta(1+\eta)-\eta+x^*)} - 1 < 0;$$

ie $\delta(1+\eta) < (1+x^*)(\delta(1+\eta)-\eta+x^*) = \delta(1+\eta)+x^*\delta(1+\eta)+(1+x^*)(x^*-\eta)$;

$$\text{or } 0 < x^*\delta(1+\eta)+(1+x^*)(x^*-\eta). \qquad (12)$$

For $\sigma = 0.2$, $\alpha = 0.4$, $\beta = 0.2$, $A = 1.5$, $\delta = 0.05$, $\gamma = 0.04$, $\eta = 0.02$, one obtains $x^* = 0.019564$ in (11), slightly less than the population growth rate, 0.02. This value of x^* and parameter values $\delta = 0.05$ and $\eta = 0.02$ satisfy (12). At this solution (x^*, and the corresponding m^* and n^*) the positive flow of skilled workers "down" (0.00033825) is slightly smaller than the corresponding flow of unskilled workers "up" (0.00042929). In the Appendix we report on the stability of our illustrative balanced growth path.

3. Concluding Remarks

Our model admits balanced growth paths with distinct population growth rates for each type of worker and this suggests using such a model to characterize a path of development for say England over the period 1701 to 1875. We might start with a balanced growth path with the savings rate "low" and the growth rate of unskilled workers "high". Development would be the path of transition to a new balanced growth path with a higher savings rate and a lower rate of population growth for unskilled workers (the demographic revolution). The principal outcome of the transition would be the emergence of a relatively large middle class comprising skilled workers and a "high" ratio of durable capital to the number of unskilled workers. Maddison [2007] emphasizes high ratios of capital per worker as a central mechanism of income improvement for workers. Mokyr [2009] reports "... the best numbers we have today about the proportion of gross investments in GDP indicate that it increased from 8.6 percent in the 1760s to 13.3 percent in the 1840s... the increase in the investment ratio is consistent with the acceleration in the growth of the labor force (new workers needed more equipment and houses to live in)..." (p. 260) Maddison has rates of accumulation of non-residential capital of about 5.5% for the US from 1820 to 1913 (Table 8.3). Comparable rates for the UK are about one half the US rates. Maddison has Japan accumulating capital at rates twice those in the US from 1913 to 2003. (The capital he is considering is the machine and structures type, net of human capital.) Other observers emphasize fertility decline after 1850 being a major factor in contributing to the rise in the wage of workers in England. Galor [2005] argues that fertility decline in England was linked to parents aiming for quality in their offspring (educated children) instead of quantity. This of course links labor-augmenting technical progress to fertility decline.

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4. Appendix: Stability analysis

Consider the system

$$m_{t+1} = \frac{(1+\eta)}{(1+x_t)}\left(m_t(1-\delta) + \gamma\frac{x_{t-1}}{1+x_{t-1}}\right)$$

$$n_{t+1} = \frac{(1+\eta)}{(1+x_t)}\left(n_t + \delta m_t - \gamma x_t\right)$$

$$x_t = \sigma A^\beta \left[(1-\delta)m_t + \gamma\frac{x_{t-1}}{1+x_{t-1}}\right]^\beta \left[n_t + \delta m_t - \gamma x_t\right]^{1-\alpha-\beta}$$

Define

$$y_{t+1} = x_t$$

Then we have the system

$$F(y_{t+1}, m_{t+1}, n_{t+1}, y_t, m_t, n_t) = (1+y_{t+1})m_{t+1} - m_t(1-\delta)(1+\eta) - \gamma(1+\eta)\frac{y_t}{1+y_t} = 0$$

$$G(y_{t+1}, m_{t+1}, n_{t+1}, y_t, m_t, n_t) = (1+y_{t+1})n_{t+1} - (1+\eta)n_t - \delta(1+\eta)m_t + \gamma(1+\eta)y_{t+1} = 0$$

$$\begin{aligned} H(y_{t+1}, m_{t+1}, n_{t+1}, y_t, m_t, n_t) &= y_{t+1} - \sigma A^\beta \left[(1-\delta)m_t + \gamma\frac{y_t}{1+y_t}\right]^\beta \left[n_t + \delta m_t - \gamma y_{t+1}\right]^{1-\alpha-\beta} \\ &= 0. \end{aligned}$$

And

$$F(y_{t+1}, m_{t+1}, n_{t+1}, y_t, m_t, n_t) - F(y^*, m^*, n^*, y^*, m^*, n^*) = 0$$

$$G(y_{t+1}, m_{t+1}, n_{t+1}, y_t, m_t, n_t) - G(y^*, m^*, n^*, y^*, m^*, n^*) = 0$$

$$H(y_{t+1}, m_{t+1}, n_{t+1}, y_t, m_t, n_t) - H(y^*, m^*, n^*, y^*, m^*, n^*) = 0.$$

Linearization gives

$$(y_{t+1} - y^*)F_1^* + (m_{t+1} - m^*)F_2^* + (n_{t+1} - n^*)F_3^* +$$

$$(y_t - y^*)F_4^* + (m_t - m^*)F_5^* + (n_t - n^*)F_6^* = 0$$

etc., where

$$F_1 = \frac{\partial F}{\partial y_{t+1}} = m_{t+1} \text{ and } F_1^* = m^*$$

$$F_2 = \frac{\partial F}{\partial m_{t+1}} = (1 + y_{t+1}) \text{ and } F_2^* = 1 + y^*$$

$$F_3 = \frac{\partial F}{\partial n_{t+1}} = 0 \text{ and } F_3^* = 0$$

$$F_4 = \frac{\partial F}{\partial y_t} = -\frac{\gamma(1+\eta)}{(1+y_t)^2} \text{ and } F_4^* = -\frac{\gamma(1+\eta)}{(1+y^*)^2}$$

$$F_5 = \frac{\partial F}{\partial m_t} = -(1-\delta)(1+\eta); \; F_6 = \frac{\partial F}{\partial n_t} = 0$$

$$G_1^* = n^* + \gamma(1+\eta); \; G_2^* = 0; G_3^* = (1+y^*); \; G_4^* = 0;$$

$$G_5^* = -\delta(1+\eta); \; G_6^* = -(1+\eta)$$

$$H_1^* = 1 + \frac{y^*(1-\alpha-\beta)\gamma}{n^* + \delta m^* - \gamma y^*}; H_2^* = H_3^* = 0;$$

$$H_4^* = -\frac{\beta y^*}{(1-\delta)m^* + \frac{\gamma y^*}{(1+y^*)}} \left(\frac{\gamma}{(1+y^*)^2} \right)$$

$$H_5^* = -(1-\delta)\frac{\beta y^*}{(1-\delta)m^* + \frac{\gamma y^*}{(1+y^*)}} - \delta \frac{(1-\alpha-\beta)y^*}{n^* + \delta m^* - \gamma y^*}; \; H_6^* = \frac{-(1-\alpha-\beta)y^*}{n^* + \delta m^* - \gamma y^*}.$$

The linearized system can then be expressed as

$$\begin{bmatrix} y_{t+1} - y^* \\ m_{t+1} - m^* \\ n_{t+1} - n^* \end{bmatrix} = \begin{bmatrix} a_{11}, & a_{12}, & a_{13} \\ a_{21}, & a_{22}, & a_{23} \\ a_{31}, & a_{32}, & a_{33} \end{bmatrix} \begin{bmatrix} y_t - y^* \\ m_t - m^* \\ n_t - n^* \end{bmatrix}$$

for

$$\begin{bmatrix} a_{11}, & a_{12}, & a_{13} \\ a_{21}, & a_{22}, & a_{23} \\ a_{31}, & a_{32}, & a_{33} \end{bmatrix} = \begin{bmatrix} F_1^*, & F_2^*, & F_3^* \\ G_1^*, & G_2^*, & G_3^* \\ H_1^*, & H_2^*, & H_3^* \end{bmatrix}^{-1} \begin{bmatrix} F_4^*, & F_5^*, & F_6^* \\ G_4^*, & G_5^*, & G_6^* \\ H_4^*, & H_5^*, & H_6^* \end{bmatrix}$$

For the following parameter values: $\alpha = 0.4$, $\beta = 0.2$, $\sigma = 0.2$, $\eta = 0.02$, $A = 1.5$, $\delta = 0.05$, and $\gamma = 0.04$, we obtain y*= 0.01956404816, m*= 0.01548326470, and n*= 0.01965048531 and the a_{ij} matrix is:

$$\begin{bmatrix} -0.009575826535, & -0.2560197266, & -0.3921611416 \\ -0.03835067024, & -0.9465182502, & 0.005955422587 \\ 0.0005677557603, & -0.03484183591, & -0.9771761469 \end{bmatrix}$$

The three eigen values for this matrix are

$$[-2.590730592 \times 10^{-12}, -0.9457593519, -0.9875108718],$$

all less than unity in absolute value. Hence the system of three difference equations is locally stable. This stability result accords with our brute force forward recursions.