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The Multinomial Option Pricing Model and Its Brownian and Poisson Limits

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The Cox, Ross, and Rubinstein binomial model is generalized to the multinomial case. Limits are investigated and shown to yield the Black-Scholes formula in the case of continuous sample paths for a wide variety of complete market structures. In the discontinuous case a Merton-type formula is shown to result, provided jump probabilities are replaced by their corresponding Arrow-Debreu prices.

A multinomial option pricing formula consistent with an Arrow-Debreu complete markets equilibrium is derived. Economic uncertainty is modeled as evolving on an $(n + 1)$-ary tree with branching occurring during a short interval of time in which there is no trading. Direct empirical implementation of such a formula is feasible, though issues associated with the identification of the number of branches and their probabilities should be addressed. This article develops limiting formulas that arise on letting the intertrading interval approach zero. Two limiting contexts are considered, one yielding continuity for the sample paths of share prices (the continuous case), while the other permits jump discontinuities (the discontinuous case). For other approaches to option valuation that usefully can be

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compared to the methods of this article, see Merton (1977), Jones (1984), Kreps (1982), and Harrison and Pliska (1981).

The limiting formula in the continuous case is found to be the Black-Scholes formula [Black and Scholes (1973)] thus generalizing the binomial result of Cox, Ross, and Rubinstein (1979) (CRR) and establishing the validity of the Black-Scholes formula for the wider context of multiple branching {or high but finite martingale multiplicity [see Duffie and Huang (1985, p. 1342)]}. For a critical examination of the relationship of the binomial model to the Arrow-Debreu theory, see Milne and Shefrin (1987).

The limiting formula in the discontinuous case requires the replacement of jump probabilities by corresponding Arrow-Debreu prices in Met-ton's (1976) formula. This makes it necessary for empirical researchers, who wish to assume such processes and account for risk aversion, to describe the underlying Arrow-Debreu structure in sufficient detail to permit identification of the relevant Arrow-Debreu prices. Procedures for the identification of these prices in two special cases are developed in Madan and Milne (1988a, 1988b).

The limits investigated extend the analysis of CRR to a wider subclass. In the continuous case the analysis shows that for a variety of primitive security prices consistent with complete markets, the limiting option value is Black-Scholes. This suggests that in the continuous case the option can be an asymptotically redundant security in certain incomplete market structures, in that a variety of differing individual valuations converge upon each other. Though this is also possible for the discontinuous case, the underlying conditions are shown to be more restrictive.

The article is divided into four sections. Section 1 develops the multinomial option pricing model for an arbitrary (n + 1)-ary branching tree. Section 2 is devoted to the continuous case. The discontinuous case is presented in Section 3, and Section 4 concludes.

1. The Multinomial Model

Suppose that the uncertainties are given by a finite set of events E with elements e, partially ordered as a tree. For simplicity each node is assumed to possess $n + 1$ successor nodes e'. Denote by $S(e)$ the $n \times 1$ vector of share prices at node e, with R (independent of e) being the associated $n \times (n + 1)$ matrix of one-period rates of return. Let r denote the return on a risk-free asset. The assumed full rank, independent of e, rate of return matrix is then given by

$$\mathbf{A} = \begin{bmatrix} R \\ r\mathbf{1}^T \end{bmatrix}$$

where $\mathbf{1}$ is an $(n + 1)$-dimensional vector of unit entries (the superscript T denoting transposition).

Following Kreps (1982), let q be the normalized vector of one-period primitive prices. A cash flow $\mathbf{y}_e$ at successor nodes e' then has an Arrow-

Debreu price at e given by $W = r^{-1}\Sigma_e q_e y_e$. In particular, the primitive prices q are defined by $r^{-1}\Lambda q = 1$.

Theorem 1. ***Let $\rho_1, \ldots, \rho_{n+1}$ denote the entries in the first row of Λ. The Arrow-Debreu price W_0 at e_0 of an option to buy the first share at time m for a price of K is given by***

$$W_0 = \sum_{\nu \in A} \begin{bmatrix} m \\ \nu_1, \ldots, \nu_{n+1} \end{bmatrix} \left\{ S_0 \prod_{j=1}^{n+1} \left(\frac{q_j \rho_j}{r} \right)^{\nu_j} - K r^{-m} \prod_{j=1}^{n+1} q_j^{\nu_j} \right\}$$

where $A = \{\nu \mid \nu = (\nu_1, \ldots, \nu_{n+1}),\ \nu_j$ an integer, $\nu_j \geq 0,\ \Sigma_j \nu_j = m,\ S_0 \rho_1^{\nu_1} \ldots \rho_{n+1}^{\nu_{n+1}} > K\}$ and S_0 is the price of the first share at e_0.

Proof. **The price of the option at e_0 is the discounted expected terminal value of the option using Arrow-Debreu prices as probabilities for the calculation of expectations. The terminal price of the share is $S_0 \rho_1^{\nu_1} \ldots \rho_{n+1}^{\nu_{n+1}}$ where ν_j is the number of times the growth factor was ρ_j. The terminal value of the option is $\max(S_0 \rho_1^{\nu_1} \ldots \rho_{n+1}^{\nu_{n+1}} - K, 0)$. Because the probability of ν is**

$$\begin{bmatrix} m \\ \nu_1, \ldots, \nu_{n+1} \end{bmatrix} q_1^{\nu_1} \cdots q_{n+1}^{\nu_{n+1}}$$

the formula follows on direct computation of discounted expected values. ∎

Observe that the expression for W_0 is precisely analogous to the CRR binomial case and that it may be summarized by writing

where $P_x(A)$ denotes the multinomial probability of the set A under the system of probabilities given by x for $x = v, q$ and $v_j = q_j \rho_j / r$. Equation (1) constitutes the multinomial option pricing model. Though the derivation here uses a state pricing approach, while CRR use an arbitrage argument, the two approaches are equivalent for a complete markets context.

2. The Continuous Case

The Black-Scholes formula is derived in this section as the limit of the multinomial option pricing model (1), obtained by letting m, the number of trading periods in a time interval of length t, tend to infinity while the return matrix Λ^m is adjusted with m to yield sample path continuity in the share prices. For this purpose let the time to maturity be t, and break up the interval $[0, t\ ]$ into m pieces of length t/m. Construct a finite rooted tree of height m with each node possessing $n + 1$ branches, a unit of time on the tree being t/m units of real time. Suppose that in addition to the riskless asset there are n shares available for trading at each node with constant mean and variance rates for log returns given by μ_i, σ_i^2 for the *ith* share. We need to define for each m the one-tree period matrix $\mathbf{\Lambda}^m$. Noting the compounding of returns in Equation (1) motivates the specification of log returns $\mathbf{Z}^m$ ($\mathbf{Z}_{ij}^m = \ln R_{ij}^m$ is the log return for security i along the *jth* branching state) and this is

$$\mathbf{Z}^m = \Delta(\eta^m)\mathbf{C} \tag{2}$$

where C is a fixed $n \times (n + 1)$ matrix and $\Delta(\eta^m)$ is a diagonal matrix dependent on m containing scaling factors for each share. The purpose of the scaling factors is to reduce the jump magnitudes, as the number of jumps in a fixed interval of time increases towards infinity, with a view to preserving finiteness of the variance of the sum of all the jumps over the interval. Let p^m denote the probabilities of the $n + 1$ branches at each node *e*. Given C, the requirement that the mean and variance rates be μ_i and σ_i^2 completely determines p^m and n^m from the following equations:

$$\begin{bmatrix} \Delta(\eta^m) & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{C} \\ \mathbf{1}^T \end{bmatrix} p^m = \begin{bmatrix} \dfrac{\mu t}{m} \\ \mathbf{1} \end{bmatrix} \tag{3}$$

$$\eta_i^{m^2}(\Sigma_j p_j^m C_{ij}^2) = \frac{\sigma_i^2 t}{m} + \frac{\mu_i^2 t^2}{m^2} \qquad \text{for } i = 1, \ldots, n \tag{4}$$

with Equation (4) being an evaluation of the second moment about zero.

Theorem 2 establishes the existence of a well-defined underlying multinomial probability structure that approximates arbitrarily closely any multidimensional Brownian motion with drift for the vector process of share prices. The motivation behind this construction is Merton's (1982) analysis suggesting, for limiting sample path continuity, the choice of Z^m as $\Delta(\eta^m)\mathbf{C}$, where n^m is organized to be of order $\sqrt{1/m}$, in that $\eta^m/\sqrt{1/m}$ tends to a vector of positive constants.

Threorem 2. If $(C^T, 1)$ is invertible, and $\exists \pi > 0$, $C\pi = 0$, then Equations (3) and (4) can be solved for p^m, η^m with m sufficiently large.

Proof. **See the Appendix. ■**

Corollary 1. A sequence $p^m > 0$ of solutions to Equations (3) and (4) converges to π.

Proof. From Equation (4), observe that $\eta_k^{m^2} m$ converges to $\sigma_k^2 t$ and hence on multiplying the first n equations in (3) by $\sqrt{m}$ and taking limits we find that the limit of p^m as m tends to infinity must belong to the null space of C. It must, of course, also satisfy the last equation in (3). Under our assumptions this limit must be the unique vector π. ■

The assumptions on C are not restrictive. First note that the invertibility of $(R^{m^T}, \mathbf{1})$ ensures a complete markets structure while the existence of π

> 0 satisfying $R^m\pi = \mathbf{1}$ is equivalent (by the Farkas theorem) to supposing that no linear combination of the risky securities is risk-free. These properties are expected to be carried down to the invertibility of $(\mathbf{C}^T, \mathbf{1})$ and the existence of $\pi > 0$ in the null space of $\mathbf{C}$ on noting that $R^m - \mathbf{U}$ is approximately equal to $\Delta(\eta^m)\mathbf{C}$ for large m, where U is the $n \times (n + 1)$ matrix with all entries unity. The rows of C may, without loss of generality, be resealed to ensure that $\Sigma_j\pi_j C_{ij}^2 = 1$, and in this case the limiting correlation matrix for the log returns is $\mathbf{C}\Delta(\pi)\mathbf{C}^T$. Given π, $\mathbf{C}$ therefore may be partially identified from estimated correlations.

Three steps are involved in showing that the Black-Scholes formula may be obtained as the limit of the Arrow-Debreu option valuation formula of Equation (1). The first step reduces the multinomial probability $P_x(A)$ in the limit to that of a standard normal probability $N(z)$ for a suitably constructed point z. The second step establishes the limit of z as m tends to ifinity. It is here that the continuous and discontinuous cases show their difference, essentially because a Taylor series analysis, valid for the continuous case, fails for the jumps. Finally, one substitutes the limiting value of z into $N(z)$ and then into Equation (1) for the appropriate $P_x(A)$ to obtain the Black-Scholes formula.

The application of Lemma A1 (see Appendix) yields the following asymptotically valid approximation for $P_{x^m}(A)$, $x^m = q^m, v^m$:

$$P_{x^m}(A) \approx N(z^m)$$

where

$$z^m = [\ln S_0/K + mp^{m^T}\zeta^m + m(x^m - p^m)^T\zeta^m]/w^m$$
$$w^{m^2} = \sqrt{m}(\zeta^m - \zeta^m_{n+1})^T V^m \sqrt{m}(\zeta^m - \zeta^m_{n+1})$$
$$V^m = x^m - x^m x^{m^T}$$
$$\zeta_k = \ln \rho_k$$

The Black-Scholes formula is obtained in the limit by showing that the limit of z^m as m tends to infinity for $x^m = q^m$, ι is the appropriate Black-Scholes argument of the N function. An important step in this regard is the behavior of $m(x^m - p^m)^T\zeta^m$, the first row of $mZ^m(x^m - p^m)$ and its limit is determined in Theorem 3.

Theorem 3. The limit as m tends to infinity of q^m is π, and

$$[\mathbf{Z}^m(q^m - p^m)]_i = \frac{(\ln r - \mu_i - \sigma_i^2/2)t}{m} + o\left(\frac{1}{m}\right) \tag{5}$$ *

while the corresponding equation for v^m is

$$[\mathbf{Z}^m(v^m - p^m)]_i = \frac{(\ln r - \mu_i - \sigma_i^2/2)t}{m} + \frac{\sigma_1\sigma_i t}{m}\Sigma_j\pi_j C_{1j}C_{ij} + o\left(\frac{1}{m}\right) \tag{6}$$

* $o(x)$ is the order notation indicating a function converging to zero faster than x, in that $o(x)/x$ tends to zero as x tends to zero [Merton (1982, p. 24)].

and the limit of v^m as m tends to ingfinity is also π.

Proof: The normalized primitive prices q (see 'the observation preceding Theorem 1) are defined by

$$\begin{bmatrix} r^{-t/m}R^m \\ \mathbf{1}^T \end{bmatrix} q^m = \mathbf{1} \tag{7}$$

Let $\mathbf{U}$ denote a conformable matrix with all entries unity. Then (7) may be written as

$$\begin{bmatrix} r^{-t/m}R^m - \mathbf{U} \\ \mathbf{1}^T \end{bmatrix} q^m = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{8}$$

From Equation (2), we have $r^{-t/m}R^m - \mathbf{U} = \exp(-t/m \ln r\mathbf{U} + \Delta(\eta^m)\mathbf{C})$ - $\mathbf{U}$ (where the exponentiation is entrywise). Then, noting that η^m behaves like a nonzero multiple of $1/\sqrt{m}$, a Taylor series expansion yields

$$r^{-t/m}R^m - \mathbf{U} = \Delta(\eta^m)\mathbf{C} - \frac{t}{m}\ln r\mathbf{U} + \frac{1}{2}[\Delta(\eta^m)^2\mathbf{C}*\mathbf{C}] + o\left(\frac{1}{m}\right) \tag{9}$$

where $\mathbf{C}*\mathbf{C}$ denotes entrywise multiplication. Substituting (9) into the first n equations of (8) yields (noting $\mathbf{U}q^m = 1$)

$$\Delta(\eta^m)\mathbf{C}q^m = \frac{t}{m}\ln r\mathbf{1} - \frac{1}{2}[\Delta(\eta^m)^2\mathbf{C}*\mathbf{C}q^m] + o\left(\frac{1}{m}\right) \tag{10}$$

Subtracting from (10) the first n equations of (3) and writing $\mathbf{Z}^m$ for $\Delta(\eta^m)\mathbf{C}$ yields

$$[\mathbf{Z}^m(q^m - p^m)]_i = \frac{(\ln r - \mu_i - \sigma_i^2/2)t}{m} - \frac{1}{2}\{[\Delta(\eta^m)^2\mathbf{C}*\mathbf{C}q^m]_i - \sigma_i^2 t/m\} + o\left(\frac{1}{m}\right) \tag{11}$$

The Appendix completes the proof by showing that the term in curly brackets is also $o(1/m)$.

In the Appendix, the analogous Equation (6) for the primitive prices v^m is also established. ■

Equation (5), established by Theorem 3, relates log return expectations under pseudoprobabilities q to those under p. The negative of the difference is the risk premium, which in the limit is given by the approximation $(\mu_i + \sigma_i^2/2 - \ln r)t/m$ for security i.

Noting that ζ^m is the first row of Z^m, Equations (5) and (6) yield

$$\lim_{m\to\infty} m(q^m - p^m)^T\zeta^m = \left(\ln r - \mu_1 - \frac{\sigma_1^2}{2}\right)t \tag{12}$$

$$\lim_{m\to\infty} m(v^m - p^m)^T\zeta^m = \left(\ln r - \mu_1 + \frac{\sigma_1^2}{2}\right)t \tag{13}$$

Equations (12) and (13) capture the effects of shifting from the original probabilities to the normalized Arrow-Debreu prices or pseudoprobabilities in the computation of expectations. Under universal risk neutrality, Equation (12) must equal zero, and Equation (13) is then just the variance $\sigma_1^2 t$. Observe that (12) is the covariance of ζ^m with q^m/p^m, and as the latter is precisely the density of the Arrow-Debreu measure with respect to the probability system given by the p's, it follows that (12) is the negative of the risk premium for log returns on the first security. On the other hand, (13) is the negative of the risk premium plus the risk measured by $\sigma_1^2 t$.

Lemma A2 (see Appendix) shows that $mp^{m^T} \zeta^m = \mu_1 t$ and $\lim w^m = \sigma_1 \sqrt{t}$. The substitution of these limits along with (12) and (13) into the expression for z^m yields the Black-Scholes formula

$$\lim_{m \to \infty} W_0 = S_0 N(d_1) - Kr^{-t} N(d_2)$$

$$d_1 = \frac{\ln(S_0/K)}{(\sigma_1 \sqrt{t})} + \left(\frac{\ln r}{\sigma_1} + \frac{\sigma_1}{2}\right)\sqrt{t}$$

$$d_2 = d_1 - \sigma_1 \sqrt{t}$$

An important observation about this result is that the limiting option value is independent of the particular choice of **C**. In complete markets the option is a redundant security with the completeness obtained from the full rank of **C** and the associated unique q. Because the limiting option price does not depend on the entries of **C**, this shows that the Black-Scholes formula can be relevant to a variety of complete market economies.

In incomplete markets, there would typically be a set of primitive prices associated with the marginal rates of substitution of different individuals. These could be used to obtain personal valuations of a European call option on any share, but in general these personal valuations will differ, reflecting the potential gains from trade. Of course, the introduction of the option as a traded security in the incomplete markets equilibrium will result in a redetermination of marginal rates of substitution that result in an equalization of personalized option valuations to its price. The variation of personal valuations in an incomplete markets equilibrium before its introduction as a traded security reflects the nonredundancy of the option in incomplete markets. However, using the above results, an asymptotic redundancy is possible if one structures the incomplete markets equilibria such that the personalized option valuations converge on each other and the Black-Scholes formula. In order to ensure that the various **C** matrices are consistent with the same economy, they must imply the same state probabilities and hence the same π. In this case, all primitive prices are convergent to each other and to π.

3. The Discontinuous Case

We show in this section that in the discontinuous case, the limiting multinomial formula is precisely Merton's (1976) formula with one change:

jump probabilities must be replaced by corresponding Arrow-Debreu prices. For the discontinuous case, the log return matrix Z^m is partitioned as follows:

$$\mathbf{Z}^m = (\Delta(\eta^m)\mathbf{C}, \mathbf{G}) \tag{14}$$

where C and G are fixed matrices of dimension $n \times c$ and $n \times d$, respectively, $c + d = n + 1$, and the columns of $\mathbf{G}$ correspond to the jump states with entries (some of which may be zero) giving the effects on the logarithm of the corresponding security prices. Partitioning p^m into (p^{cm}, p^{dm}), the equations defining p^m, η^m; analogous to Equations (3) and (4) now are

$$\begin{bmatrix} \Delta(\eta^m)\mathbf{C} & \mathbf{G} \\ \mathbf{1}^T & \end{bmatrix} \begin{bmatrix} p^{cm} \\ p^{dm} \end{bmatrix} = \begin{bmatrix} \frac{\mu t}{m} \\ 1 \end{bmatrix} \tag{15}$$

$$\sum_{j=1}^{c} p_j^{cm} C_{ij}^2 \eta_i^{m^2} + \sum_{k=1}^{d} p_k^{dm} G_{ik}^2 = \frac{\sigma_i^2 t}{m} + \frac{\mu_i^2 t^2}{m^2} \tag{16}$$

Assume the columns of $\mathbf{G}$ and the rows of $\mathbf{C}$ are not zero. Consider multiplying Equation (16) by m and observe that the convergence of p^{cm} to π^c implies the existence of convergent subsequences for $m\eta_i^{m^2}$, mp_k^{dm} with respective limits, say, $\tilde{\sigma}_i^2 t$, π_k^d. It also follows on multiplying the first n equations of (15) by $\sqrt{m}$ and taking limits, provided $\tilde{\sigma}_i^2 > 0$ for all i, that π^c belongs to the null space of $\mathbf{C}$. Hence, one can structure the limits as in the continuous case, choosing $\mathbf{C}$ with a one-dimensional null space containing $\pi^c > 0$ with the rows of $\mathbf{C}$ scaled to ensure $\Sigma_{j=1}^{c} \pi_j^c C_{ij}^2 = 1$. $\tilde{\sigma}_i^2$ then has the interpretation of the limiting variance rate of the continuous component, being the limit of $\Sigma_{j=1}^{c} p_j^{cm} C_{ij}^2 \eta_i^{m^2} m/t$, while the mean $\mathbf{C}\pi^c = 0$. On the other hand, $\tilde{\pi}_k^d = \pi_k^d/(\mathbf{1}^T\pi^d)$ is the limiting probability of the kth jump given that there is a jump, with $(\mathbf{1}^T\pi^d)t/m$ being the limiting probability of a jump in an interval of length t/m.

To construct a system of discrete trees with a well-defined limiting process, Equations (15) and (16) must be shown to have convergent solutions as m tends to infinity. To assist in this construction, consider the limiting equations obtained on rewriting the equations as (note $\mathbf{C}\pi^c = 0$, $\mathbf{1}^T\pi^c = 1$)

$$\sqrt{\frac{m}{t}}\Delta(\eta^m)\mathbf{C}\sqrt{\frac{m}{t}}(p^{cm} - \pi^c) + \mathbf{G}\frac{m}{t}p^{dm} = \mu$$

$$\mathbf{1}^T\left(\sqrt{\frac{m}{t}}\right)(p^{cm} - \pi^c) + \mathbf{1}^T\sqrt{\frac{t}{m}}\frac{m}{t}p^{dm} = 0$$

$$\frac{[1 + \sqrt{t/m}\Sigma_j\sqrt{m/t}(p_j^{cm} - \pi_j^c)C_{ij}^2]\eta_i^{m^2} m}{t} + \sum_k \frac{G_{ik}^2 m}{t} p_k^{dm} = \sigma_i^2 + \frac{\mu_i^2 t}{m} \tag{17}$$

which in the limit yield

$$\begin{aligned}\Delta(\tilde{\sigma})C\theta + G\pi^d &= \mu\\ 1^T\theta &= 0\\ \tilde{\sigma}_i^2 + \sum_1 \pi_k^d G_{ik}^2 &= \sigma_i^2\end{aligned} \tag{18}$$

Let x denote the vector $(\theta, \pi^d, \tilde{\sigma})$ and x_m denote the vector $[\sqrt{m/t}(p^{cm} - \pi^c), m/tp^{dm}, \sqrt{m/t}\eta^m]$. Equations (18) may be written $\Phi(\mathbf{x}) = (\mu, 0, \sigma^2)^T$ while (17) asserts $\Phi_m(\mathbf{x}_m) = (\mu, 0, \sigma_m^2)^T$, where $(\sigma_m^2)_i = \sigma_i^2 + \mu_i^2 t/m$. Since $\Phi_m(\mathbf{x}) - \Phi(\mathbf{x}) = [0, 1^T\sqrt{t/m}\pi^d, (\Sigma_j\sqrt{t/m}C_{ij}^2\theta_j)\tilde{\sigma}_i^2]$, follows that Φ_m converges uniformly on compact subsets of $\mathbf{x} = (\theta, \pi^d, \tilde{\sigma})$ to Φ. Solutions of Equations (17) therefore are related to those of (18). Now note that the dimension of the space $(\theta, \pi^d, \tilde{\sigma})$ is $c + d + n = 2n + 1$ and the equation 13 = 0 restricts choices to a 2n-dimensional linear subspace in which the inequalities $\pi^d > 0$ and $\tilde{\sigma} > 0$ constitute 2n-dimensional relatively open conditions. It follows that the restriction of Φ to $\mathcal{U} = \{(\theta, \pi^d, \tilde{\sigma}) \mid 1^T\theta = 0, \pi^d > 0, \tilde{\sigma} > 0\}$ i is a smooth map from an open subset of a smooth manifold of dimension 2n into the 2n-dimensional space of triples $(\mu, 0, \sigma^2)$. By Sard's theorem [Milnor (1965)] the set of critical values of F has Lebesgue measure 0, and so for almost all choices of $(\mu, 0, \sigma^2)$ in the range of Φ, $\nabla\Phi$ has rank *2n.*

Theorem 4 is analogous to Theorem 2 in that it establishes for the discontinuous case the existence of an underlying multinomial probability structure that approximates in the limit a Poisson jump process for the vector of share prices.

Theorem 4. *If $(\mu\ 0, \sigma^2)$ is a regular value of the range of Φ, with $\Phi(x_0) = \Phi(\theta_0, \pi_0^d, \tilde{\sigma}_0) = (\mu, 0, \sigma^2)$, then there exists a neighborhood Ω of $\theta_0, \pi_0^d, \tilde{\sigma}_0$ such that, for sufficiently large m, Equations (17), $\Phi_m(x_m) = (\mu, 0, \sigma_m^2) = y_m$, possess a unique solution x_m in Ω.*

Proof. See the Appendix. ■

The local uniqueness offered by Theorem 4 is all that is desired, as the intention is to construct trees for arbitrarily large *m* with $\mathbf{x}_m$ convergent to $\mathbf{x}_0$.

The equations defining the primitive prices, analogous to (7) and employing Equation (2), are

$$\begin{bmatrix} r^{-t/m}e^{\Delta(\gamma^m)C} & r^{-t/m}e^{G} \\ 1^T & 1^T \end{bmatrix}\begin{bmatrix} q^{cm} \\ q^{dm} \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \tag{19}$$

where $(e^G)_{ik} = e^{G_{ik}}$. It is shown in Lemma A3 (see Appendix) that q^{cm}, q^{dm} are convergent to $\pi^c, 0$: and analogous to Φ, and the equations defining the limiting values λ, ϕ^d of $\sqrt{m/t}(q^{cm} - \pi^c), m/tq^{dm}$ are

$$\begin{bmatrix} \Delta(\tilde{\sigma})\mathbf{C} & e^{G} - \mathbf{U} \\ \mathbf{1}^{T} & 0 \end{bmatrix} \begin{bmatrix} \lambda \\ \phi^{d} \end{bmatrix} = \begin{bmatrix} \ln r - \tilde{\sigma}^{2}/2 \\ 0 \end{bmatrix} \tag{20}$$

where U is again a conformable matrix of unit entries. Comparison of Equation (20) with the first two sets of (18) shows that π^{d} and ϕ^{d} are not likely to be identical. However, like $\tilde{\pi}^{d}$, $\tilde{\phi}^{d} = (\mathbf{1}^{T}\phi^{d})^{-1}\phi^{d}$ gives the Arrow-Debreu prices for the events of particular jumps conditional on there being a jump, while $\omega = \mathbf{1}^{T}\phi^{d}$ is the Arrow-Debreu Poisson rate of jumps.

The equations relating the expectations of continuous component log returns under primitive prices q^{m}, n^{m} and probabilities p^{m} are given by the following theorem, proved in a manner analogous to Theorem 3. Theorem 5 is used in a manner analogous to the use of Theorem 3 in deriving a Merton (1976) type generalization of the Black-Scholes formula.

Theorem 5. The expected log returns using q^{m} and p^{m} for the continuous components are related by

$$[\Delta(\eta^{m})\mathbf{C}(q^{cm} - p^{cm})]_{i} = \{\ln r - [\Delta(\tilde{\sigma})\mathbf{C}\theta]_{i} - \tilde{\sigma}_{i}^{2}/2 - \omega\,[(e^{G} - \mathbf{U})\tilde{\phi}^{d}]_{i}\}\,t \,/m + o\left(\frac{1}{m}\right) \tag{21}$$

The analogous result for v^{m} is

$$[\Delta(\eta^{m})\mathbf{C}(v^{cm} - p^{cm})]_{i} = \{\ln r - [\Delta(\tilde{\sigma})\mathbf{C}\theta]_{i} - \tilde{\sigma}_{i}^{2}/2 - \omega[(e^{G} - \mathbf{U})\tilde{\phi}^{d}]_{i}\}\,t \,/m + \tilde{\sigma}_{1i}\,\Sigma_{j}\mathbf{C}_{1j}\mathbf{C}_{ij}\pi_{j}^{c} + o\left(\frac{1}{m}\right) \tag{22}$$

Proof. See the Appendix. ■

We can now establish the Arrow-Debreu form for Merton's generalization of the Black-Scholes formula in the presence of jump discontinuities.

Theorem 6. The limiting Arrow-Debreu option value for the jump case is given by

$$\lim_{m\to\infty} W_{0}^{m} = \sum_{b=0}^{\infty} \frac{(\omega t)^{b} e^{\omega t}}{b!} E_{Y_{b}}[Y^{b}N(d_{1}^{b}) - Kr^{-t}N(d_{2}^{b})]$$
$$d_{1}^{b} = \frac{\ln(Y^{b}/K)}{\tilde{\sigma}_{1}\sqrt{t}} + \left(\frac{\ln r - \omega[(e^{G} - \mathbf{U})\tilde{\phi}^{d}]_{1}}{\tilde{\sigma}_{1}} + \frac{\tilde{\sigma}_{1}}{2}\right)\sqrt{t} \tag{23}$$
$$d_{2}^{b} = d_{1}^{b} - \tilde{\sigma}_{1}\sqrt{t}$$

where $Y^{b} = S_{0}\Sigma_{r=1}^{b} X_{r}$, the X_{r} are i.i.d with probability $\tilde{\phi}_{k}^{d}$ of being $e^{G_{1k}}$, and $E_{Y_{b}}$ is the expectation operator with respect to the density of Y^{b}.

Proof. The limit of W_{0}^{m} is obtained by conditioning Equation (1) on the number and magnitude of jumps. Essentially, as a result of the multiplicative nature of compounding rates of returns and the instantaneity of the jumps in the limit, the jumps may be accounted for at the start and are incorporated into the variable Y^h for h jumps. The probabilities of the jump/no jump partition are given by the binomial, and this converges by standard arguments to the Poisson with parameter w. The rest of the analysis is identical to the continuous case, with expressions (21) and (22) replacing the roles of (5) and (6) and hence the extra term $\omega[(e^G - \mathbf{U})\bar{\phi}^d]_1$ in the argument for $N(x)$. The role of μ_t in the continuous case in expressions (5) and (6) is taken over by $\Delta(\tilde{\sigma})\mathbf{C}\theta$, which from Equation (17) is like a continuous component mean and disappears from the final expressions for reasons identical to those associated with the absence of μ_t in the Black-Scholes formula. ■

Unlike the continuous case, the limiting option value does depend on the choice of **C** and **G**, but only through the lower dimensional entities $\tilde{\sigma}_1, \phi^d$: and the associated jump magnitudes of the particular share. Hence, there could be asymptotic redundancy of the option value with jumps and incomplete markets, but this would be less likely.

4. Conclusion

The Cox, Ross, and Rubenstein binomial model was generalized to the multinomial case. Limits were investigated and shown to yield the Black-Scholes formula in the case of continuous sample paths for a wide variety of complete market structures. In the discontinuous case a Merton-type formula was shown to result, provided jump probabilities were replaced by their corresponding Arrow-Debreu prices.

Appendix

Proof of Theorem 2

Since $\pi > 0$ and $[\mathbf{C}^T, \mathbf{1}]^T\pi = [0, 1]^T$ there exists a bounded closed convex neighborhood V of π strictly interior to $\mathscr{R}^+_{n+1}$ that is mapped one-to-one onto a neighborhhod Ω of $[0, \mathbf{1}]^T$. The existence of a solution to Equations (3) and (4) is established using a fixed point for a function F on V. To define F, take p arbitrary in V, use (4) to solve for η (possible by strict nonnegativity of p) then observe that (3) rewritten as $[\mathbf{C}^T, \mathbf{1}]^T p' = \Delta(\eta, 1)^{-1}[\mu t/m, 1]^T$ has for sufficiently large m a right-hand side that is in Ω so the solution p' is in V and $p' = F(p)$ defines F whose fixed point is a solution to (3) and (4). ■

Lemma A1. Reduction of $P_x(A)$ to $N(z)$.

Proof. Define the (n + 1)-dimensional random vector $\mathbf{u}^b = (..u_j^b..)$ by $\mathbf{u}_j^b = 1$, if at time ht/m the event associated with the jth branching occurs, 0 otherwise. $\mathbf{u}^b$ satisfies $\mathbf{1}^T\mathbf{u}^b = \mathbf{1}$ and so its covariance matrix is degenerate.

For any vector s let the notation $\hat{\mathbf{s}}$ denote the truncated vector obtained from s by deleting its last entry. The covariance matrix of $\hat{\mathbf{u}}^b$ for branch probabilities given by $\hat{\mathbf{x}}$ is $V = \Delta(\hat{\mathbf{x}}) - \hat{\mathbf{x}}\hat{\mathbf{x}}^T$ and this is nonsingular for x > 0. $P_x(A)$ is the probability that $\nu \in A$ and $\nu = \Sigma_{b=1}^m \mathbf{u}^b$, which for large m is [Bhattacharya and Rao (1976, p. 184, Corollary 18.3)] distributed as a multivariate normal vector with the appropriate mean and covariance matrix, $\hat{\nu}$ being nondegenerate. Simple manipulations of the definition of A yield that $\nu \in A$ just if

$$\sqrt{m}(\hat{\zeta} - \zeta_{n+1}\mathbf{1})^T(1/\sqrt{m})(\hat{\nu} - m\hat{\mathbf{x}}) > -\ln(S_0/K) - mp^T\zeta - m(\mathbf{x} - p)^T\zeta$$

where $\zeta_j = \ln \rho_j$ for all j. The result follows noting the normality of the left-hand side. ■

Proof of Theorem 3 completed

Multiplying Equation (11) of the article by $\sqrt{m}$ and letting m tend to infinity one observes that the limit of q^m is in the null space of $\mathbf{C}$ and so must be π (note $q^m > 0, \mathbf{1}^Tq^m = 1$). Now multiply the term in curly brackets in Equation (11) by m, take limits, noting $m\eta_1^{m^2}$ convergent to $\sigma_1^2 t$, q^m convergent to π, and $\Sigma_j \pi_j C_{1j}^2 = 1$. Substitution of these facts into the limit yields the result. n

Demonstration of Equation (6) and the limit of v^m

As $v^m = r^{-t/m}\Delta(\rho^m)q^m$, substitute for q^m, $r^{t/m}\Delta(\rho^m)v^m$ in Equation (7) and conduct a Taylor series expansion analogous to that for Theorem 3 to derive, on substraction of the first n equations of (3),

$$[\Delta(\eta^m)\mathbf{C}(v^m - p^m)]_i = \frac{(\ln r - \mu_i)t}{m} - \frac{1}{2}[\Delta(\eta^m)^2\mathbf{C}^*\mathbf{C}v^m]_i$$
$$+ \eta_1^m \Delta(\eta^m)\Sigma_j C_{1j} C_{ij} v_j^m + o\left(\frac{1}{m}\right)$$

The result follows on an argument analogous to that of Theorem 3, noting on multiplication of the above equation by $\sqrt{m}$ and taking limits that v^m tends to π.

Lemma A2.

Proof: *(a)* $mp^{m^T}\zeta^m = \mu_1 t$ by construction for all m. *(b)* For the limit of w^m, note $\sqrt{m}\zeta$ tends to $\sigma_1\sqrt{t}C_{1j}$ and V tends to $\Delta(\hat{\pi}) - \hat{\pi}\hat{\pi}^T$. The limit of w^{m^2} is therefore $\sigma_1^2 t\psi^T[\Delta(\hat{\pi}) - \hat{\pi}\hat{\pi}^T]\psi$ where $\psi_j = C_{1j} - C_{1,n+1}$. The result follows on showing that the quadratic form is unity. For this calculation the reader is referred to Madan and Milne (1987). ■

Proof of Theorem 4

Let $x_0 = (\theta_0, \pi_0^d, \tilde{\sigma}_0)$, supposing $\Phi(x_0)$ is regular for Φ restricted to $\mathbf{1}^T\theta = 0$ and considered as a map into the space of vectors $(\mu, 0, \sigma^2)$. Let $\mathbf{A}$ denote the $2n \times (2n + 1)$ matrix of derivatives of Φ so viewed. The Jacobian&)

of Φ at x_0 is the determinant of A augmented by the row $(\mathbf{1}^T, 0, 0)$ and this must be nonzero for otherwise there exists $z = (\theta', \pi^{d\prime}, \tilde{\sigma}_0)$, $\mathbf{1}^T\theta' = 0$ and $Az = 0$ contradicting the supposed regularity of $\Phi(x_0)$. By the Corollary to Theorem 21 (pp. 276, 277) of Buck (1965), Φ is one-to-one in a neighborhood Ω of x_0. In order to solve $\Phi_m(x_m) = y_m$ parameterize m by $u = 1/m$ and write the equation to be solved generally as $\Phi(x, u) = y(u)$ where the dependence of y on m is incorporated in the expression $y(u)$. Now let $\Psi(x, u)$ be defined as $\Phi(x, u) - y(u)$, note that $\Psi(x_0, 0) = 0$ and by the regularity of y_0 that $\Psi_x = \Phi_x$ is nonsingular. The result follows on use of the implicit function theorem [Spivak (1965, p. 41)], treating u as the parameter space. ■

Lemma A3. Supposing $e^G - U$ is of full rank, q^{cm}, q^{dm} are convergent to π^c, 0 and $\sqrt{m/t}(q^{cm} - \pi^c)$, m/tq^{dm} are convergent to λ, ϕ^d defined by Equation (20) of the article.

Proof. Proceed as in the proof of Theorem 3 to deduce that

$$\sqrt{\frac{m}{t}}\Delta(\eta^m)\mathbf{C}q^{cm} + \sqrt{\frac{m}{t}}(e^G - \mathbf{U})q^{dm} = o\left(\frac{1}{\sqrt{m}}\right)$$

One must therefore have in the limit that $\mathbf{C}q^{cm}$ tends to zero as does $(e^G - \mathbf{U})q^{dm}$ whence it follows that q^{cm} tends to π^c and q^{dm} to zero. The Taylor series expansion used in deriving the above equation gives in greater detail that

$$\sqrt{\frac{m}{t}}\Delta(\eta^m)\mathbf{C}\sqrt{\frac{m}{t}}(q^{cm} - \pi^c) + \frac{(e^G - \mathbf{U})m}{t}q^{dm}$$
$$= \ln r(\mathbf{1}^T q^{cm}\mathbf{1} + e^G q^{dm}) - \frac{1}{2}\left(\frac{m}{t}\right)\Delta(\eta^m)^2\mathbf{C}^*\mathbf{C}q^{cm} + \left(\frac{m}{t}\right)o\left(\frac{1}{m}\right)$$

The first set of equations follow on taking limits. The second is a limiting consequence of

$$\mathbf{1}^T\sqrt{\frac{m}{t}}(q^{cm} - \pi^c) + \sqrt{\frac{t}{m}}\mathbf{1}^T\left(\frac{m}{t}\right)q^{dm} = 0$$

which is a rewrite of the condition that the q^m*s* and π^c*s* add to unity. ■

Proof of Theorem 5

Subtracting the first set of Equations (15) from the Taylor series approximation used in Lemma A3 one obtains that

$$\Delta(\eta^m)C(q^{cm} - p^{cm}) + (e^G - U)q^{dm} - Gp^{dm} = \left(\frac{t}{m}\right)(1^T q^{cm})\ln r1$$
$$- \frac{1}{2}\Delta(\eta^m)^2 C^* C q^{cm}$$
$$+ \left(\frac{t}{m}\right)(\ln r)e^G q^{dm} - \frac{\mu t}{m} + o\left(\frac{1}{m}\right) \qquad \text{(A1)}$$

Now note that

1. $(m/t)(t/m)e^G q^{dm}\ln r$ tends to zero so $(t/m)e^G q^{dm}\ln r$ is $o(1/m)$.
2. $(m/t)[\Delta(\eta^m)^2 C^* C q^{cm} - \bar{\sigma}^2 t/m]$ tends to zero so $\Delta(\eta^m)^2 C^* C q^{cm} - \bar{\sigma}^2 t/m$ is $o(1/m)$.
3. $m/t[(t/m)1^T q^{cm} - t/m]$ tends to zero so $(t/m)1^T q^{cm} - t/m$ is $o(1/m)$.
4. $m/t[Gp^{dm} - (t/m)G\pi^d]$ tends to zero so $Gp^{dm} - (t/m)G\pi^d$ is $o(1/m)$.
5. $m/t[(e^G - U)q^{dm} - (e^G - U)\phi^d(t/m)]$ tends to zero so $[(e^G - U)q^{dm} - (e^G - U)\phi^d(t/m)]$ is $o(1/m)$.

Replacing the terms in 1 to 5 above by their limits up to order l/m in Equation (A1) yields the result. ■

Demonstration of Equation (22)

Using $v^m = r^{-t/m}\Delta(\rho^m)q^m$ and following an argument analogous to that used for Equation (6) to deduce on subtracting the first n equations of (15) that

$$[\Delta(\eta^m)C(v^{cm} - p^{cm})]_i = (1^T v^{cm})\ln r - \frac{1}{2}\eta_i^{m^2}\Sigma_j C_{ij}^2 v_j^{cm}$$
$$+ \eta_1^m \eta_i^m \Sigma_j C_{1j} C_{ij} v_j^{cm}$$
$$+ [(e^G - U)\Delta(e^{-G1k})v^{dm}]_i - \frac{\mu_i t}{m} + (Gp^{dm})_i + o\left(\frac{1}{m}\right) \qquad \text{(A2)}$$

Note that the limits 1 to 4 in the proof of Theorem S above hold with v's replacing q's. For the limit 5, note that since the limit of m/tv_k^{dm} is equal to $e^{G1k}[\lim(m/t)q_k^{dm}]$ which equals $e^{G1k}\phi_k^d$ that $(m/t)[(e^G - U)\Delta(e^{-G1k})v^{dm} - (e^G - U)\phi^d t/m]$ tends to zero so $[(e^G - U)\Delta(e^{-G1k})v^{dm} - (e^G - U)\phi^d t/m]$ is $o(1/m)$. Also note that as $(m/t)[\eta_1^m\eta_i^m\Sigma_j C_{1j}C_{ij}v_j^{c} - \bar{\sigma}_1\bar{\sigma}_i\Sigma_j C_{1j}C_{ij}\pi_j^c t/m]$ tends to zero $[\eta_1^m\eta_i^m\Sigma_j C_{1j}C_{ij}v_j^{cm} - \bar{\sigma}_1\bar{\sigma}_i\Sigma_j C_{1j}C_{ij}\pi_j^c t/m]$ is $o(1/m)$. The result follows on substitution of these limits into (A2).

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