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Incomplete Diversification and Asset Pricing

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Abstract

Investors in equilibrium are modeled as facing investor specific risks across the space of assets. Personalized asset pricing models reflect these risks. Averaging across the pool of investors we obtain a market asset pricing model that reflects market risk exposures. It is observed on invoking a law of large numbers applied to an infinite population of investors that many personally relevant risk considerations can be eliminated from the market asset pricing model. Examples illustrating the effects of undiversified labor income and taste specific price indices are provided. Suggestions for future work on asset pricing include a need to focus on identifying and explaining investor specific risk exposures.

1 Introduction

Asset pricing theories explain risk premia on financial assets as compensating investors for risk exposures or risks that investors cannot diversify. The theories differ in their specification of these undiversifiable or systematic risks. In the Sharpe (1964), and Lintner (1965) Capital Asset Pricing Model investors are only exposed to the risks of the market portfolio. The Arbitrage Pricing Theory of Ross (1976) has investors exposed to a finite set of factor risks. While in the consumption beta model of Merton (1973) and Breeden (1975) investors face real consumption risk. In these theories investors reduce risk by diversifying their portfolios across the universe of assets.

This paper focuses on another dimension of diversification and the resulting asset pricing model, more akin to insurance. By aggregating across the risk exposures of a large number of investors we derive an asset pricing model that averages out many investor specific concerns. Hence even if investors have to take positions in many specific risks and diversification is incomplete at the individual investor level, many of these nondiversified risks need have no impact on the market prices of assets. For emphasis and exactness we model an economy with *infinitely many investors,* in which each single investor is insignificant.

Formally, we define, in a general and abstract setting, the concept of *investor specific risk exposures* in equilibrium in terms of measurability with respect to an appropriate investor specific $\sigma-$algebra of events. We then identify these investor specific risk exposures and relate them to *personalized asset pricing models.* An asset pricing model for the economy or a *market asset pricing model* is obtained by aggregating personalized asset pricing models across investors. The risk exposures compensated for in the market asset pricing model are similarly defined in terms of measurability and termed *market risk exposures*. A precise relationship between market and investor specific risk exposures modeled as $\sigma-$ algebras is also developed. The general model we present is abstract, but includes a range of examples. The general procedure of focusing on investor specific risk exposures and the associated personalized asset pricing models and then averaging over the set of investors is suggestive and offers considerable guidance for future research into asset pricing. The important insight gained from our analysis is precisely the proposition that future work on asset pricing needs to focus on identifying and explaining investor specific risk exposures across the pool of investors in addition to the more traditional focus on asset returns.

Section 2 presents the economic model. Investor specific risk exposures and the associated personalized asset pricing models are defined in Section 3. The relationship between investor specific and market risk exposures and personalized and market asset pricing models is presented in Section 4. Examples illustrating the theory for important special cases are presented in Section 5. Section 6 concludes.

2 The Economic Model

The economic model follows Milne(1988). There are two dates and an abstract set I of investors. The space of time 1 contingent cash flows over which investors have preferences is modeled as a separable Hilbert space V. Let $V = L^2(\Omega, \mathcal{F}, P)$ be the space of finite second moment random variables defined on the probability space $(\Omega, \mathcal{F}, P)$, where Ω is the set of events, $\mathcal{F}$ is a $\sigma-$ algebra of events and P is a probability measure. For generality we suppose that preferences are defined over an attainable convex set $X_i \subset V$.

Each investor $i \in I$ is supposed to have a monotone increasing, continuous and quasi-concave utility function u_i defined on X_i.[1] Cash flows at time 1 are obtained by holding assets at time 0. There are a finite set J of assets indexed by j, with claims to time 1 state contingent cash flows $Z_j \in V$ for all $j = 1, \cdots, J$.[2] Let the vector a_i denote investor i's holding of the J assets, the associated time 1 cash flow is given by the linear operator

$$Z[a] = \sum_{j=1}^{J} Z_j a_{ji},$$

that maps $\mathbb{R}^J$ into V. Each investor also has an initial endowment of assets of $\overline{a}_i$.

It is easily shown that the set of feasible asset portfolios for i, $A_i = Z^{-1}\left(X_i \cap Z[\mathbb{R}^J]\right)$, is convex. Define induced preferences on A_i by

$$u_i^*(a) = u_i(Z[a]).$$

These induced preferences inherit the properties of being continuous and quasi-concave from u_i and the linearity and continuity of the operator Z. Furthermore, we also suppose nonsatiation of u_i^* or the absence of bliss points.

Consider an economy with a countable infinity of investors.[3] One may therefore suppose, without loss of generality, that I, the index set for the investors, is the set of

[1] The utility function could represent the utility of consuming the entire cash flow at time 1 or it could represent the optimized utility of a dynamic program beginning at time 1. The utility function could also be used to represent the immediate one period objectives of institutional investors, firms or other members of the investing community.

[2] Extending the results of this paper to the case of infinitely many assets ia an interesting and useful research problem. There are however technical difficulties associated with the double infinity of assets and investors.

[3] We restrict to a countable infinity of individuals since the law of large numbers does not hold for a continuum, (See Judd(1983), Feldman and Gilles (1985)).

all natural numbers or positive integers. Since we wish to model individual investors as insignificant in the infinite economy, we follow Aumann (1964) and Ostroy (1984), by modeling investors as having zero measure. Accordingly we take the space of investors to be a finitely additive non-atomic measure space $(I, \mathcal{A}, \mu)$, where I is the set of positive integers, $\mathcal{A}$ is the algebra of all subsets of I and μ is a finitely additive measure that gives measure zero to all finite sets.[4] Only large and in particular infinite sets of individuals have positive measure. We normalize the population of investors and suppose that $\mu(I) = 1$, whereby $\mu(A)$ is the proportion of investors in the set A.

The specific choice of μ among the class of finitely additive non-atomic measures is not important for the general results we obtain. Different choices correspond to different limit economies. The measures of sets may be constructed as limits of weighted investor memberships in the set, with the limit taken over economies with finitely many investors, as the population size approaches infinity and simultaneously the weight of individual investors approaches zero. For further details on the construction of such finitely additive non-atomic measures on a countable set, the reader is referred to the Appendix. The theory of integration with respect to such measures is presented in Dunford and Schwartz (1988) and Leader (1953). We note here that all bounded functions are integrable in the L^1 sense. Equilibrium theory for exchange economies in this context is studied by Weiss (1981). We suppose that the endowment function $\overline{a} : I \to \mathbb{R}^J$ defined by the endowments $\overline{a}_i$ is μ integrable in i.

Equilibrium allocations for the limit economy, with finitely additive non-atomic measure space of investors, are unique up to perturbation by a null function. Formally, allocations are determined within equivalence classes with two allocations being equivalent if their difference is a null function. A function $h(i)$ is said to be a null function if it has a norm integrating to zero, i.e. $\int \|h(i)\| \, d\mu(i) = 0$. In this regard note that there do exist strictly positive null functions. A function $h(i)$ is null if, for all $\varepsilon > 0$, the measure of the set of investors for which $\|h(i)\|$ exceeds ε is zero. So for example the function $h(i) = 1/i$ is strictly positive and null. Null perturbations have no effect on the limits of average allocations taken over a sequence of economies with a population tending to infinity and the weighting of single investors approaching zero. It is precisely for this reason that, from the perspective of the limit economy, such perturbations are admissible without disturbing the limit equilibrium.

The definition of equilibrium used by Weiss (1981) is in terms of these equivalence classes of allocations. Equilibria have the property that investors may deviate from their utility maximizing allocations by a null function without disturbing the market clearing condition of the limit economy. A competitive equilibrium for the asset exchange economy over the infinite set of investors I is defined as follows:

Definition 1 *An* attainable allocation *is a μ integrable function $a : I \to \mathbb{R}^J$ such that $a_i \in A_i$ for all i and*

$$\int_I a_i d\mu(i) = \int_I \overline{a}_i d\mu(i).$$

Definition 2 *An attainable allocation is* budget feasible for the price system $p \in \mathbb{R}^J$ *if there exists a subset $A \subseteq I$ with $\mu(A) = \mu(I)$ and a null function $h : I \to \mathbb{R}^J$ such that*

$$p'(a_i - h_i) = p'\overline{a}_i \text{ for all } i \in A.$$

[4] The measure space is atomic if some subset of positive measure cannot be split into two sets of strictly lower measure. We shall take our measure space of individuals to be nonatomic, and hence single individuals have zero measure.

The definition of budget feasibility permits individual exceptions to the budget constraint for a null set of investors and for a non-null set by a null aggregate.

Definition 3 *A* competitive equilibrium *is an attainable allocation a^* and a price system p^* such that a^* is budget feasible for p^* and for some subset $A \subseteq I$,$\mu(A) = \mu(I)$ and null functions $h^* : I \to \mathbb{R}^J$, $k^* : I \to \mathbb{R}^1$*

$$u_i^*(a_i^* - h_i^*) \geq u_i^*(a_i^0) - k_i$$

where

$$a_i^0 = \underset{a_i}{ArgMax}\left[u_i^*(a_i)\,|p^{*\prime}a_i = p^{*\prime}\overline{a}_i\right].$$

This definition of competitive equilibrium permits null function, h^*, perturbations from a utility maximizing allocations with the resulting utilities also simultaneously perturbed by, k^*, another null function. Such perturbations have no effect on the limit economy and cannot be detected in the limit economy.

For the existence of such an equilibrium we require an assumption of uniform continuity on the utility functions across both i and the allocations. Such an assumption would be satisfied if marginal utilities were bounded in absolute value.

Assumption 1. For each $\varepsilon > 0$, there exists $\delta > 0$ such that $\|a - b\| \leq \delta$ implies that $|u_i^*(a) - u_i^*(b)| \leq \varepsilon$ for all $i \in I$.

Theorem 4 *Under assumption 1, and supposing that J is finite there exists a competitive equilibrium.*

Proof. (See Appendix) ■

3 Personalized Asset Pricing Models

Consider an Asset Exchange Economy in equilibrium as described in Section 2. For the purposes of this section we assume an appropriate differentiability of u_i. In fact, we suppose that u_i is the restriction to X_i of a function that is Fréchet differentiable on an open set containing X_i. This assumption enables us to define personalized marginal rates of substitution and identify, in Lemma 5 below, the structure of state contingent marginal utilities.

Lemma 5 *For each i, there exist random variables $\psi^i(\omega)$, $\psi_0^i(\omega) \in V^i$, $\psi^i, \psi_0^i \geq 0$ a.e. in Ω with respect to P, such that,*

$$\frac{\partial u_i^*(a_i^*)}{\partial aij} = \int \psi^i(\omega) Z_j(\omega) P(d\omega) \quad (1)$$

$$\frac{\partial u_i^*(a_i^0)}{\partial aij} = \int \psi_0^i(\omega) Z_j(\omega) P(d\omega) \quad (2)$$

Proof. See Appendix. ■

The random variables ψ^i and ψ_0^i are the marginal utilities of state contingent cash flows evaluated at the cash flows arising from the equilibrium and optimal asset holdings a_i^* and a_i^0 respectively.

Theorem 6 *For all i, the market price of traded assets p_j^* satisfies*

$$p_j^* = E^P\left[\lambda_0^i Z_j\right]. \tag{3}$$

Proof. Since u_i^* is maximized for all i with respect to the budget constraint, the first order condition implies that

$$\frac{\partial u_i^*(a_0^i)}{\partial a_{ij}} = \gamma_0^i p_j^*. \tag{4}$$

The result follows from (2) on defining $\lambda_0^i = \frac{\psi_0^i}{\gamma_0^i}$. ■

Define the valuation operators, $\Phi_0^i[x], \Phi^i[x]$ by

$$\Phi_0^i[x] = E^P[\lambda_0^i x] \text{ for } x \in V \tag{5}$$

$$\Phi^i[x] = E^P[\lambda^i x] \text{ for } x \in V \tag{6}$$

where $\lambda^i = \frac{\psi^i}{\gamma_0^i}$.

The random variables λ_0^i, λ^i are state price functions (Duffie (1988)) and define the state contingent discount to be applied to future or time 1 cash flows in determining their contribution to current values. The linear operator Φ^i defined by (6) provides a personalized valuation of x by measuring the sacrifice in terms of time 0 wealth that will compensate investor i for giving up a marginal unit of the time 1 cash flow x.

These personalized valuations of traded cash flows Z_j, given by $\Phi^i[Z_j]$, do not in general equal the market price $p_j^* = \Phi_0^i[Z_j]$. The next section presents sufficient conditions implying that personalized valuations and market prices differ by a null function. Under these conditions, for all but finitely many investors, personalized valuations and market prices are arbitrarily close. We therefore have in the linear operators Φ^i a sequence of personalized asset pricing linear operators. We now wish to propose, in a general and abstract setting, a definition for the concept of *investor specific risk exposures* in equilibrium. The basic intuition motivating this definition is that we may treat investor i as unconcerned about the risk, or risk neutral in equilibrium, if it is the case that $\Phi^i[x] = E^P[x]$. In this case there are no risks that particularly concern investor i, we have risk neutrality at the margin for valuation.

More generally, we say that the $\sigma-$algebra $\mathcal{G}^i$ characterizes investor $i's$ risk exposure in equilibrium if for all x, $\Phi^i[x] = \Phi^i[E^P[x\left|\mathcal{G}^i\right.]]$, or investor i is indifferent at the margin between x and $E^P\left[x\left|\mathcal{G}^i\right.\right]$. Hence, if we have risk neutrality at the margin conditional on $\mathcal{G}^i$, then investor specific risk concerns are characterized by $\mathcal{G}^i$.

A simple example illustrates the situation further. Define a tree representing the uncertainty resolution of two equally likely Bernoulli outcomes. The first represents good(G) or bad (B) health for the individual and the second represents cloudy (C) or sunny (S) weather conditions in some distant country. There are in all four equally states labeled GC, GS, BC, and BS respectively. Suppose a cash flow x pays the amounts $1, 2, 3$ and 4 thousand dollars in these four states. If the investor's equilibrium state price function turns out to be insensitive to weather conditions in the distant country but responsive to her state of health, with the state price function taking on for example the values 0.1, 0.1, 0.3, and 0.3 in the states GC, GS, BC and BS then the weather in the distant country is not a risk concern while her state of health is. The personal valuation of x, $\Phi^i[x]$ is the same as the personal valuation of $E^P\left[x\left|\text{state of health}\right.\right]$, or the cash flow 1.5, 1.5, 3.5, and 3.5. The investor may be thought of as first averaging out events with respect to which no risk adjustment turns out to be necessary in equilibrium, and then prices the resulting cash flow, taking account of personally required risk compensations.

Definition 7 *The $\sigma-$algebra $\mathcal{G}^i$ defines investor i's risk exposure in equilibrium if $\mathcal{G}^i$ is the smallest $\sigma-$ algebra satisfying*

$$\Phi^i[x] = \Phi^i\left[E^P\left[x\left|\mathcal{G}^i\right.\right]\right].$$

If the value of x to i, at the margin equals the value to i of the expectation of x conditional on $\mathcal{G}^i$, then investor i is marginally, $\mathcal{G}^i$ conditionally, risk neutral. Hence investor $i's$ risk concerns or relevant risk exposures are captured in the $\sigma-$algebra $\mathcal{G}^i$. The example motivating this definition suggests that $\mathcal{G}^i$ is related to the sensitivity of equilibrium marginal rates of substitution to events. This suggestion is confirmed in Theorem 8 below. Specifically let $\mathcal{F}^i = \sigma(\lambda^i)$ be the smallest $\sigma-$ algebra with respect to which λ^i is measurable.

Theorem 8 $\mathcal{G}^i = \sigma(\lambda^i)$.

Proof. We first show that $\mathcal{G}^i \subseteq \mathcal{F}^i$. This is accomplished by showing that $\mathcal{F}^i$ satisfies the defining condition of $\mathcal{G}^i$.

$$\begin{aligned}
\Phi^i[x] &= E^P\left[\lambda^i x\right] \\
&= E^P[E^P\left[\lambda^i x\left|\mathcal{F}^i\right.\right]] \\
&= E^P[\lambda^i E^P[x\left|\mathcal{F}^i\right.]] \\
&= \Phi^i[E^P[x\left|\mathcal{F}^i\right.]]
\end{aligned}$$

For the inclusion in the other direction, it follows from the definition of $\mathcal{G}^i$, that for all x

$$\Phi^i[x] = \Phi^i[E^P[x\left|\mathcal{G}^i\right.]]$$

However, this implies that for all x, as $\mathcal{G}^i \subseteq \mathcal{F}^i$ and λ^i is $\mathcal{F}^i$ measurable that

$$\begin{aligned}
E^P[\lambda^i x] &= E^P\left[\lambda^i E^P[x\left|\mathcal{G}^i\right.]\right] \\
&= E^P[E^P[\lambda^i\left|\mathcal{G}^i\right.]E^P[x\left|\mathcal{G}^i\right.]] \\
&= E^P\left[E^P[\lambda^i\left|\mathcal{G}^i\right.]x\right]
\end{aligned}$$

or that equivalently

$$E^P\left[\left(\lambda^i - E^P[x\left|\mathcal{G}^i\right.]\right)x\right] = 0$$

for all x.Taking for x the random variable $\left(\lambda^i - E^P[x\left|\mathcal{G}^i\right.]\right)$ we conclude that λ^i is $\mathcal{G}^i$ measurable or that $\mathcal{F}^i \subseteq \mathcal{G}^i$. ■

Risk exposures of concern to investors in equilibrium are most generally and abstractly given by the $\sigma-$ algebra of events $\mathcal{G}^i$. Theorem 8 shows that this is precisely the $\sigma-$ algebra generated by the single random variable λ^i, that essentially describes equilibrium personalized state prices of events. The key to understanding equilibrium asset pricing in terms of traditional factor model representations lies in describing the measurability of λ^i in terms of some factors. Both linear and nonlinear factor representations of linear pricing rules given by λ^i are possible.

Consider first linear representations. Invoking the separability of V, let $Q = \{q_k, k = 1, 2, \cdots\}$ be a countable orthonormal basis for V. Since V is self dual, λ^i is in the closed linear span of Q and we may write that

$$\lambda^i = \sum_{k=1}^{\infty} \phi_{i,k} q_k \tag{7}$$

Define

$$Q^i = \left\{q_k\left|\phi_{i,k} \neq 0\right.\right\}$$

as the set of basis elements that is actually required to span λ^i. For purposes of simplification or empirical approximation we may suppose that Q^i is finite. Standard arguments now enable us to derive the personalized asset pricing model

$$\mu = \gamma_0^i + \beta^i \gamma_1^i \tag{8}$$

where μ is the vector of asset mean returns on the traded assets, β^i is the matrix of assets betas with respect to the elements of Q^i and γ_0^i, γ_1^i are constants. Expression (8) is written as an approximation for this this economy on two counts. First, asset prices are approximately given by the operators Φ^i, with the difference being arbitrarily small for all but finitely many investors, and second an approximation may be involved in getting Q^i to be finite.

The number of factors involved in the linear representation (7) may be unduly large if λ^i is in fact a nonlinear function of a few factors, say

$$\lambda^i = \lambda(S_1, S_2, \cdots, S_{K(i)}) \tag{9}$$

where $S_1, S_2, \cdots, S_{K(i)}$ are the $K(i)$ factors needed to describe nonlinearly the variations in the measurability of λ^i. Equation (8) provides us with a $K(i)$ dimensional nonlinear representation of λ^i. This may be further reduced to a linear model by introducing as separate factors the products of powers of the primary factors in the nonlinear representation. The representation (9) clearly subsumes (7) and allows for more powerful dimensional reductions of λ^i at the cost of more complex associated asset pricing models as one loses the representation (8).[5]

In summary, investor specific risk concerns defined by $\mathcal{G}^i$ are identified as $\sigma(\lambda^i)$ and result in the linear in factors personalized asset pricing models given by equation (8) or equivalently the nonlinear in factors representation (9). These asset pricing models hold approximately for almost all investors in equilibrium for economies with a large number of investors.

4 The Systematic Risk Exposures

The risk exposures relevant to investor i in equilibrium are given by the $\sigma-$ algebras $\mathcal{F}^i$ and yield approximate personalized asset pricing models with respect to the factors Q^i for all i. The risks Q^i are relevant for assessing the personalized valuations of cash flows by individuals. On the other hand market risk exposures or systematic risks are risks that are relevant for assessing the market prices of securities or traded state contingent cash flows. These risks will be shown to define a $\sigma-$ algebra $\mathcal{M}$ that is identified and related to the $\mathcal{F}^{i\prime}s$ or $\mathcal{G}^{i\prime}s$ in this section. In particular we shall observe that $\mathcal{M}$ can be considerably smaller than the union of the $\mathcal{F}^{i\prime}s$. Alternatively, the factors present in the market asset pricing model can be smaller than the union of the $Q^{i\prime}s$. Many risk concerns relevant to particular investors in equilibrium can be eliminated from relevance for market valuation by the law of large numbers applied to the average of the personalized valuations procedures. In this sense the market can be viewed as an implicit insurer of personalized risk exposures and this insurer does not face the multitude of specific risk concerns affecting the diverse population of the insured, by essentially an application of the law of large numbers.

It is first established that the average of personalized values equals market prices. This is done by showing that the operator $\Phi^i - \Phi_0^i$ is a null operator in that for all x, the function $\Phi^i[x] - \Phi_0^i[x]$ is a null function of i. For this theorem we employ a condition on

[5] We shall follow the more traditional representations (7) and (8) in the rest of this paper.

the norm boundedness of the first and second Fréchet differentials of the utility functions u_i.

Assumption 2. Suppose that u_i is twice Fréchet differentiable and that there exists a constant C such that, $\|Du_i[x;\cdot]\|$ and $\|D_2u_i[x;\cdot]\|$ is uniformly bounded by C for all x and i.

Theorem 9*Assumption 2 implies that $\Phi^i - \Phi^i_0$ is a null operator.*

Proof. See Appendix ■

Suppose Assumption 2 and let Φ be the average of the operators Φ^i, more precisely

$$\Phi[x] = \int_I \Phi^i[x] d\mu(i).$$

The norm boundedness of Φ^i under assumption 2 implies that Φ is a continuous linear functional on V and hence there exists λ such that

$$\Phi[x] = E^P[\lambda x]. \tag{10}$$

Define $\mathcal{M}$ to be the smallest $\sigma-$ algebra with respect to which λ is measurable. We will show that unlike the operators Φ^i, Φ agrees with market prices for traded assets. Furthermore, there is a precise relationship between the $\sigma-$ algebra $\mathcal{M}$ and the $\sigma-$algebras $(\mathcal{F}^i, i \in I)$, whereby $\mathcal{M}$ is considerably smaller than the union of the $\mathcal{F}^{i\prime}s$. Hence, $\mathcal{M}$ is a candidate for a relatively parsimonious specification of market risk exposures or systematic risks.

Theorem 10*For all $j \in J$*

$$p^*_j = \Phi[Z_j] = E^P[\lambda Z_j] \tag{11}$$

Proof. For all traded assets j

$$p^*_j = \Phi^i_0[Z_j]$$

Since $\Phi^i - \Phi^i_0$ is a null operator by theorem (4), it follows that

$$\begin{aligned} p^*_j &= \int_I \Phi^i_0[Z_j] d\mu(i) \\ &= \int_I \Phi^i[Z_j] d\mu(i) \\ &= \Phi[Z_j] \\ &= E^P[\lambda Z_j]. \end{aligned}$$

■

Theorem 11*$\mathcal{M}$ is contained in the asymptotic or tail algebra associated with the sequence $\mathcal{F}^i$*

$$\mathcal{M} \subseteq \bigcap_{n=1}^{\infty} \bigcup_{k \geq n} \sigma(\lambda^k) \tag{12}$$

Proof. Let n be an integer and set

$$\mathcal{H}_n = \bigcup_{k \geq n} \sigma(\lambda^k)$$

Since the finite set of integers $1, \cdots, n$ has zero measure, it follows that for all x

$$\begin{aligned}\Phi[x] &= \int_{k\geq n} E^P\left[\lambda^k x\right] d\mu(k) \\ &= \int_{k\geq n} E^P\left[\lambda^k E^P\left[x\left|\mathcal{H}_n\right.\right]\right] d\mu(k) \\ &= \Phi\left[E^P\left[x\left|\mathcal{H}_n\right.\right]\right].\end{aligned}$$

Hence,

$$\begin{aligned}E^P[\lambda x] &= E^P\left[\lambda E^P\left[x\left|\mathcal{H}_n\right.\right]\right] \\ &= E^P\left[E^P\left[\lambda\left|\mathcal{H}_n\right.\right] E^P\left[x\left|\mathcal{H}_n\right.\right]\right] \\ &= E^P\left[E^P\left[\lambda\left|\mathcal{H}_n\right.\right] x\right]\end{aligned}$$

or that for all x we have

$$E^P\left[\left(\lambda - E^P\left[\lambda\left|\mathcal{H}_n\right.\right]\right) x\right] = 0$$

which implies that λ is $\mathcal{H}_n$ measurable for all n and so the result follows. ■

The inclusion (12) provides the relationship between market and systematic risks and the risk exposures relevant to individual investors. Formally, $\mathcal{M}$, the $\sigma-$algebra of the market risks is contained in the tail algebra of the risks relevant to individual investors. The tail algebra can be considerably smaller than the union of the individual $\sigma-$ algebras $\sigma\left(\lambda^i\right)$. Hence many risk factors relevant to individual investors need not be important in the market place for pricing assets. A sufficient condition useful in providing examples where $\mathcal{M}$ is considerably smaller is given by the following theorem.

Theorem 12 *If $\mathcal{D} \subseteq \mathcal{F}^i$ for all i and the sequence of $\sigma-$algebras $\mathcal{F}^i$ are conditionally independent, conditional on $\mathcal{D}$ Then $\mathcal{M} = \mathcal{D}$.*

Proof. This is a consequence of the conditional zero-one law (See Appendix). ■

The variables defining $\mathcal{D}$ measurability can be linked to risks accounted for in determining insurance premiums, the additional variables needed to define $\mathcal{F}^i$ measurability are personal risks that the insurer avoids through aggregating across the pool of insurers. Hence, life insurance premiums may vary with smoking habits as this has been isolated as an important part of $\mathcal{D}$, while many other factors affecting personal life risks, elements of $\mathcal{F}^i$, are ignored for the purpose of setting life insurance premiums.

If we define by Q^M the basis elements needed to span λ in equation (11) then by standard arguments we may derive the exact asset pricing model

$$\mu = \gamma_0 + \beta\gamma_1 \tag{13}$$

where β is now the matrix of asset beta's with respect to the elements of Q^M. Unlike expression (8), equation (13) is exact as the operator (11) gives asset prices exactly. Since $\mathcal{M}$ is contained in the tail algebra of the $\mathcal{F}^{i\prime}s$ the number of factors represented in (13) is expected to be considerably smaller than the union of all factors represented in the personalized asset pricing models.

5 Asset Pricing Examples

Our examples focus on two fundamental sources of investor specific risk exposure in equilibrium. These are i) the effects of non-traded assets or endowment effects and ii) the effects of direct utility based variables. We specialize the probability space and suppose that $\mathcal{F}$ is generated by the following:

1.a set of K random variables, denoted $S = (S_1, \cdots, S_K)$,

2.a sequence of investor specific real random variables y_i for all $i \in I$,

3.a sequence of investor specific vector random variables v^i for all $i \in I$.

4.a finite set of real random variables u_j for $j \in J$.

For the individual utility functions, we suppose that each investor i has a state preference utility function of the form

$$U_i = u_i(w_i, S, v^i) \tag{14}$$

where w^i denotes wealth at time 1 attained as a consequence of the portfolio held at time 0, and S,v^i are state variables that affect the utilities of investor i.

Each individual is also endowed with an initial holding of traded assets $\overline{a}_i \in \mathbb{R}^J, i \in I$. We suppose that time 1 wealth reflects the effects of both portfolio holdings and non-traded assets and so

$$w_i = Z[a_i] + y_i. \tag{15}$$

Define the linear projection of asset cash flows on the space generated by the random variables S, by

$$Z_j = \alpha_j + \beta_j' S + u^j \tag{16}$$

where α_j is a constant, β_j is a $K-$dimensional vector. The operator $Z[a]$ can then be written

$$Z[a] = \alpha' a + S' B a + u' a$$

where α is the vector of coefficients α_j and B is a matrix with K rows $\beta_j, j = 1, \cdots, K$. The single investor's utility function may now be written as

$$U_i = u_i(\alpha' a_i + S' B a_i + u' a_i + y_i, S, v^i).$$

It follows from the specification of u_i and the Fréchet differentiability of u_i with respect to the traded time 1 cash flow w that the Fréchet differential of $u_i, \delta u_i(w, h)$ takes the form

$$\delta u_i(w, h) = \int_\Omega \psi_i(w, S, v^i) h(\omega) P(d\omega)$$

from which it follows that investor $i's$ state price function has the form

$$\lambda^i = \Lambda^i(\alpha' a_i + S' B a_i + u' a_i + y_i, S, v^i)$$

where $\Lambda^i = \psi_i / \gamma_0^i$.

The risk factors priced by investor i in equilibrium are therefore given by

$$\mathcal{F}^i = \sigma(\lambda^i) \subseteq \sigma\left(S, v^i, u' a_i + y_i\right) = \mathcal{K}^i$$

where $\sigma(X)$ refers to the smallest $\sigma-$ algebra with respect to which the vector of variables X is measurable. Within this general framework we can discuss a number of special cases that have received attention in the literature.

First consider models in which both v^i and y_i are absent. For example, Ross (1976), Connor(1984), Milne (1988) discuss the diversification of the idiosyncratic components $u' a_i$ by essentially setting out conditions under which each $u' a_i$ is zero for each i. The factors then reduce to S with no necessity of invoking a law of large numbers. The associated conditions on preferences and asset returns are however quite strong. Milne

(1988) also discusses approximate asset pricing models with $u'a_i$ approaching zero as the number of assets approaches infinity.

Another line of attack is exemplified by the work of Grinblatt and Titman (1983) and Dybvig (1983). This approach obtains approximate asset pricing models by bounding the sensitivity of λ^i to $u'a_i$ via preference restrictions but allows $u'a_i$ to be possibly non-zero.

In the light of theorem 11, however, and recognizing that one may aggregate over the set of investors in obtaining asset pricing models we observe that it is not necessary that investors be able to diversify away firm specific risks completely to the point that $u'a_i$ is equal to zero. In fact letting $\varepsilon_i = u'a_i + y_i$ we have that

$$\lambda^i = \Lambda^i(\alpha' a_i + S'Ba_i + \varepsilon_i, S, v^i) \tag{17}$$

and provided (v^i, ε_i) are conditionally independent given S, theorem 7 implies that λ is S measurable, $\mathcal{M} = \sigma(S)$ and we may write

$$\lambda = \Lambda(S). \tag{18}$$

A particularly interesting special case is obtained on specializing further. Suppose that L_i represents the effects on wealth of labor income, a typically undiversifiable component for most investors. Let the utility function be a indirect utility of nominal wealth w_i and a personalized price index p_i that takes account of personalized expenditure tastes. Hence we may write that

$$\lambda^i = \Pi^i(Z[a_i] + L_i, p_i) \tag{19}$$

In order to invoke theorem 7 we need to specify the factors S. For this purpose consider a vector of portfolios with returns S that are useful in predicting investor specific labor incomes and price indices by the regression models,

$$\begin{aligned} L_i &= \chi_i + \zeta_i' S + y_i \\ p_i &= \kappa_i + \xi_i' S + v_i \end{aligned}$$

Substituting back into (19) we obtain

$$\lambda^i = \Pi^i(Z[a_i] + \chi_i + \zeta_i' S + y_i, \kappa_i + \xi_i' S + v_i) \tag{20}$$

Now perform the regression (16) and substitute into (20) to obtain the form (17). To derive (18) we require conditional independence of $(u'a_i + y_i, v_i)$ conditional on S. This might require us to expand s to include portfolios that are useful in predicting Z_j in the regression (16) even though they may not be significant in explaining L_i or p_i. Under multivariate normality of (S, u, y, v) the conditional independence follows from the orthogonality of (u, y, v) and S obtained on the three regressions for Z_j, L_i, p_i.

The factors relevant for asset pricing suggested by our model of an asset exchange economy include those factors that explain the cross sectional variation across investors of effects on marginal utilities or the investor specific duals λ^i. This may usefully be contrasted with the more traditional approach of focusing solely on explaining the cross sectional variation across assets of asset returns. The important insight into asset pricing gained from our analysis is precisely the proposition that empirical work on asset pricing needs to focus on factors relevant in explaining the investor specific pricing duals λ^i across i in addition to identifying factors explaining Z_j across the set of assets.

Once we have established the validity of (18) for some set of factors S, a traditional K factor approximate asset pricing model may be derived by invoking a first order approximation to the function Λ using familiar arguments (See Breeden (1979), Grossman and Shiller (1982), Madan (1988), Milne (1988) and Back (1991)).

Theorem 13 *The market pricing operator* $\Phi[Z]$ *may be approximated by*

$$\Phi[Z] \approx \delta E^P[Z] + \sum_k \theta_k Cov^P(S_k, Z) \tag{21}$$

Proof. By the definition of Φ,

$$\Phi[Z] = E^P[\lambda Z] = E^P[\Lambda(S_1, \cdots S_K)Z]$$

Now write the expectation of the product ΛZ as the product of the expectations plus the covariance of Λ and Z to get

$$\Phi[Z] = E^P[\Lambda]E^P[Z] + Cov^P[\Lambda(S_1, \cdots S_K), Z] \tag{22}$$

Now approximate Λ by a first order Taylor series expansion about 0 the expectation of the $S_i^{\prime}s$,

$$\Lambda(S_1, S_2, \cdots, S_K) \approx \Lambda(0, \cdots, 0) + \sum_k \Lambda_k S_k \tag{23}$$

where Λ_k refers to the partial of Λ with respect to S_k. The result follows on substituting (23) into (22) and noting that $\delta = E^P[\Lambda]$ and $\theta_k = \Lambda_k$. ■

Theorem 8 provides a $K-$ systematic factor asset pricing model in which covariances with the variables $S_1, \cdots, S_K$ determine risk premia. By the usual arguments this may be written as a beta pricing model for expected asset returns in which just the beta's with respect to $S_1, \cdots, S_K$ are priced.

The approach to obtaining asset pricing models as averages of personalized asset pricing models can hopefully be used to derive relatively parsimonious but successful asset pricing models. At an empirical level, the approach is suggestive and offers guidance into directions that may fruitfully be taken in future research on empirical asset pricing models.

6 Conclusion

Asset pricing in equilibrium is conducted by investors who are in agreement about the prices of traded assets. The reasons for this agreement however, are varied, typically reflecting investor specific risk exposures that arise from incomplete diversification of personal risks across the space of assets. Personalized investor specific asset pricing models reflect the multitude of these risks. By averaging across the pool of investors, in a manner akin to how insurers average risks across the pool of the insured, market risk exposures and asset pricing models are derived. It is observed on invoking a law of large numbers applied to an infinite population of investors that many personally relevant risk considerations can be eliminated from the market asset pricing model.

Examples illustrating the effects of undiversified labor income and taste specific price indices are provided. An important insight into asset pricing gained from our analysis is the proposition that work on asset pricing needs to focus on identifying and explaining investor specific risk exposures cross sectionally across the pool of investors in addition to explaining the variation of asset cash flows. In this sense the approach outlined here is jointly focused on both the pricing dual and the primal aspects of asset cash flows.

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7 APPENDIX

7.1 Details on The Construction of Non-Atomic Finitely Additive Measures on the set of Natural Numbers.

Two general strategies for the construction of finitely additive measures with zero measure for finite sets are described in Madan and Owings (1988). We present here two examples that illustrate these constructions.

7.1.1 Example 1.

Let A_k be a sequence of pairwise disjoint infinite subsets of the set of natural numbers whose union is the set of all natural numbers. For example A_1 could be the even numbers while the union of all the other sets is the odd numbers. One could then partition the odd numbers into two infinite sets A_2 and the complement of $A_1 \cup A_2$. The sequence A_k can then be generated by repeated partitioning. Now let p_k be any positive sequence of numbers that satisfies $\sum_{k=1}^{\infty} p_k = 1$. Define the binary valued sequence of set functions δ_k on the collection of all subsets of I by

$$\delta_k(A) = 1 \text{ if } A_k - A \cap A_k \text{ is finite and } 0 \text{ otherwise}$$

that is $\delta_k(A) = 1$ just if A contains all but finitely many elements of A_k. Now define

$$\mu(A) = \sum_{k=1}^{\infty} p_k \delta_k(A)$$

Since no finite set can contain all but finitely many elements of any A_k $\delta_k(A)$ is zero for all finite sets for all k, hence $\mu(A) = 0$. However, though finite sets have zero measure, this measure μ has the feature that for example, A_1 with $\mu(A_1) = p_1$, cannot be partitioned into sets of measure less than p_1 and hence the measure is atomic with the sets A_k serving as the atoms.

7.1.2 Example 2.

In this construction we obtain a finitely additive non-atomic measure that is the limit of measures relevant for finite economies and reflects the limits of averages.

We first define a sequence of finitely additive measures μ_n on the set of all subsets of I as follows:

$$\mu_n(A) = \frac{|A \cap L(n)|}{n}$$

where $L(n) = \{k \,|1 \le k \le n\}$, $|X|$ denotes the cardinality of the set X, and μ_n is the proportion of elements less than or equal to n that belong to A. it is clear that μ_n is a finitely additive measure on the set of all subsets of I. Since μ_n is a function from $\mathcal{A}$ the set of all subsets of I into the unit interval I, we may think of μ_n as an element of the set $I^{\mathcal{A}}$. If we endow $I^{\mathcal{A}}$ with the product topology of the Euclidean topology on I then $I^{\mathcal{A}}$ is a compact set by Tychonoff's theorem. Therefore the set $\{\mu_n \,|n \in I\}$ has an accumulation point μ. Note that $\mu(A) = \lim_n \mu_n(A)$ whenever this limit exists. Hence, since for all finite sets A, $\lim_n \mu_n(A) = 0$ the measure μ is zero on all finite sets.

For the finite additivity of μ, suppose that A_1 and A_2 are two disjoint sets with $A = A_1 \cup A_2$. Since μ is an accumulation point there exists a subsequence μ_{n_k} such that $\lim_{n_k} \mu_{n_k}(A_1) = \mu(A_1), \lim_{n_k} \mu_{n_k}(A_2) = \mu(A_2)$ and $\lim_{n_k} \mu_{n_k}(A) = \mu(A)$. Now by the finite additivity of μ_n we have that for all k

$$\mu_{n_k}(A) = \mu_{n_k}(A_1) + \mu_{n_k}(A_2)$$

and it follows on taking limits that $\mu(A) = \mu(A_1) + \mu(A_2)$.

To observe that μ is non-atomic, observe that for each m we may define sets $C_1, C_2, \cdots C_m$ such that $k \in C_i$ just if $i = 1 + k \text{ mod}(m)$. For each i and n equal to mN, $\mu_n(C_i) = \frac{1}{m}$, while for n exceeding mN, we have that

$$\frac{N}{mN+m-1} \leq \mu_n(C_i) \leq \frac{N+1}{mN+1}$$

Since as n and N tend to infinity these upper and lower bounds converge to $\frac{1}{m}$, it follows that $\mu_n(C_i)$ converges to $\frac{1}{m}$ and so $\mu(C_i) = \frac{1}{m}$ for all i. For and A with $\mu(A) = p > 0$ consider the sets $A \cap C_i$ and note that

$$p = \mu(A) = \sum_{i=1}^{m} \mu(A \cap C_i)$$

where all the sets $\mu(A \cap C_i)$ have measure bounded by $\frac{1}{m}$ which for large enough m is less than $\frac{p}{2}$. Hence A has two disjoint subsets with positive measure strictly less than p and hence A is not an atom.

The measure $\mu(A)$ reflects the asymptotic proportion of the population in the set A. In fact if $\mu(A) = p > 0$ then for any $\varepsilon > 0$ it must be the case that for infinitely many n $\frac{|A \cap L(n)|}{n}$ is within ε of p.

7.2 Proof of Theorem 4.

The theorem is proved by establishing the equivalence under assumption 1 between a competitive equilibrium and the Weiss (1981) definition of a competitive equilibrium, termed here a WCE for a Weiss competitive equilibrium. The proof is completed by noting that a WCE exists (Weiss (1981) modified along the lines of Milne (1976) to account for short sales).

We first define a WCE:

Definition 14*A Weiss attainable allocation for a subset B,$\mu(B) > 0$. is a $\mu-$integrable function $a^B : B \to \mathbb{R}^J$, such that $a_i^B \in A_i$, for all i and*

$$\int_B a_i^B d\mu(i) = \int_B \overline{a}_i^B d\mu(i)$$

We now define preference orderings and budget feasibility for positive measure subgroups of individuals. Null functions are used in the definitions below to ensure that they apply to equivalence classes of allocations.

Definition 15*For two allocations $a^B, a^{B'}$ for subsets $B.B'$ with $S \subseteq B \cap B', \mu(S) > 0$, we define the relation a^B is preferred to $a^{B'}$ by the subset S written $a^B \succ_S a^{B'}$, if for all null functions h, g on I with values in $\mathbb{R}^J$*

$$u_i^*(a_i^B + h_i) > u_i^*(a_i^{B'} + g_i)$$

for almost all $i \in S$ for which $a_i^B + h_i$ and $a_i^{B'} + g_i$ belong to A_i.

Definition 16*An allocation is budget feasible for the group B for prices p if there exists a real valued null function h such that for almost all $i \in B$,*

$$pa_i^B + h_i \leq p\overline{a}_i.$$

Definition 17 *An allocation a^B is preference maximal for B if,*
a) a^B is budget feasible for group B,
b) for every allocation c^B of B, if $c^B \succ_B a^B$ then there exists $S \subseteq B, \mu(S) > 0$, such that the restriction of c^B to S is not budget feasible for S.

Definition 18 *A Weiss competitive equilibrium (WCE) for an Asset Exchange Economy is a price vector $p^* \in \mathbb{R}^J$, and an allocation a^* such that*
1) for all $B \subseteq I$, $\mu(B) > 0$, the restriction of a^* to B is preference maximal for B;
2) $\int a_i^* d\mu(i) = \int \overline{a}_i d\mu(i) = \overline{a}$.

The existence of competitive equilibrium for such an economy can be established using a modification of the arguments in Weiss to account for short sales along the lines of Milne (1976).

We now establish the equivalence between a competitive equilibrium and a WCE under assumption 1.

Suppose first that we have a competitive equilibrium. Therefore there exists $A \subseteq I, \mu(A) = \mu(I)$ and h_i^*, k_i null functions satisfying

$$\begin{aligned} i)\ p^*\left(a_i^* - h_i^*\right) &= p^*\overline{a}_i \\ ii)\ u_i^*(a_i^* - h_i^*) &\geq u_i^*(a_i^0) - k_i \end{aligned}$$

Define the real values function $h_i = p^* h_i^*$ and note that as h_i^* is null, so is h_i. it follows from property i) that

$$p^* a_i^* - h_i \leq p^* \overline{a}_i$$

for all $i \in A \cap B$, which is almost everywhere in B for all B of positive measure. Hence a^* is budget feasible for all $B, \mu(B) > 0$.

For preference maximality, suppose that $\widetilde{a} \succ_B a^*$ for some B, $\mu(B) > 0$. It follows from definition 15 in particular that for all null functions g, t

$$u_i^*(\widetilde{a}_i - g_i) > u_i^*(a_i^* - h_i^* + t_i)$$

for almost all i in B. Since one can not find t_i null such that

$$u_i^*(a_i^* - h_i^* + t_i) = u_i^*(a_i^* - h_i^*) + k_i$$

it follows from property ii)

$$u_i^*(\widetilde{a}_i - g_i) > u_i^*(a_i^0).$$

Because a_i^0 is utility maximal in $i's$ budget set we must have that,

$$p^*\widetilde{a}_i - p^* g_i > p^* a_i^0$$

Assumption A.6 and lemma 2 of Milne (1976) (convexity of u_i^*) imply that $p^* a_i^0 = p^* \overline{a}_i$ and therefore

$$p^*\widetilde{a}_i - p^* g_i > p^* \overline{a}_i$$

for almost all $i \in B$. This contradicts the budget feasibility of $\widetilde{a}$ in B. Hence a^* is a WCE.

On the other hand suppose that a^* is a WCE. Define

$$\Gamma = \left\{ h, k \,\middle|\, p^*(a_i^* - h_i) \leq p^* \overline{a}_i \text{ and } u_i^*(a_i^* - h_i) = u_i^*(a_i^0) - k_i \right\}$$

We wish to show that there exist null functions h, k that belong to Γ.

First let g_i be smallest in norm such that $p^*(a_i^* - g_i) \leq p^*\overline{a}_i$. It follows from principles of distance minimization that $g_i = \lambda_i \alpha$, where a is the unique directional vector orthogonal to the null space of p^*. Of course if a_i^* is itself budget feasible for i then $g_i = 0$. Since a^* is budget feasible for I there exists a null function k such that

$$p^* a_i^* - k_i \leq p^* \overline{a}_i$$

By the minimization,

$$p^* g_i = p^* a_i^* - p^* \overline{a}_i \leq k_i$$

or that

$$|\lambda_i| \, |p^* \alpha| \leq |k_i| \, .$$

It follows that

$$\|g_i\| = |\lambda_i| \, \|\alpha\| \leq |k_i| \, \frac{\|\alpha\|}{|p^* \alpha|}$$

and hence g is null.

Therefore there exist null functions h that satisfy the first of the clauses for entry into Γ. For any such h, let k be defined by

$$k_i = u_i^*(a_i^0) - u_i^*(a_i^* - h_i).$$

We wish to show that if k is not null then a^0 contradicts preference maximality of a^* for some set of positive measure.

Suppose that k is not null. Since $a_i^* - h_i$ is budget feasible for i, k_i is non-negative. k is not null implies that there exists a set of positive measure B such that k_i exceeds a constant c for all i in B. Consider now the restrictions to B of a^0 and $(a^* - h)$, that we denote $a^0 \,|_B$ and $(a^* - h)\,|_B$ respectively. For all $i \in B$, $u_i^*(a_i^0) > u_i^*(a_i^* - h_i) + c$. By Assumption 1, choose δ such that $|a - b| \leq \delta$ implies that $|u_i^*(a) - u_i^*(b)| \leq c/4$. Since for null functions s_i and t_i the norms are almost everywhere less than δ, we have that

$$u_i^*(a_i^0 - s_i) > u_i^*(a_i^* - h_i + t_i)$$

for almost all $i \in B$. Equivalently,

$$a^0 \succ_B (a^* - h) \, .$$

As the points a^* and $a^* - h$ are in the same equivalence class modulo null functions this implies that

$$a^0 \succ_B a^*$$

However, $a^0 \,|_B$ is budget feasible for all subsets S of B, and so we have a contradiction of a^* being preference maximal for B. Therefore k must be null.

7.3 Proof of Theorem 5

Since $u_i^* = u_i(Z[a_i])$, the differential of u_i^* with respect to a_i is the Fréchet differential of u_i evaluated at $x_i^* = Z[a_i^*]$ applied to the differential of Z with respect to a_i, which is Z_j. The Fréchet differential of u_i evaluated at x_i^* is a linear operator which by the self duality of V is given by an element of V that we denote $\lambda^i(\omega)$, with the application to Z_j being as described in (1). Nonnegativity of λ^i follows from $A.4$. The construction of $\lambda_0^i(\omega)$ is similar, except that we now work with a^0 in place of a^*.

7.4 Proof of Theorem 9

Let a^* be an equilibrium allocation. By definition of equilibrium one can make a null function movement h^* and reach for almost all i, a point that is both budget feasible with utilities within a null function k of the individual utility maximizing point a^0.

Define a' from the definition of a competitive equilibrium by $a' = a^* - h^*$. Construct Φ_m^i analogously to Φ^i, except that we employ the Fréchet differential $\psi_m^i(\omega)$ of u_i at $Z[a_i']$ in place of $\psi^i(\omega)$, and $\lambda_m^i = \psi_m^i/\gamma_0^i$. The result follows on showing that both $\Phi^i - \Phi_m^i$ and $\Phi_0^i - \Phi_m^i$ are null operators.

For $\Phi^i - \Phi_m^i$, it is sufficient to show that $\left\|\psi^i - \psi_m^i\right\|$ is a null function over i. Let $x_i^* = Z[a_i^*]$, and $x_i' = Z[a_i']$, applying Result 42 (Neustadt (1976) I.7 page 55) to the derivative of u_i we obtain that

$$\begin{aligned} \left\|\psi^i - \psi_m^i\right\| &= \left\|Du_i[x_i^*;\cdot] - Du_i[x_i';\cdot]\right\| \\ &\leq C\left\|x_i^* - x_i'\right\| \end{aligned}$$

where C is used here as a bound for the second derivative. Since $\|x_i^* - x_i'\|$ is bounded by $\|Z\|\,\|h_i^*\|$, which is a null function, it follows that $\left\|\psi^i - \psi_m^i\right\|$ is a null function.

For the proof of $\Phi_0^i - \Phi_m^i$ being null, let Δ^i be the set of all a_i satisfying

$$\begin{aligned} p^* a_i &\leq p^* \overline{a}_i \\ u_i^*(a_i) &= u_i^*(a_i^0). \end{aligned}$$

Each element of Δ^i is a utility maximizing point and any point in Δ^i can play the role of a_i^0 in theorem 4. Define the point to set, distance function $d_i(y)$ for $y \in \mathbb{R}^J$ by

$$d_i(y) = \inf\left\{\|y - a\| \,\middle|\, a \in \Delta^i\right\}.$$

Define the functions $\Psi_i(c)$ by

$$\Psi_i(c) = u_i^*(a_i^0) - Max\left\{u_i^*(a)\,|p^* a \leq p^* \overline{a}_i \text{ and } d_i(a) \geq c\right\}$$

By construction $\Psi_i(0) = 0, \Psi_i(c) > 0$, and Ψ_i is monotone increasing.

Let $a_i' = a_i^* - h_i^*$ with a^* and h^* satisfying the conditions of theorem 4. Since

$$u_i^*(a_i^0) - u_i^*(a_i') \leq k_i$$

it follows that

$$\Psi_i(d_i(a_i')) \leq u_i^*(a_i^0) - u_i^*(a_i') \leq k_i$$

Now $d_i(a_i') \geq \alpha$ implies that $\Psi_i(d_i(a_i')) \geq \Psi_i(\alpha)$ as Ψ_i is monotone increasing. It follows then that $k_i \geq \Psi_i(\alpha)$. Since $\Psi_i(\alpha)$ is positive for positive α, $d_i(a_i')$ not null implies k_i is not null. But as by theorem 1, k_i is null, we must have that $d_i(a_i')$ is null.

Now choose a_i^0 in Δ^i so that $\left\|a_i' - a_i^0\right\| \leq d_i(a_i') + \frac{1}{i}$ and theorem 4 holds for a_i^0 with $a_i' - a_i^0$ being a null function. This implies that $Z[a_i'] - Z[a_i^0]$ is a null function and by an argument similar to that used for $\Phi^i - \Phi_m^i$ we have that $\Phi_0^i - \Phi_m^i$ is a null operator.

7.5 Proof of Theorem 12

Consider a complete probability space $(\Omega, \Im, P)$ and a complete $\sigma - field\ \mathcal{D} \subseteq \Im$. Suppose that $X_1, X_2, \cdots$ are random variables on $(\Omega, \Im, P)$ which are conditionally independent given $\mathcal{D}$. Write for $m > n$

$$\Im_n^m = \sigma\left(X_n, X_{n+1}, \cdots X_m\right)$$

and

$$\mathcal{C} = \bigcap_{n=1}^{\infty} \Im_n^{\infty}$$

Theorem 19$\mathcal{C} = \mathcal{D}$

Proof. Suppose $A \in \mathcal{C}$. Then $A \in \Im_1^{\infty} = \bigcup_{m=1}^{\infty} \Im_1^m$. By the monotone class theorem there exist sets $A_m \in \Im_1^m$ such that

$$P\left(A \Delta A_m \,|\mathcal{D}\right) > 0$$

as m tends to infinity. That is,

$$\lim_m P\left(A_m \,|\mathcal{D}\right) = P(A\,|\mathcal{D})$$

and

$$\lim_m P\left(A \cap A_m \,|\mathcal{D}\right) = P(A\,|\mathcal{D})$$

But $A \in \mathcal{C} \subseteq \Im_{n+1}^{\infty}$, and so A and A_n are conditionally independent given $\mathcal{D}$. Therefore, $P(A\,|\mathcal{D})P\left(A_n\,|\mathcal{D}\right) > P(A\,|\mathcal{D})$ and so $P\left(A\,|\mathcal{D}\right)^2 = P\left(A\,|\mathcal{D}\right)$ for any $A \in \mathcal{C}$.

That is,

$$E\left[I_A\,|\mathcal{D}\right](\omega)^2 = E\left[I_A\,|\mathcal{D}\right](\omega)$$

and so the random variable $E\left[I_A\,|\mathcal{D}\right](\omega)$ takes only the values 0 and 1. Write

$$B = \{\omega\,|E\left[I_A\,|\mathcal{D}\right](\omega) = 1\}$$

Then $B \in \mathcal{D}$ and $E[I_A\,|\mathcal{D}](\omega) = I_B(\omega)$.

Now

$$\begin{aligned} E\left[(I_A - I_B)^2\,|\mathcal{D}\right] &= E\left[I_A^2 + I_B^2 - 2I_B I_A\,|\mathcal{D}\right] \\ &= E\left[I_A + I_B - 2I_B I_A\,|\mathcal{D}\right] \\ &= I_B + I_B - 2I_B I_B = 0 \end{aligned}$$

Therefore

$$E\left[(I_A - I_B)^2\right] = 0$$

and so $I_A = I_B$ almost surely. Consequently (modulo a set of measure zero, and $\mathcal{D}$ is complete)

$$A = B \in \mathcal{D}.$$

That is $\mathcal{C} = \mathcal{D}$. ■