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Should the Samuelson Rule Be Modified to Account for the Marginal Cost of Public Funds?

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Abstract: Disputes over the marginal cost of public funds may be about its magnitude in any given time and place or about its role in cost-benefit analysis. This paper is about the latter. The Samuelson rule was devised for an omnipotent, omniscient and benevolent government. This paper is about how the Samuelson rule should be modified to take account of the impact upon total deadweight loss in the tax system from the required an increase in the tax rate to finance public projects as well as from the appearance of the projects themselves. A very simple device is employed to analyze these questions.

> "...social welfare will rise (indeed, there will be a Pareto improvement) whenever the basic cost-benefit test is met."
>
> Kaplow (1998, 117)

> "...an expenditure program will be efficient only if its benefits exceed the direct cost by an amount as least as large as the additional welfare cost of the funds."
>
> Browning (1976, 283)

> "...the conventional rule.... may be an over- or an under- estimate of the incremental benefits of a public good"
>
> Atkinson and Stern (1974, 126)

These three quotations are about rules for deciding whether or not to undertake given public sector projects. Kaplow would undertake projects if and only if the sum of the benefits exceeds the sum of the costs in accordance with the original Samuelson rule. Browning would impose a wedge between benefit and cost to account for the marginal deadweight loss in taxation.[1] Atkinson and Stern argue that, depending on the circumstances, the appropriate wedge

[1]"Browning", in this context, is representative of a wide range of literature on the estimation of the marginal cost of public funds from information about the tax structure and the tax-induced distortions in the economy. Browning (1976) himself, postulating a distortion in the labour-leisure choice, based his estimate of the marginal cost of public funds upon the estimated elasticity of labour. At about the same time, Campbell (1975), looking at the entire tax system as a set of unequal excise taxes, based his estimate on the assumption that an increase in public revenue would be financed by proportionally equal increases in all tax rates. Later on, Feldstein

may be greater or less than 1. The claim in this paper is a) that the authors are to some extent talking at cross-purposes, b) that all three are correct in some circumstances, but c) that as a matter of practice, cost-benefit analysis should be formulated in a way that makes Browning's recommendation appropriate. The paper also employs a mode of analysis that is in my opinion infinitely simpler and no less instructive than what is usually encountered in the analysis of the marginal cost of public funds.

Let us be clear what we are talking about. We are talking about rules for cost-benefit analysis, rules for the government to follow in deciding which roads to build, which schools to build, whether the army needs new helicopters, ans so on. The starting point for our investigation is the Samuelson rule to undertake projects if and only if the sum of the benefits to all citizens exceeds the cost of the project. I define projects broadly to include public programs of all kinds as well as public goods that can be provided in larger or smaller amounts (so that the rule becomes to supply public goods up to the point where the sum of the marginal benefits is just equal to marginal cost). The rule itself is to undertake projects if and only if

$$\Sigma\, b_i > c \qquad (1)$$

where b_i is the benefit of the project to the i^{th} person and c is the cost of the project. For our purposes, it is convenient to rewrite the equation as

$$\Sigma\, (b_i /c) > 1 \qquad (2)$$

meaning that the sum of the benefits per dollar of public expenditure must equal 1. It is important fore what follows to emphasize the context in which the rule was formulated. Samuelson derived the rule for an omnipotent, omniscient and benevolent planner with complete control of the private sector as well as of the behaviour of the government. The planner was confronted with a production constraint for the economy as a whole, but there was no taxation because all resources could be assigned directly to their appropriate use and the national income could be allocated accordingly. Think of equations (1) and (2) as part of a much larger system of equations describing the optimal administration of the economy as a whole.[2]

The three quotations at the outset of this paper can be thought of as pertaining to whether and how equation (2) ought to be modified for an economy where the government chooses

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(1995) employed the observed change in tax revenue following an actual change in the tax rate to derive an estimate of the elasticity of tax base to tax rate from which the marginal cost of public funds could be derived. For a review of attempts to measure the marginal cost of public funds, see Ballard and Fullerton (1992). For a critique of Feldstein's method of estimation, see Goolsbee (2000).

[2]For convenience of exposition, the entire analysis in this paper is static, as though all benefits and costs accrued simultaneously. For a crude analysis of the relation between interest rates and the marginal cost of public funds, see Usher (1982). For a thorough treatment of the subject, see Liu (2003).

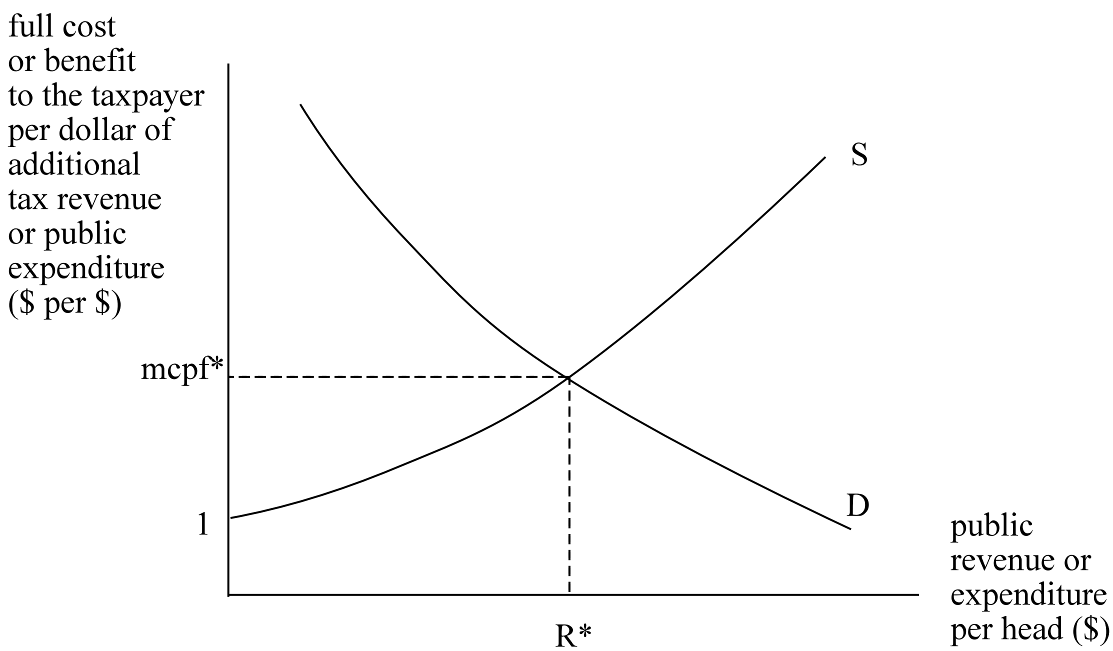
projects, sets taxes to finance these projects but cannot control the taxpayer's response to the government's choice of projects and taxes to finance them. Browning would place a wedge between benefits and costs in equation (1), converting equation (2) to

$$\Sigma (b_i /c) > \text{mcpf} \tag{3}$$

were mcpf (the marginal cost of public funds) can be computed from characteristics of the economy, is not project-specific and is always greater than 1. Atkinson and Stern also recognize that there is a wedge, but they see the wedge as project-specific and they deny that it is invariably greater than 1. Kaplow sees no wedge at all.

It is argued in this paper 1) that all three authors are correct in the sense that there are circumstances where their claims turn out to be true, 2) that much depends on how exactly the terms in these equations are defined, and 3) that the choice of a rule to replace the original Samuelson rule when governments are less than omnipotent is as much a matter of administrative arrangements as of pure economic analysis.

Figure 1: Demand and Supply Curves for Public Revenue



Equation (3) can be looked upon as describing the crossing of a demand curve and supply curve of public expenditure as shown in figure 1. The horizontal axis shows total public expenditure, and is graduated as "$". The vertical axis shows demand and supply prices of public expenditure, and is graduated, somewhat paradoxically, as "$ per $". The demand curve for public expenditure reflects an ordering of all possible public projects in accordance with their benefits per dollar of public expenditure. For any amount of public revenue, R, the corresponding demand price (the height of the demand curve) is

$p^D(R)$ = "combined benefits to people" per "additional dollar of expenditure by the government"

and the corresponding supply price (the height of supply curve) is the

$p^S(R)$ = "total cost to people" per "additional dollar of revenue acquired by the government"

where "total cost" in this context is the sum of the extra dollar tax people pay and the dead extra deadweight loss from the slight increase in tax rates required to finance the extra dollar's worth of public expenditure, and where, at this level of generality, public revenue and public expenditure are one and the same. Graduated as $ per $, both prices are of public expenditure with dollars' worth to people as the numeraire. The demand price shows what an additional dollar of public revenue is worth to people, and the supply price shows what they must give up to procure it. Optimal public revenue and the corresponding common value of the demand and supply prices are designated as R* and mcpf*.

Several observations about this diagram:

a) The connection between the diagram and the Samuelson cost-benefit rules in equation s (1), (2) and (3) is that, for any given amount of public expenditure, the height of the demand curve is the cut-off value of $\Sigma(b_i /c)$, such that projects which can be financed with that expenditure all yield higher values of $\Sigma(b_i /c)$ while additional projects all yield lower values of $\Sigma(b_i /c)$.

b) That being so, the left and right sides of the equations representing the Samuelson rule are the demand and supply prices of public expenditure where the curves cross.

c) In Samuelson's original formulation - with an omnipotent government and no deadweight loss - the supply curve of public funds must be flat at a height of 1 above the horizontal axis, validating equation (1).

d) The supply curve begins at 1 when public expenditure is 0, and it rises steadily as public expenditure increases. By construction, the height of the supply curve is $1 + \Delta$, where Δ the additional deadweight loss per additional dollar of expenditure. The curve is constructed on the assumption that additional deadweight loss per additional dollar of expenditure begins at 0 and rises steadily thereafter. [There are actually two supply prices for each quantity of public expenditure, a positive price, as shown on the figure, and a negative price for an economy on the wrong side of the Laffer curve.]

e) In drawing the demand and supply curves for public expenditure, we need not commit ourselves about the source of deadweight loss in the tax system. It may originate in the labour-leisure choice as commonly supposed in much of the literature on the marginal cost of public funds, but it may also originate in the tax-induced switch from paid work to do-it-yourself activities, or in the consumption-investment decision, or the extra incentive for tax avoidance and evasion. One may think of all of these incentives as combined in the supply curve of public expenditure. [In this respect, the demand and supply diagram is actually more general than much of the more precise algebraic analysis of the marginal cost of public funds which tends to concentrate on only one source deadweight loss.]

f) The marginal cost of public funds can be interpreted *either* as the height of the intersection of the demand and supply curves *or* as the height of the supply curve regardless of whether or not public expenditure is optimal. The former interpretation is common in the theoretical literature. The latter is more common in the empirical literature which is sometimes oriented to showing that actual public expenditure is inappropriately large.

g) A semblance of the original Samuelson rule could be restored by a reinterpretation of "$" on the horizontal axis and of "$ per $" on the vertical axis. Replace the "total expenditure" on the horizontal axis with "total sacrifice" interpreted as the sum of total revenue and the corresponding total deadweight loss in taxation, and construct a new more-inclusive demand curve the height of which is extra dollars of benefit per dollar of sacrifice. By construction the new demand price is p^D/p^S which is necessarily equal to 1 at the optimal public expenditure, so one could imagine this new demand curve cutting a horizontal supply curve at a height of 1 above the horizontal axis.

There is another difficulty. Our discontent with the original Samuelson rule stems from its failure to account for tax payers' response to public expenditure and taxation. The rule was derived for an omnipotent, omniscient and benevolent government, a government which, by definition, need not consider people's responses to its actions. Drop that assumption, restrict government to the choice of tax rates and public expenditures, and the response to its actions must be taken into account. But there are two such responses, of which only one has been considered so far. Think of tax paid as a positive externality to private behaviour, an externality that is relatively large for some forms of activity and relatively low for others. For example, the externality is relatively large for labour and relatively small for leisure. Deadweight loss in taxation is the loss of a part of that externality as a consequence of people's response to tax increases. That is what we have been considering so far under the heading of the "marginal cost of public funds". There is a corresponding impact of public expenditure itself. Just as taxation affects people's behaviour and tax paid, so too do public projects, but with this exception: that the appearance of a given project may serve, depending on the nature of the project, either to increase or to decrease the total tax paid. For example, the appearance of a new park may lead to a reduction in total tax paid because people work less, spend less on costly entertainment and, like Ferdinand, spend more time smelling the flowers. By contrast, better roads may increase the productivity of labour, raising gross incomes and increasing the revenue from the income tax. The effect of the appearance of public projects on privately-advantageous but publicly detrimental manoeuvres to lower one's tax bill are incorporated in what is commonly called the "shadow price of public expenditure". Formally, the shadow price of public expenditure is a multiplier attached to $\Sigma\ (b_i /c)$ in equation (2), a multiplier that is greater than 1 if the effect of the appearance of the project is to augment tax revenue or to reduce total deadweight loss in the system as a whole, and is less than 1 if the effect of the appearance of the project is to reduce tax revenue or to increase total deadweight loss in the tax system as a whole. In other words, to take account of the shadow price of public expenditure is to replace the original Samuelson rule in equation (2), not just with the modified rule in equation 3, but as follows

$$\Sigma\,(b_i\,/c)(sp) > mcpf \qquad (4)$$

where sp, the shadow price of public expenditure, is greater than 1 if the effect of a project is to reduce deadweight loss and sp < 1 if the effect of a project is to increase deadweight loss from the pre-existing tax system.

The locus classicus of the correction to the Samuelson rule for both the marginal cost of public funds and the shadow price of public expenditure is the article by Atkinson and Stern from which the quotation at the outset of this paper was taken. For an economy with identical people whose utility is a function of one consumption goods, x, hours of work, L, and one public good, G, the appropriate modification of the Samuelson rule turns out to be

$$N(u_G\,/u_x\,) \;=P_G\{1/[1+\epsilon_{Lt}]\}\{1-\epsilon_{LG}\} \qquad (5)$$

where $N(u_G\,/u_x\,)$ is Atkinson and Stern's equivalent of Σb_i in equation 4, P_G is their equivalent of c, $P_G\{1/[1+\epsilon_{Lt}]$ is their derived value of the marginal cost of public funds, mcpf, and $1/[1-\epsilon_{LG}]$ is their derived value of the shadow price of public expenditure, sp. For easy reference, their derivation of equation (5) is included in this paper as Appendix B. Note that the connection between ϵ_{Lt} and ϵ_{LG} with the marginal cost of public funds and the shadow price of public expenditure respectively is appropriate because the one shows the response of the tax base to the tax rate and the other show the response of the tax base to a change in the provision of the public good, which is the sole object of public expenditure. Note also that marginal cost of public funds is greater than 1 as long as ϵ_{Lt} is negative, signifying that the tax base shrinks as the tax bas increases, but that there is no particular reason for deciding whether the shadow price is greater or less than 1. Note finally, that equation (5) reduces to the original Samuelson rule - vindicating Kaplow as quoted above - in equation (1) if the two elasticities, ϵ_{Lt} and ϵ_{LG} , or, more interestingly, if the absolute values of the marginal cost of public funds and the shadow price of public expenditure turn out to be the same. To complete the equivalence, equation (5) may be rewritten as

$$[N(u_G\,/u_x\,)/P_G\,]/[1/(1-\epsilon_{LG}) = 1/[1+\epsilon_{Lt}] \qquad (5^*)$$

so that the marginal cost of public funds stands alone on the right hand side of the equation, exactly as in Browning's formulation.[3]

With this machinery in hand, we are in a position to reconsider the three views at the outset of this paper. Atkinson and Stern are clearly and unambiguously right in asserting that,

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[3]The distinction between $\{1/[1+\epsilon_{Lt}]\}\{1-\epsilon_{LG}\}$ in equation (2) above and $\{1/[1+\epsilon_{Lt}]\}$ in equation (3) is close to the Mayshar's (1991) distinction between alternative definitions of the marginal cost of public funds, particularly MCF^N in his equation (3) and MCF^{M1} in his equation (4). His discussion of the most efficient measure of the marginal cost of public funds for use in cost-benefit analysis foreshadows our discussion at the end of this article about the separation of the marginal cost of public funds from the shadow price of public expenditure. See also, Liu (2004).

depending on the magnitude of the elasticities, the optimal supply of the public good may be greater or less than as indicated by the original Samuelson rule; the expression $\{1/[1 + \epsilon_{Lt}]\}\{1 - \epsilon_{LG}\}$ in equation (5) may, in general be greater than or less than 1.

Kaplow is right in at least two conditions: i) when the elasticities ϵ_{Lt} and ϵ_{LG} are both equal to 0 and, more generally, ii) when the marginal cost of public funds and the shadow price of public expenditure cancel out. Something like the first condition arises when public goods can be financed by personalized lump sum taxation. The possibility of personalized lump sum taxation puts the government is essentially the same position as the omnipotent, omniscient and benevolent government that is postulated in the derivation of the original Samuelson rule, so it is hardly surprising that its cost-benefit rule is the same. In effect, that condition amounts to assuming the entire problem away, for the problem arises when and only when government is constrained to employ distortionary taxation.

The second condition is much more interesting because there is one important circumstance where the condition obtains. The marginal cost of public funds and the shadow price of public expenditure cancel out when i) the public good is intermediate in the production of the private good *and* ii) deadweight loss is restricted to the labour-leisure choice as is often assumed in the analysis of the marginal cost of public funds. Formally, what is required is that utility be a function of consumption, x, and labour, L, alone, that income, Y, be produced with inputs of labour and a public good, G, and that, once produced, income can be apportioned by the government between consumption and the public good. A proof of this proposition is presented below as Appendix C. However, both conditions are required for this result to emerge. The marginal cost of public funds and the shadow price of public expenditure fail to cancel out even though the public good is intermediate is the source of deadweight loss is a tax-induced increase in tax avoidance or tax evasion.

Browning can be rendered right by definition. He can be interpreted as talking about equation (5*) rather than equation (5), and, if he does not have much to say about the left-hand side of the equation, he need not be interpreted as claiming the shadow price of public expenditure to be irrelevant. More importantly, there is a strong reason for keeping the shadow price of public expenditure on the left-hand side of the equation, a reason that tends to be overshadowed by the working assumption in much of this literature that the government's only choice on the expenditure side of the budget is of the quantity of a supposedly homogeneous public good rather than among literally thousands of individual projects each with its own shadow price of public expenditure. The reason is that the shadow price is project-specific, while the marginal cost of public funds is what it is for the economy as a whole. Referring to the demand and supply schedules in figure 1 above, the marginal cost of public funds is the height of the supply curve, while a shadow price of public expenditure can be attached to each and every project so that the ordering of projects in the construction of the demand curve depends upon its value of $\Sigma\,(b_i/c)(sp)$ rather than upon its value of $\Sigma\,(b_i/c)$ alone. Browning can be said to be talking about the height of the supply curve in figure 1, and about that he is surely correct.

There is also an administrative consideration. The form of the cost-benefit rule identifying the optimal public and discrimination among worthy and unworthy projects should be a reflection of the responsibilities of the different ministries involved. The marginal cost of public funds as defined here and the supply curve of public expenditure are the responsibility of the Ministry of Finance which sets one unique marginal cost of public funds as a guide for the rest of the government. Then the separate ministries - Health, Transport, Education and so on - choose projects which all things considered yield a premium of benefit over cost in excess of the marginal cost of public funds. Yes, every ministry must know the shadow prices of its projects, but ministries can be expected to have the expertise to do so or they may rely on the advice of the Ministry of Finance. Yes, the Ministry of Finance must be ready to adjust the marginal cost of finance, up or down the supply curve of public expenditure, in response to the volume of projects, but this is unavoidable. The important consideration is to define the marginal cost of public funds so that it is not project-specific and to shove information about shadow prices onto the left-hand side of the equation.

Appendix A: Derivation of the Original Samuelson Rule

The original Samuelson rule identifies the socially oprimal supply of public goods in an economy with N identical people with utility functions

$$u = u(x, L, G) \qquad \text{(A1)}$$

where x is a person's consumption per unit of time, L is his supply of labour and G is the total supply of a public good. The technology of this economy is represented by a production possibility frontier

$$Nx + P_G G = NwL \qquad \text{(A2)}$$

where w is the real wage (consumption goods produced per hour of labour) and P_G is the cost of the public good (graduated in terms of the consumption good).

An omnipotent, omniscient and benevolent planner would choose x and L as well as G to maximize utility in equation (3) when constrained by the production possibility frontier in equation (4). The planner's choice is in accordance with the original Samuelson rule in equation (1) because no such planner need tolerate tax-induced distortions. The planner chooses x, L and G to maximize u(x, L G) subject to the constraint that $Nx + P_G G = NwL$. The first order conditions of this constraint are

$$-u_L/u_x = w \qquad \text{(A3)}$$

and

$$N(u_G/u_x) = P_G \qquad \text{(A4)}$$

which is what we are calling the original Samuelson rule as derived in Samuelson (1950). This equation is the precise representation of equation (1) in the text, where "benefit" refers to $N(u_G/u_x)$ and "cost" refers to P_G.

Appendix B: Atkinson and Stern's Derivation of the Wedge within the Samuelson Rule to account for the Marginal Cost of Public Funds and the Shadow Price of Public Expenditures.

The original Samuelson rule pertains to the choice of x, L and G by an omnipotent, omniscient and benevolent planner. The rule might be modified to take account of the taxpayers' response to taxation when the planner cannot choose G, x and L directly, but must accept the taxpayers' choice of x and L corresponding to any public choice of G together with whatever tax, t, is required to pay for it.

The modified Samuelson is derived in two stages. In the first stage, the taxpayer may be though of as choosing x and L to maximize his utility in response to *his* budget constraint and to the government's choice of G and t, giving rise to a pair of functions x(G, t) and L(G, t) that the government must respect in its choice of G and t. In the second stage, the government maximizes utility in its choice of G and t. recognizing *its* budget constraint as well to the taxpayers' response functions x(G, t) and L(G, t).

In the first stage of the analysis, for any given t and G, the taxpayer chooses x and L to maximize utility subject to *his* budget constraint

$$x = wL(1 - t) \qquad \text{(B1)}$$

where t is the rate of the income tax and where the wage, w, is an externally-given parameter.. Specifically, the taxpayer chooses x and L to maximize the Lagrangian

$$\mathcal{L} = u(x, L, G) - \alpha[x - wL(1-t)] \qquad \text{(B2)}$$

The first order conditions become

$$u_x = \alpha \quad \text{and} \quad - u_L = \alpha w(1-t) \qquad \text{(B3)}$$

from which it follows that

$$- u_L/u_x = w(1 - t) \qquad \text{(B4)}$$

which differs from the planner's first order condition in equation (A3) because the rate of trade-off in production between x and L differs from the taxpayer's rate of trade-off in use.

From equations (B1) and (B4) it follows that

$$x = x(G, t) \quad \text{and} \quad L = L(G, t) \qquad \text{(B5)}$$

because u_x and u_L are both functions of x, L and G.

Since the taxpayer's budget constraint (B1) is true for any and every t and G,

$$x_t = w(1 - t)L_t \; - wL \quad \text{and} \quad x_G = w(1 - t)L_G \tag{B6}$$

In the second stage of the analysis, the government - recognizing how taxpayers respond to its choice of t and G - chooses t and G to maximize its Lagrangian

$$\mathcal{L} = u(x, L, G) - \beta[P_G G - twLN] \tag{B7}$$

subject to its budget constraint[4]

$$P_G G = twLN \tag{B8}$$

The first order condition with respect to t becomes

$$u_x\, x_t \; + \; u_L\, L_t \; + \beta[twNL_t \; + wLN] = 0 \tag{B9}$$

From equations (B3) and (B9), it follows that

$$u_x\, x_t \; + \; u_L\, L_t \; = \alpha\, x_t \; - \; \alpha w(1\text{-}t)L_t \; = \; - \alpha wL, \tag{B10}$$

allowing equation (B9) to be rewritten as

$$(\beta N/\alpha)[1 + (t/L)L_t\,] = 1 \tag{B11}$$

where $(t/L)L_t$ can be interpreted as the elasticity, ϵ_{Lt}, of leisure with respect to the tax rate.

Similarly, the first order condition of the government's Langragian with respect to G is

$$u_x\, x_G \; + u_L\, L_G \; + u_G \; -\beta[P_G \; - \; twNL_G\,] = 0 \tag{B12}$$

which, recognizing that $u_x\, x_G \; + u_L\, L_G \; = 0$ and dividing both sides of the equation by N/u_x, simplifies to

$$N(u_G\, /u_x) \; = (N\beta/\alpha)[P_G \; - \; twNL_G\,] \tag{B20}$$

Replacing the expression $N\beta/\alpha$ in equation (19) by its value in equation (16) yields the modified Samuelson rule.

$$N(u_G\, /u_x\,) \; = \; P_G\{1/[1 + (t/L)L_t\,]\}\{1 - (G/L)L_G\}$$

$$= P_G\{1/[1 + \epsilon_{Lt}]\}\{1 - \epsilon_{LG}\} \tag{B21}$$

where ϵ_{Lt} and ϵ_{LG} are the elasticities of labour supply with respect to the tax rate and the supply

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[4]Note that , together, the equations for the production possibility frontier $[Nx + P_G G = NwL]$ and the budget constraint of the taxpayer $[x = wL(1\text{-}t)]$ imply equation (14).

of public goods. Equation (B21) is the Atkinson and Stern modification of the Samuelson rule in equation (A4) in the preceding appendix.. The original Samuelson rule is modified by the imposition of a wedge between "benefit" and "cost".

The wedge itself is the product of two expressions.[5] The first, called "the marginal cost of public funds", is the correction for the effect of taxation on the supply of labour; this expression must be greater than 1 as long as L_t is negative, as it would be whenever leisure is a normal good. This expression reflects the taxpayer's response to taxation regardless of the amount of public goods supplied. Generalizing somewhat, we would expect the expression to remain as it is regardless of the nature and composition of the projects, programs or activities of the government. The second expression, called "the shadow price of the public goods", is the correction for the effect of the provision of the public good upon the taxpayers' supply of labour; it may be greater or less than 1 depending on whether an increase in the supply of the public good augments or diminishes the taxpayer's willingness to work.

Five aspects of Atkinson-Stern's formula should be noted.

1) The Identical Twins Assumption: All citizens are identical and are treated identically by the government. This assumption supplies a clear and unambiguous criterion for public policy with no need for interpersonal comparison and no recognition of the possibility that the best policy for one person is not the best policy for others.

2) The Proportional Income Tax Assumption: All public revenue is raised by a proportional income tax.

3) The Double Correction: The original Samuelson Rule is modified in two respects, for the marginal cost of public funds - represented by the term $[1/(1 + (t/x)\delta x/\delta t)]$ - and for the shadow price of public expenditure - represented by the term $[1 - N(t/P_G)\delta x/\delta G]$. The modified Samuelson rule is reduced to the original rule if the marginal cost of public funds and the shadow price of public expenditure can be ignored (that is, if they are both equal to 1) or if they cancel out.

4) The Single Public Good: The model abstracts from the nearly infinite diversity of public expenditure. The assumption is important in this context because different items of public expenditure have different degrees of substitutability with labour. Some public expenditures induce a switch in private usage of time from labour to leisure, accentuating the tax-induced distortion associated with the marginal cost of public funds. Other public expenditures do just the opposite

5) The Single Tax-induced Distortion: In this model, the modified Samuelson rule differs from the original rule because, and only because, the increase in the supply of the public good requires an increase in the tax rate generating an increase in the tax-induced diversion of taxpayer's time from taxed labour to untaxed leisure. Abstracted away in this formulation are the

[5]Equation (20) is a rearrangement of equation (3) in Atkinson and Stern, *op cit.*

tax-induced diversions of time from labour to do-it-yourself activities, of time from productive labour to schemed for tax evasion or tax avoidance and of expenditure from saving to current consumption.

The assumptions of the Atkinson-Stern model are quite restrictive in one sense but significantly less so in another. The assumptions, as stated above, wipe away much that is relevant for the conduct of cost-benefit analysis, but the formula in equation (B21) generalizes well beyond the narrow context in which it is derived. The formula shows the original Samuelson rule modified in accordance with two elasticities, the elasticity, ϵ_{Lt} , of tax base to tax rate in the marginal cost of public funds, and the elasticity ϵ_{LG} , of tax base to the supply of the public good. Strictly speaking, the tax base is income, wL, rather than labour supply, L, but the two are the same for all practical purposes the same because the wage, w, is assumed to be invariant. The modified Samuelson rule is in practice considerably broader than its derivation might suggest:

The modified Samuelson rule applies with modest changes to all sources of distortion in the tax system: the choice between work-for-pay and do-it-yourself activity, the choice between consumption and investment in human and physical capital, the choice of how much tax to evade, legally or illegally, when avoidance is for one reason or another costly. Any tax-induced contraction of the tax rate gives rise to a modified Samuelson rule similar to that in equation (19).

The modified Samuelson rule boils down to the original Samuelson rule when either a) the tax base (whatever it may be) is invariant to the tax rate and the supply of public goods, or b) the elasticities ϵ_{Lt} and ϵ_{LG} are such that the corrections to the original Samuelson rule cancel out. If $\epsilon_{Lt} = 0$, then the marginal cost of public funds is equal to 1. If $\epsilon_{LG} = 0$, then the shadow price of the public good is equal to 1. If $(1 + \epsilon_{Lt}) = (1 - \epsilon_{LG})$, then the original Samuelson rule is restored.

The formula remains approximately valid when there are many different kinds of taxes and many different kinds of public goods, but the two extra expressions have to be interpreted differently. The marginal cost of public funds would be the same for all sorts of taxes because the tax payer is made as well off as he can be when marginal distortions per additional dollar of revenue acquired are the same. The shadow price of public projects and public goods would differ from one project to the next because projects have different impacts upon the tax base. Some projects induce taxpayers to devote time to the acquisition of highly taxed goods. Other projects induce taxpayers to devote time to the acquisition of less taxed or untaxed goods. The greater the effect of the project on the taxpayer's propensity to acquire highly taxed goods, the larger ϵ_{LG} and te smaller the shadow price of the project must be.

It is in the light of the potential generalization of the modified Samuelson rule in equation (B21) that one should assess the mismatch, bordering on blatant contradiction, between the identical twins assumption (1) and the proportional income tax assumption (2). If people really were identical, they would not be so stupid as to raise public revenue through a proportional income tax. Rather they would levy a lump sum tax for which there really is no deadweight loss, and for which the original Samuelson rule would be strictly valid. That however is no real criticism of the model where the identical twins assumption is introduced for analytical convenience, and the proportional income tax assumption is employed because we know full-well that the identical twins assumption is never true and that tax systems must be restricted to what might be appropriately imposed in a world where people are far from identical.[6] It is hardly a valid criticism of this model that the identical twin assumption and the proportional income tax assumption are unrealistic, for *all* economic analysis - indeed all clear thinking - is based on strategic assumptions that are imperfectly descriptive of the world.

Appendix C: Why the Wedge Vanishes when a) the Public Good is an Input in the Production of the Private Good and b) the Deadweight Loss in Taxation Arises from a Distortion in the Labour-leisure Choice.[7]

Now we introduce a change in the role of the public good. Now the public good is no longer an ingredient of the utility function. Instead, it serves to augment the productivity of labour. Now, the utility function is u(x, L) rather than u(x, L, G), but the wage is an increasing function, W(G), of the supply of public goods rather than a parameter, w, as in the preceding appendices. This change in the role of public goods is sufficient to restore the original Samuelson rule.

In showing this to be so, it may be helpful to begin by applying our new assumptions about the role of G in the economy to the derivation in Appendix A of the Samuelson rule for an omniscient, omnipotent and benevolent planner. Once again, the planner chooses x, L and G to maximize the typical person's utility subject to a production constraint and the production possibility frontier for the economy as a whole. The economy-wide production possibility curve is

$$Nx + P_G G = NW(G)L \qquad \text{(C1)}$$

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[6]Another consideration becomes important when a social-welfare-maximizing government chooses a tax structure for a two class society to maximize the welfare of the poor for any given level of welfare of the rich and any given provision of the public good. In that case the original Samuelson rule must be modified for differences between rich and poor in their marginal valuations of the public good in terms of the private good. Broadly speaking, the sum of the marginal valuations of the public good should exceed the marginal cost if and only if the marginal valuation of the poor exceeds the marginal valuation of the rich. For a thorough analysis of this proposition and its qualifications, see Boadway, R. and Keen, M., (1993).

[7]This proof is a special case of a earlier result: if utility (u) is a function of private goods (x), public goods (G) and hours of work (h) and if h is separable from x and G in the utility function, that is, if there is some function v(x, G) such that u = u(h, v(x, G)), then the original Samuelson rule remains valid, with no requirement for a correction to account for the marginal cost of public funds. See Diamond and Mirlees (1971) and, especially, Boadway and Keen (1993).

which is the same as that in appendices A and B except for the replacement of w with W(G).

The Lagrangian and its first derivatives are

$$\mathcal{L} = u(x, L) - \alpha[Nx + P_G G - NW(G)L] \tag{C2}$$

$$\mathcal{L}_x = u_x - \alpha N = 0 \tag{C3}$$

$$\mathcal{L}_L = u_L - \alpha W(G)N = 0 \tag{C4}$$

and

$$\mathcal{L}_G = -\alpha[P_G - NW'(G)L] = 0 \tag{C5}$$

from which it follows immediately that

$$-u_L / u_x = W(G) \tag{C6}$$

and

$$N[W'(G)L] = P_G \tag{C7}$$

which reproduces the Samuelson rule in equation (A4) because the rate of trade-off in production between x and G and their rate of trade-off in use must be the same. Since u depends here on x but not on G, the rate of trade-off in use is

$$\{du/dG\}/\{du/dx\} = \{(du/dx)(dx/dG)\}/\{du/dx\} = dx/dG = W'(G)L \tag{C8}$$

as long as u_G is reinterpreted as the total derivative of u with respect to G.[8]

Now reconsider the double maximization procedure where, for any t and G, the taxpayer chooses x and L to maximize utility, and then the welfare-maximizing planner, taking account of the taxpayer's behavour, maximizes the taxpayer's utility in the choice of t and G. The taxpayer choosing x and L in an environment where t and G are looked upon as invariant. He chooses x and L to maximize u(x, L) subject to a budget constraint

$$x = rL \tag{C9}$$

where r is his net after-tax wage

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[8]Differentiating the Lagrangian in equation (25) with respect to G yields the equation

$$du/dG = \alpha\, NW'(G)L = (u_x / N)NW'(G)L$$

from which it follows that $(du/dG)/u_x = w'(G)L$

$$r = w(G)(1 - t) \tag{C10}$$

which he looks upon as invariant regardless of his choice of x and L. Utility becomes

$$u(x, L) = u(rL, L) \tag{C11}$$

so that
$$u_x r + u_L = 0 \tag{C12}$$

or equivalently
$$- u_L/u_x = r \tag{C13}$$

From these equations it follows that

$$x = x(r),\ \ L = L(r), \tag{C14}$$

and
$$\delta r/\delta G = w'(G)(1 - t), \quad \delta r/\delta t = - w(G),$$

$$\delta x/\delta G = (\delta x/\delta r)(\delta r/\delta G), \qquad \delta L/\delta G = (\delta L/\delta r)(\delta r/\delta G),$$

$$\delta x/\delta t = (\delta x/\delta r)(\delta r/\delta t) \quad \text{and} \quad \delta L/\delta t = (\delta L/\delta r)(\delta r/\delta t).$$

Recognizing the effects of t and G upon the taxpayer's choice of x and L, the government now chooses t and G to maximize

$$\mathcal{L} = u(x, L) - \beta[Nx + P_G G - NW(G)L] \tag{C15}$$

which is the same as the objective function of the omniscient planner except for the substitution of β for α as the Lagrange multiplier. Differentiating with respect to t,

$$\mathcal{L}_t = u_x \delta x/\delta t + u_L \delta L/\delta t - \beta N[\delta x/\delta t - w(G)\delta L/\delta t] = 0 \tag{C16}$$

Differentiating with respect to G,

$$\mathcal{L}_G = u_x \delta x/\delta G + u_L \delta L/\delta G - \beta N[\delta x/\delta G - w(G)\delta L/\delta G] + \beta\{N[w'(G)L] - P_G\} = 0 \tag{C17}$$

which, using properties of the taxpayer's optimization problem, implies that

$$\mathcal{L}_G = \mathcal{L}_t \{(\delta r/\delta G)/(\delta r/\delta t)\}\{- w'(G)(1 - t)/ w(G)\} + \beta\{N[w'(G)L] - P_G\} = 0 \tag{C18}$$

Since $\mathcal{L}_t = 0$, equation (40) reduces to

$$N[w'(G)L] = P_G \tag{C19}$$

which is precisely the original Samuelson rule already derived as equation (C7) above for the omniscient. omnipotent and benevolent planner.

Appendix D: Why the Wedge Persists, even if the Public Good is an Input into the Production of the Private Good, when the Deadweight Loss in Taxation Arises from Tax Avoidance or Tax Evasion

Consider an economy where tax evasion is costly but foolproof. Taxpayers can conceal income from the tax collector with no risk of discovery, but there is a cost of concealment, C(E), where E is the amount of income concealed from the tax collector and C is a progressively increasing function of E. Then, tax is evaded up to the point where the marginal cost of hiding a dollar is just equal to the marginal tax that would otherwise be paid. With a tax, T, levied as an increasing function, T(B), of declared income, B, and with a cost of concealment, the taxpayer with an income of Y chooses E to minimize the sum of T(Y - E) and C(E). He chooses E such that $C' + T' = 0$.

More generally, think of taxpayer as devoting a fraction α of his incomes to concealing a portion $f(\alpha)$ of his incomes from the tax collector, reducing his tax paid from tw(G)L to $tw(G)L(1 - \alpha)[1 - f(\alpha)]$, and reducing his net - post tax and post cost-of-concealment - income from $(1 - t)w(G)L$ to $w(G)L - \alpha w(G)L - tw(G)L[(1 - \alpha)(1 - f(\alpha)]$. Given t and G, the taxpayer chooses L and α to maximize his net income. The maneuver can only be advantageous to the taxpayer when $f(\alpha)$ is significantly greater than α when α is close to 0 and when $f'(\alpha) > 0$. The amount of tax evasion can only be constrained when $f'' < 0$, so that additional tax evasion becomes progressively costly to the taxpayer. The more effort one puts into the concealment of income, the more of one's income is concealed, but successive increments of concealment effort have successively smaller impacts on observed income, ensuring that concealment remains well short of 100%. Alternatively, the taxpayer can then be thought of as choosing his income net of tax evasion, Y and his tax base, B - where $Y = Y(G, t)$, $B = B(G, t)$, $Y_G > 0$, $Y_t < 0$, $B_G > 0$ and $B_t < 0$.

Since $Y = w(G)L(1 - \alpha)$, it might be assumed that both L and B are affected by t and G, but to focus on tax evasion and to show how tax evasion differs from the labour-leisure choice in its affect on the wedge in the Samuelson rule, we shall assume from here on that L is invariant, that it is affected by neither t nor G. With that assumption, what differentiates the model in this appendix from the model in Appendix C is that a tax-sensitive L is being replaced with a tax-sensitive B. Furthermore, since L is invariant and since G is an input into the private good, the taxpayers' objective function is nothing more than his private consumption, Y - tB where Y is only affected by t because, the larger t the greater the proportion of earned income that is wasted in tax evasion.

Thus, the government seeks to maximize Y - tB with respect to its budget constraint

$$P_G G = NtB \tag{D1}$$

It can be seen as maximizing the Lagrangian

$$\mathcal{L} = NY - NtB - \lambda[P_G G - NtB] \tag{D2}$$

The first order condition with respect to G becomes

$$\mathcal{L}_G = NY_G - NtB_G - \lambda[P_G - NtB_G] = 0 \qquad \text{(D3)}$$

so that

$$NY_G = \lambda P_G - (1 - \lambda)NtB_G \qquad \text{(D4)}$$

which would boil down to the original Samuelson rule if $\lambda = 1$, for, in that case,

$$NY_G = P_G \qquad \text{(D5)}$$

But the parameter is not equal to 1 because Y and B are both dependent on t as well as on G. The first order condition of the Lagrangian with respect to t is

$$\mathcal{L}_T = NY_t - N(tB_t + B) + \lambda N[(tB_t + B)] = 0 \qquad \text{(D6)}$$

from which it follows that

$$\lambda = [Y_t - (tB_t + B)]/[(tB_t + B)] \qquad \text{(D7)}$$

which must necessarily be greater than 1 because, by construction, $Y_t < 0$.

Returning to equation (D4), we see at last that, since $\lambda > 1$,

$$NY_G > P_G \qquad \text{(D8)}$$

which is consistent with equation (3) in the text rather than with equation (1).If the public good is an input into the private good, then the labour-leisure choice can have no bearing on the optimal supply of the public good. For any given supply of labour, the amount of the public good is chosen to maximize the net output of the private good. Tax evasion is different. It always interposes a spread between the cost of an extra dollar of public expenditure to the taxpayer and to the government, a spread that should be reflected as a wedge between “benefit” and “cost” when the Samuelson rule is employed in cost-benefit analysis. .

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