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Binomial R&D Races and Growth

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Abstract

In each period, we have an R&D race among N competitive R&D firms, each with probability π of discovering a successful new technique for producing an intermediate good used in producing the economy's final consumption good. The winner of a race earns a monopoly profit over a generally uncertain interval. Each R&D firm faces distinctive "lottery" and "duration" uncertainty in each period. Numerical examples illustrate the growth behavior of the economy linked to the R&D sector.

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1 Introduction

A firm in an R&D race with similar competitors faces two sorts of risk: whether it will be the winner in the current race ("lottery uncertainty"), and how long the current race persists ("duration uncertainty"). We are able to work with these two sorts of risk separately in a discrete-time model based on a binomial specification of the race. A limiting case has only "lottery uncertainty" facing each R&D firm. In general, each of N firms has probability π of making a successful R&D hit in the current period leaving $(1-\pi)^N$ the probability that the current race moves on to the next period with no new winner.[1] The incumbant or most recent winner continues to hold the proceeds of being a winner. In the event of multiple "successes" in the current period, the winner is drawn at random from the pool of successful "hitters". We graft this simple formulation of an R&D race onto the simple model in Aghion and Howitt [1992] of an economy with one consumer good, one intermediate good used in the consumer goods sector, and an R&D sector. The current winner of the race reaps a monopoly profit[2] in supplying the intermediate good to the consumption goods sector. Growth turns on the split of the fixed supply of skilled labor between the R&D sector and the consumer goods sector. The economy's knowledge stock is incremented with the successful termination of each R&D race. Hence investment is in knowledge capital and the amount of current investment is a function of the size of the R&D sector. Being the current winner or incumbant is the way knowledge gets capitalized by a market-like process. Thus capitalization does not satisfy the usual arbitrage condition for marginal investment expenditure. Since in general the duration of incumbancy as the current winner of a race is a random variable, knowledge capitalization by a private agent is an uncertain process.

There are two central issues here. First how does the exogenous increment in knowledge capital "determine" current aggregate R&D "effort" or in the size of the R&D sector and secondly, how do parameters feed into the determination of the rate of expected growth?

[1] The incumbant turns out to be doing no R&D in this model and n is the number of R&D firms. Hence our N here will be $n-1$ below when we fill in details.

[2] Since the monopoly is linked to an uncertain duration, the winner is not being granted a patent, in the sense patent is usually defined.

Of interest is that for the same increment in knowledge capital, one can have larger values for both π and the size of the R&D sector, and these larger values correspond to a higher expected growth rate for the economy. Aghion and Howitt based their R&D races on a Poisson process for the arrival of R&D successes, the number of successes in an interval being scaled up by the number and effort of R&D firms. We are able to map the form of our solution to the model onto theirs without "forcing". Hence we end up with the Aghion-Howitt model with a simpler formulation of basic R&D race, central to the analysis. Three new propositions we arrive at are: "duration uncertainty" shows up in a simple risk-adjusted discount rate; a higher growth rate $\gamma - 1$ is associated with a lower value of π; and we observe the limiting case of "certain" growth across consecutive periods with each outside R&D firm incurring only "lottery uncertainty". We simplify matters by restricting attention to the so-called "linear case" in Aghion-Howitt; that is, each R&D worker is an R&D firm. This allows us to shift attention to other intricacies of the analysis.

2 The R&D Race

Each R&D worker is a distinct R&D firm, here. This makes the cost of "production" for a firm in a period simply the prevailing wage, w_t for a skilled worker.[3] Consider research firm i, among N such firms in period t. Each firm is doing research and before the period ends some succeed and some do not. Given probability π of success for a firm, we have $\binom{N}{k}\pi^k(1-\pi)^{N-k}$ as the probability of k firms having a successful research hit in the period. The probability that firm i is among these k winners is k/N. Hence $(k/N) * \binom{N}{k}\pi^k(1-\pi)^{N-k}$ is the probability that there are k winners and firm i is among them.

Assumption 1: The monopoly-right is allocated by lot to one of the k current winners.

Hence, contingent on their being k winners and firm i is among them, the probability

[3] It seems unobjectionable to be assuming a minimum optimal scale for an R&D firm, i.e. one researcher per firm. We must be more careful about the possibility of researchers combining for form a "more powerful" R&D enterprise. We can rule this out by statistical independence across workers in R&D worker effort and prospective success. Each worker has probability π of success in the current period, independent of the activity of other workers. This suffices to rule out "large", more efficient R&D enterprises.

that firm i gets the monopoly-right is

$$\begin{aligned} & (1/k) * (k/N) * \binom{N}{k} \pi^k (1-\pi)^{N-k} \\ = \quad & (1/N) * \binom{N}{k} \pi^k (1-\pi)^{N-k}. \end{aligned}$$

Hence for firm i to win the current race, it must be among the k successful "hitters" at R&D activity and it must also be drawn from among the $k-1$ other currently successful firms. Now in period t there are either 0 winners, 1 winner, 2 winners, etc. Hence the probability that firm i gets the monopoly-right in period t is

$$\begin{aligned} & 0 + \left[\frac{1}{N}\right] \{ \binom{N}{1} \pi^1 (1-\pi)^{N-1} + \binom{N}{2} \pi^2 (1-\pi)^{N-2} + ... + \binom{N}{N} \pi^N \} \\ = \quad & \left[\frac{1}{N}\right] \Pi \end{aligned}$$

for $\Pi \equiv 1 - [1 - \Pi] = 1 - [(1-\pi)^N]$. Π/N tends to $(1/N)$ for N relatively large and π near unity since then $1 - \Pi \cong 0$. Central to the analysis is the case of none of the N active R&D firms being successful in the current period. This event occurs with probability $1 - \Pi$. Hence

Π is the probability that a winner of a race emerges in the current period.

Assumption 2: The state is realized at the end of the period and each firm in a race spends w_t at the beginning of period t. The winner of the race in period t receives current monopoly profit P_{t+1} at the start of the next period.

The incumbant in period t was a winner in period $t-1$ and hence is reaping current profit P_t. Expected discounted profit of the incumbant in period t is

$$P_t^e = P_t - w_t + \rho \frac{1}{n_t} \Pi_t P_{t+1}^e + \rho \frac{n_t - 1}{n_t} \Pi_t P_{t+1}^{el} + \rho \left(1 - \Pi_t\right) P_t^e \tag{1}$$

where $\frac{1}{n_t}\Pi_t$ is the probability[4] that she is the current winner of the current R&D race taking place in period t, $\frac{n_t - 1}{n_t}\Pi_t$ is the probability that she is the loser of the current race, and $(1 - \Pi_t)$ is the probability that she remains the incumbant because the current race has no winner (no successful developer of the new technology). P_{t+1}^{el} is the expected discounted profit accruing to the loser. ρ $(= 1/(1+r))$ is the discount factor and r is the discount rate.

[4] The subscript on Π indicates that n_t can be changing in the process of convergence to the solution with n_t unchanging.

Equation (1) clearly only makes sense if P_t^e on the right hand side is a correct "extrapolation" to period $t+1$ from the value of P_t^e, on the left hand side, in period t. This expectations critique applies also to the "extrapolation" of P_{t+1}^e and P_{t+1}^{el} from the agent's position in period t. Roughly speaking agents must be correct in some sense at predicting future values both along "steady state" paths (n_t unchanging) and along transient paths (n_t changing). However, the presence of the same P_t^e on both sides indicates the implicit assumption that agents are on the "steady state" path, one with n_t unchanging.[5] There is a sense in which a correct up-dating by agents is easier along the steady state path than along "off equilibrium" paths. Hence we proceed under

Assumption 3: We consider only the case of "steady state" (essentially n_t unchanging) behavior of the model. We assume that the expectations of agents support the "steady state" behavior of the model.[6]

Assumption 4: One of n_0 R&D firms is randomly selected at time zero to be the monopoly-right holder with current profit, P_0.

For the outsider (non-incumbant current "winner"), expected discounted profit is

$$\begin{aligned} P_t^{eL} &= \{-w_t\} + \rho\frac{1}{n_t}\Pi_t P_{t+1}^e \\ &\quad +\rho\left(\frac{n_t-1}{n_t}\right)\Pi_t P_{t+1}^{eL} + \rho\left(1-\Pi_t\right)P_t^{eL} \end{aligned} \tag{2}$$

where $\frac{1}{n_t}\Pi_t$ is the probability that this outsider wins the current race; $\left(\frac{n_t-1}{n_t}\right)\Pi_t$ is the probability that she loses the current race; and $(1-\Pi_t)$ is the probability that she remains an outsider because the current race has no winner.

Assumption 5: Firms are risk neutral and thus the conditions in (1) and (2) characterize "profit-maximizing" equilibria for R&D firms along the equilibrium growth path.

The concept of **zero expected profit for an R&D firm** works as follows. Suppose in equilibrium one new firm enters the R&D sector from a sector with wage w_t.[7] In

[5] I am indebted to Peter Howitt for drawing my attention to this matter.
[6] Some of the more interesting propositions in Aghion and Howitt involved off "steady-state" paths in our sense. We are concerned that Aghion and Howitt may not have specified fully an expectations machinery that "supported" the hypothesized off "steady-state" motion of the model.
[7]

R&D this firm must earn w_t dollars. Hence current expected earnings from entry, namely $\rho\frac{1}{n}\Pi P^e_{t+1} + \rho\left(\frac{n-1}{n}\right)\Pi P^{eL}_{t+1} + \rho\left(1-\Pi\right)P^{eL}_t$ must equal w_t. This equality implies $P^{eL}_t = 0$ in (2). And we infer

Proposition 1: $P^{eL}_t = 0$ for all t. Hence the basic zero profit entry condition for an outside R&D firm reduces to

$$0 = \{-w_t\} + \rho\frac{1}{n_t}\Pi_t P^e_{t+1} \text{ for } (1-\Pi_t) = (1-\pi)^{n_t}. \tag{3}$$

(1) reduces to $P^e_t = P_t - w_t + \rho\frac{1}{n_t}\Pi_t P^e_{t+1} + \rho\left(1-\Pi_t\right)P^e_t$. And this, in light of (3) implies that the incumbant is indifferent whether to engage in current R&D activity or not. This leads to

Assumption 5: The incumbant does no R&D.[8]

There is a literature on incumbants doing less research the outsiders (see references in Aghion and Howitt [1992]). The standard argument is that the value of an extra dollar of prospective profit is higher for an outsider than it is for the incumbant since the incumbant is currently reaping some profit. The arguments usually lead the incumbant putting in less R&D effort than an outsider. Here R&D effort is "integerized" so that marginally less R&D effort for the incumbant translates into zero effort. It is this integerization of R&D effort which also gives us the seemingly novel result that the incumbant is only indifferent between engaging in R&D activity or not so engaging. Since the incumbant gains zero expected profit from currently doing R&D, we treat her as doing none.

We now have just $n_t - 1$ R&D firms engaged in R&D activity in each race; the *nth* firm being the incumbant. Hence (3) becomes

$$0 = \{-w_t\} + \rho\frac{1}{n_t - 1}\Pi_t P^e_{t+1} \text{ for } (1-\Pi_t) = (1-\pi)^{n_t - 1}. \tag{4}$$

This would be a skilled worker leaving consumer goods production. Skilled workers are scarce and work in one of two sectors: R&D or consumer goods production. More on this below. We gloss over the integer issue. Thus if n^* is the current equilibrium number of R&D firms, we continue to work with n^* firms after the entry of the marginal firm, rather than $n^* + 1$ firms.

[8] We note that an incumbant has nothing to gain by hiring an outside firm to do R&D in the hope that in so doing her incumbancy could be extended. She might as well being doing such R&D herself, which we have seen, has no positive expected profit associated with it.

And (1) becomes $P_t^e = P_t - w_t + \rho \frac{1}{n_t - 1} \Pi_t P_{t+1}^e + \rho (1 - \Pi_t) P_t^e$. Hence (4) and Proposition 1 imply that (1), reduces to $P_t^e = P_t / \{1 - (1 - \Pi)\rho\}$. Now we note that $1 - (1 - \Pi)\rho = \{r + \Pi\}/\{1 + r\}$. Hence, on substituting for P_t^e in (4), one gets the **basic zero profit entry relation**[9]

$$w_t = \frac{1}{n_t - 1} \left[\frac{1}{1 + \frac{r}{\Pi_t}} \right] P_{t+1} \text{ for } \Pi_t = 1 - (1 - \pi)^{n_t - 1}. \tag{5}$$

The "duration risk" associated with Π_t now turns up in a risk-adjusted discount rate $\frac{r}{\Pi_t}$. Note that duration risk increases the gap between w_t and P_{t+1}, other things being the same. "Lottery risk" associated with participation in an R&D race remains at $\frac{1}{n_t - 1}$. (5) resembles a reduced form rule for an outside R&D firm: invest w_t at the beginning of each period in the hope of winning a lottery with a prize of current value, $\left[\frac{1}{1 + \frac{r}{\Pi_t}} \right] P_{t+1}$. A firm in the race in "balanced growth" is chained to this open-ended mode: buy a lottery ticket at the beginning of each period and await the outcome which "is announced" at the end of the period. This "reduced form" formulation is associated with the illusion that a winner "is selected" at the end of each period. The ultimate probability of there being no winner among outsiders in the current period (probability $1 - \Pi_t$) is showing up as a premium in P_{t+1}, gleaned by a winner. With no duration risk (e.g. π near unity and n_t "large" and Π_t very near unity), there is at least one successful R&D hit in each period, almost surely. Growth across each consecutive period is "certain", though the identity of the current R&D winner in each period remains uncertain. Then (5) reduces to $w_t = \frac{1}{n_t - 1} \left[\frac{1}{1 + r} \right] P_{t+1}$. More on this limiting case below.

We now turn to a model of a simple economy with one final consumer goods sector, one R&D sector, and one sector linked to an intermediate good used in the consumer goods sector. This model is constructed so that the economy expands with successful R&D in a special proportion ("balanced growth"). The bottom line is a solution with growth with an unchanging number of workers in consumer goods activity and an unchanging number of R&D firms. We follow Aghion and Howitt [1992]. Given the appropriate model of the economy, we solve for n in (5) above.[10]

[9] With our notation, this compares with Aghion and Howitt (3.1), $w_t \geq \frac{1}{n_t} \left[\frac{1}{1 + \frac{r}{\lambda n_t}} \right] P_{t+1}$. We have inserted n_t for their $\varphi(n_t)$ and our Π_t is their $\lambda \varphi(n_t)$, their so-called "poisson arrival rate". We have $\frac{1}{n_t - 1}$ in place of their $\frac{1}{n_t}$. A "poisson arrival rate" is the average number of events or arrivals per unit time, in the long run, given an underlying Poisson stochastic process. Implicit in Aghion-Howitt was an arrival rate, less than unity. This seems "intuitive". Their limiting case would be an arrival rate of unity.

[10]

3 Output, Profit and Growth

The are L skilled workers, with n in R&D activity and $x = L - n$ working to produce an intermediate good used in final goods production. We leave out the time subscripts on n and x for the moment. For fixed L the problem becomes one of solving for the equilibrium split of L between the two competing activities: goods production and knowledge production. x can be viewed as the quantitiy of intermediate goods currently being produced and $y = AF(x)$ is the current amount of final (consumption) good being produced. The effect of successful R&D is to increase A in $A_t = \gamma A_{t-1}$, with γ the growth factor and $\gamma - 1$ the growth rate for the economy. A special case of our model has almost certain growth between periods, but in general growth is random because there is the possibility of no R&D firm in the current race being successful in the current period.

We turn to the determination of skilled-wage w and current profit P. (We follow Aghion-Howitt here.) The inverse demand curve facing the intermediate goods producing monopolist is $A_tF'(x_t)$. Hence price charged is

$$p_t = A_tF'(x_t).$$

The monopolist chooses her x_t to maximize $[A_tF'(x_t) - w_t]x_t$, taking as given, A_t and the wage w_t of skilled labor. Since the wage and profit level will grow at the balanced growth rate $\gamma - 1$ in equilibrium, we define a stationary wage, $\omega_t = w_t/A_t$ and the "marginal revenue function" as $\widetilde{\omega}(x) = F'(x) + xF''(x)$. We assume that the marginal revenue is downward sloping and satisfies Inada-type regularity conditions. Then for any positive ω_t the monopolist's choice of output x_t is given by the first order condition

$$\omega_t = \widetilde{\omega}(x_t)$$

or

$$x_t = \widetilde{x}(\omega_t),$$

where $\widetilde{x}(\omega_t)$ is the inverse function, $\widetilde{\omega}^{-1}(x_t)$. The monopoly profit in a period is

$$P_t = A_t\widetilde{\pi}(\omega_t),$$

In Hartwick [1991] optimality in R&D intensity occurred when the expected value of the prize for being the current winner in the race, to an entrant, was equal to the social value of the marginal entrant advancing the expected date of R&D success.

where $\widetilde{P}(\omega) = -(\widetilde{x}(\omega))^2 F''(x(\omega))$. $\widetilde{x}$ and $\widetilde{P}$ are each strictly positive-valued and strictly decreasing for all positive ω_t. We can thus express $P_{t+1} = \frac{A_{t+1}}{A_t} P_t$. Recall the presence of P_{t+1} in (5) above.

For the case of $F(x) = x^\alpha$, $0 < \alpha < 1$, one obtains

$$p_t = \frac{w_t}{\alpha},\ P_t = \left(\frac{1-\alpha}{\alpha}\right) w_t x_t,\ \text{and } x_t = \left(\frac{\omega_t}{\alpha^2}\right)^{1/(\alpha-1)}. \tag{6}$$

Hence wage w_t, profit level P_t, and price p_t each rise at rate $\gamma - 1$between two periods with probability Π_t. In the "steady state", x_t and n_t $(= L - x_t)$ remain unchanging and all key magnitudes increase at rate $\gamma - 1$.

4 "Steady State" Growth

We turn to solving the for the case of growth with n_t unchanging. There is a straightforward sketch of the "steady state" solution. We have the stationary, unchanging inverse demand schedule for the consumer good, $\frac{p_t}{A_t} = \alpha x^{\alpha-1}$. Corresponding to this is the demand schedule for labor in consumer goods production, $\alpha\frac{p_t}{A_t} = \frac{w_t}{A_t} = \alpha^2 x^{\alpha-1}$. This labor demand is a constant fraction α of the inverse demand shedule for the consumer good. Recall that x is both current consumer goods produced as well as the current number of skilled workers in the consumer goods sector. Solving involves finding an x such that monopoly profit to the current R&D "winner", P_t/A_t is the residual revenue in the consumer goods sector, after labor in production, $\frac{w_t}{A_t}x$ has been paid. That is, $P_t = (1-\alpha)p_t x$. Labor supply is $x = L - n$. Growth is of course stochastic since in some periods no R&D firm experiences a successful "hit". The "steady state" solution is an n satisfying

$$1 = \gamma \frac{1}{n-1}\left[\frac{1}{1+\frac{r}{\Pi}}\right]\left[\frac{1-\alpha}{\alpha}\right][L-n] \text{ for } \Pi = 1 - (1-\pi)^{n-1}.$$

We can now solve numerically, for the "steady state". We treat $P_{t+1} = \gamma P_t$ and thus restrict our attention to paths of strictly proportional expansion at rate $\gamma - 1$. For $r = 0.11$, $L = 79$, $n = 18$, $\pi = 0.11$, and $\alpha = 0.77$ we obtained $\Pi = 0.862079$ and $\gamma = 1.05205$. We perturbed parameters and re-solved. We observed that γ increased respectively with n $(= L - x)$, α and r and decreased with π.[11] This latter result is somewhat paradoxical since

[11] Note that $\frac{d\Pi}{d\pi} = (n-1)(1-\pi)^{n-2}$, which is positive for $n > 2$. One then has $\frac{d\gamma}{d\pi} = -\{Z^{-2} r \Pi^{-2} \frac{d\Pi}{d\pi}\}/Z^{-1}$ for $Z = (1 + \frac{r}{\Pi})$. Hence $\frac{d\gamma}{d\pi} < 0$.

one might expect the growth factor to be larger with R&D per firm "more effective". Not so for this model. It is true however that, holding γ constant, a simultaneous increase in n and π does result in a higher expected growth rate for the economy. (The current expected rate of growth is $g = [\gamma - 1] * \Pi$ and the variance of the growth rate is $\Pi * [[\gamma - 1] - g]^2 + [1 - \Pi] * [0 - g]^2$.) We emphasize however that the fundamental result of more rapid growth ($\gamma - 1$ increasing) requiring a larger R&D sector (larger n), other things being the same, is observed.

Capitalization of R&D in the value of an R&D firm is working as follows. In any period, the expected value of an R&D firm is $\frac{1}{n} P_t^e$ ($+ \frac{n-1}{n} P_t^{eL}$ and $P_t^{eL} = 0$). Hence the expected value of the group of n such firms is P_t^e. This value reflects the capital value in the private sector of current knowledge capital, A_t. We observed above that $P_t^e = P_t / \{1 - (1 - \Pi)\rho\} = \left\{ \frac{1+r}{\Pi + r} \right\} P_t$.[12] Hence the capital value of being the current winner exceeds current monopoly profit, P_t when $0 < \Pi < 1$. And in the special limiting case of $\Pi = 1$, $P_t^e = P_t$. The central case of $0 < \Pi < 1$ involves valuation of knowledge capital over more than one period. In the limiting case, valuation is simply intraperiod by intraperiod.

5 Almost Certain Growth and r Endogenous

For π very near unity and n "large", we have $1 - \Pi \cong 0$. Growth is almost certain because at least one of the n R&D firms will have a successful R&D hit in each period. (5) reduces to

$$w_t = \frac{1}{n_t - 1} \left[\frac{1}{1 + r} \right] P_{t+1}.$$

The length of each race for an R&D success is no longer uncertain. Clearly complexity in this world arises from "duration uncertainty". We solved for a "steady state" case of almost certain growth[13] with π increased to 0.34 from 0.11, above. Then $\Pi = 0.99914$ and $\gamma = 1.03661$.

The growth rate across periods is the certain $\gamma - 1$. This allows us to endogenize r. We can solve for the interest rate r in the Ramsay expression $r = \beta - \frac{U''(y)\dot{y}}{U'(y)}$ for $U(y)$ the aggregate utility of consumption and β the utility discount rate. Corresponding to a constant relative

[12]This is very similar to (2.12) in Aghion and Howitt, with again our Π in place of their "Poisson arrival rate". We have an extra $1 + r$ in the numerator which leads to $P_t^e = P_t$ for the limiting case of $\Pi = 1$.

[13]Notice that we simply increased π in this example relative to the previous example. This resulted in Π moving up noticeably whereas γ was reduced very little.

risk aversion utility function, we have

$$r = \beta + \frac{\sigma \dot{y}}{y}$$

with σ the positive risk aversion parameter. In "balanced growth", $\frac{\Delta y}{y} = \gamma - 1$. Hence a value for r emerges. A larger γ in this special case corresponds to a larger r. For the general case with "duration uncertainty", the growth rate is a random variable and we cannot nail down a value for the discount rate with this approach immediately above.

6 Concluding Remark

Our binomial R&D race has the merit of easy accessability. Uncertainty facing a firm in an R&D race factors into "lottery uncertainty" and "duration uncertainty" and the latter shows up in a risk-adjusted discount rate in the basic equation characterizing entry to an R&D race. A limiting case of the model with no duration uncertainty emerges with this risk adjusted discount rate becoming the familiar certain rate. In this limiting case, each R&D firm knows that one firm will be a winner each period, but not which particular firm. We also observed the interesting case of the growth increment remaining virtually unchanging while the expected growth rate rose considerably. The expected growth rate rose with a simultaneous increase in n and π. We have been restricted to an analysis of "steady states" and so the question of convergence to such states has been left open. One needs a more complete behavioral mechanism along the lines of "rational expectations" to have agents up-dating correctly along non-steady state paths. There is more interesting work to be done on this model.

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