Population Growth and Firm Dynamics*

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Abstract

Population growth has declined markedly in almost all major economies since the 1970s. We argue that this trend has important consequences for the process of firm dynamics and aggregate growth. We study a rich semi-endogenous growth model of firm dynamics, and show analytically that a decline in population growth reduces creative destruction, increases average firm size and concentration, raises market power and misallocation, and lowers aggregate growth in the long-run. We also show that lower population growth has positive effects on the level of productivity, making the short-run welfare impacts ambiguous. In a quantitative application to the U.S we find that the slowdown in population growth since the 1980s and the projected continuation of this trend accounts for a substantial share of the fall in the entry rate and the increase in firm size. The effect on aggregate growth is positive for around one decade, before turning negative thereafter. The impact on markups is modest.

Keywords: Growth, Firm Dynamics, Creative Destruction, Demographics, Dynamism, Markups

JEL Codes: O40, D22, J11, O47

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1 Introduction

Almost all major economies experienced a substantial fall in population growth in recent decades. Figure 1 shows historical population growth for a group of major world economies from 1960 to 2020. Despite different political systems, cultures and levels of development, a clear downward trend is evident for all of them. Moreover, according to the UN, this trend is projected to continue for at least the first half of the twenty-first century, driven largely by continuing falls in fertility.[1] A world of low and falling population growth looks like it is here to stay.

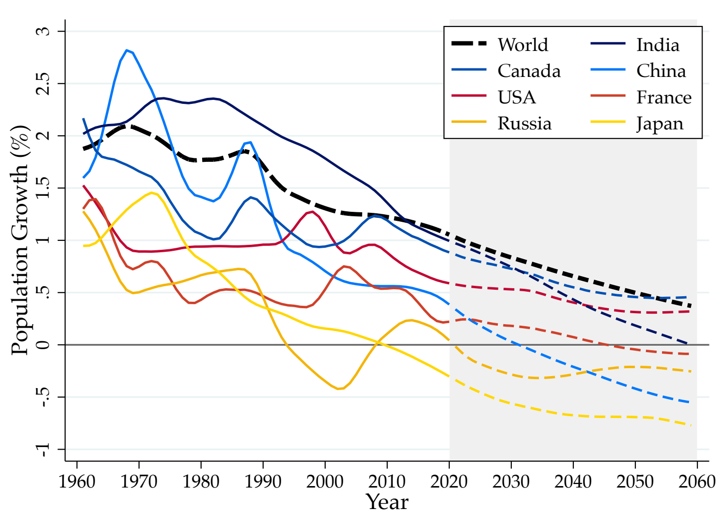
In this paper we show that this phenomenon is likely to have important implications for the process of firm dynamics and aggregate economic performance. We do so in the context of a firm-based model of semi-endogenous growth that is rich enough to rationalize many first-order features of the process of firm dynamics. The theory makes tight predictions for the likely effects of falling population growth: a slow-down in population growth reduces creative destruction and entry, increases concentration and average firm size, raises market power and lowers long-run aggregate productivity growth.

Our baseline model is an enhanced version of Klette and Kortum (2004), augmented by the possibility of population growth, new variety creation, own-innovation and a demand elasticity that exceeds unity The model has a full analytic solution and we can express the process of firm dynamics, the resulting firm size distribution and the aggregate growth rate directly as a function of population growth.The reason why falling population growth reduces entry and increases concentration and firm size is the following: Declining population growth reduces creative destruction by lowering firms' incentives to engage in product innovation. Importantly, in equilibrium this decline in creative destruction is only accommodated through a decline in entry - the rate of product creation by incumbent firms is unaffected by changes in population growth. This change in the composition of product creation implies that lower population growth increases firm growth conditional on survival and reduces incumbent firms' exit hazards. As a consequence, concentration and firm size rises and the entry rate falls.

Our theory also makes clear predictions about the relationship between population growth and income per capita growth. As in many aggregative models of semi-endogenous growth, the long-run equilibrium growth rate declines as the rate of population growth falls. However, we show that there is an important countervailing effect that makes the relationship between population growth and welfare ambiguous. By reducing creative destruction and hence the rate of firm-exit, falling population growth increases the value of firms because future profits are discounted at a lower rate. Free entry therefore requires an increase in the economy-wide level of varieties to increase competition. This raises income per-capita because of variety gains. The welfare consequences of declining population growth therefore hinge on the relative importance of these static variety gains relative to the dynamic losses due to lower growth.

[1] See Section OA-1.2 in the Appendix, where we show that birth rates are falling and projected to continue to decline. Contributions from net migration are expected to be stable, and are mostly small in most major economies.

Figure 1: Population Growth Across Major Economies



Notes: Solid lines plots historical population growth from the UN World Population Prospects 2019 for several major economies. Dashed lines plots the UN projections for population growth in the "Medium" scenario out to to 2060.

We then show that these results are robust to a variety of changes in the environment. Most importantly, we extend our model to a setting where firms compete a la Bertrand and market power is endogenous. Declining population growth interacts with firms' ability to charge high markups in an interesting way. In our theory, more productive firms post higher markup and productivity increases over the firms' life-cycle. Because creative destruction reduces firms' chances of survival, it hinders incumbents from accumulating market power and hence prevents the emergence of dominant producers. In short: creative destruction is pro-competitive. Declining population growth, by lowering creative destruction, reduces competition and increases markups and misallocation.

To quantify the strength of this mechanism, we calibrate our model to data for the population of US firms, In addition to targeting standard moments like the entry rate, average size and life-cycle growth, we also link firm-level information on sales to the U.S. Census. We can therefore measure firm-level markups for all firms in the US, and hence explicitly target the life-cycle profile of markups. Exploiting information on the evolution of both markups and size at the firm-level is an important aspect of our empirical methodology and allows us to separately identify own-innovation and variety creation at the firm-level.

With the calibrated model in hand, we ask a simple question: what are the implications of the observed and projected decline in the rate of labor force growth since 1980? Empirically, labor force growth almost halved from 2% to 1% between 1980 and 2015, and the BLS projects labor force growth to continue to decline to 0.24% after 2050. Our theory is tractable enough that we can solve for the transitional dynamics induced by this path, treating the projections of the BLS as the rational expectations of the agents in our theory. We find that this decline has quantitatively large effects. Our model can explain almost the entirety of the decline in the entry rate, the increase in average firm size and the degree of concentration. In terms of the implications for markups, the magnitudes are quantitatively limited. Our calibrated model implies that markups increase by around 1%. The effects on income growth are more subtle. While growth is bound to decline in the long-run, the static effect of variety creation can increase income growth during the transition. We find that it does. Our model implies income growth to be above trend for about one decade. However, we still find the welfare consequences of falling population growth to be negative.

Throughout the paper, we will often speak of population growth and labor force growth interchangeably. We also take movements in the size of the labor force to be exogenous to market concentration and firm dynamics. Across the developed world, falls in fertility in the 1960s and 1970s have manifested in slower rates of growth in the labor force in the 1980s and 1990s - see De Silva and Tenreyro (2017, 2020). In the U.S. in particular, slowing labor force growth also reflects an end to increasing female participation, and declining prime-age male participation. While a declining labor share and rising market power may itself have implications for worker participation, here the simplicity of taking these movements as given yields substantial insight into the changing patterns of firm dynamics we see in the data.

Related Literature. We are not the first to connect the decline in the growth rate of the labor force to changes in firm dynamics. Karahan et al. (2016) and Hathaway and Litan (2014) are early contributions that use geographic variation in the age structure of the population of the U.S. to provide direct support that a lower rate of population growth reduces the start-up rate. Recently, Hopenhayn et al. (2018) document the relationship between changes in demographics and firm dynamics in a quantitative model. Both Karahan et al. (2016) and Hopenhayn et al. (2018) perform their analysis in a model in the spirit of Hopenhayn (1992), where firm productivity and aggregate growth is exogenous and markets are competitive. Falling population growth therefore only affects the firm size distribution through changes in the age distribution of firms. By contrast, our theory builds on models with endogenous firm dynamics, and highlights that a declining rate of population growth also affects the extent of market power and aggregate productivity growth. Engbom (2017, 2020) studies the implications of population aging in the context of a search model.

On the theory side, we build on firm-based models of Schumpeterian growth in the tradition of Aghion and Howitt (1992) and Klette and Kortum (2004). We augment these models by allowing for efficiency improvements of existing firms as in Atkeson and Burstein (2010), Luttmer (2007), Akcigit and Kerr (2015) or Cao et al. (2017), the creation of new varieties as in Young (1998), and endogenous markups through Bertrand competition as in Peters (2020) or Acemoglu and Akcigit (2012). We allow for elasticities of substitution greater than unity, which requires consideration of the full joint distribution of efficiency and markups. Our model is thus akin to a version of Garcia-Macia et al. (2016), augmented by endogenous markups and endogenous innovation choices, and incorporating changes in the long run growth in the labor force. To the best of our knowledge, our paper is the first that focuses squarely on how demographic changes are likely to affect the equilibrium firm size distribution in the context of firm-based models of growth.

The relationship between economic growth and population growth has been been subject to an extensive literature. Many models of endogenous growth (e.g. Aghion and Howitt (1992), Romer (1990), Grossman and Helpman (1991) or Klette and Kortum (2004)) share the feature that economic growth depends on the level of population. By contrast, models of semi-endogenous growth (for example Jones (1995), Kortum (1997) or Young (1998)) imply that income growth is determined by the rate of population growth.[2] We conduct our analysis in a model where growth is tightly tied to the micro process of firm dynamics. In order for the firm size distribution to be stationary in the presence of a growing population, growth (generically) needs to be semi-endogenous.

In our quantitative application we focus on the case of the US. There is a growing literature highlighting the decline of dynamism and the rise of concentration in the US.This literature shows that the entry rate has fallen substantially (Karahan et al., 2015; Alon et al., 2018; Decker et al., 2014), that broad measures of reallocation have declined (Haltiwanger et al., 2015; Davis and Haltiwanger, 2014), that industries are becoming more concentrated (Kehrig and Vincent, 2017; Autor et al., 2017) and that markups are rising (Edmond et al., 2018; De Loecker and Eeckhout, 2017). See also Akcigit and Ates (2019a) for a summary.

In terms of explanations for these phenomena, the literature has proposed that improvements in IT technology raised the returns to scale for productive firms (Aghion et al. (2019); Lashkari et al. (2019)), a rise in the use of intangible capital (De Ridder (2019)), or changes in the process of knowledge diffusion (Akcigit and Ates (2019b); Olmstead-Rumsey (2020)). Our paper is complementary to these studies by highlighting that all these phenomena occurred within an environment of declining population growth, and are key implications of the theory we propose. Falling population growth might therefore be an important secular determinant of firm-dynamics and aggregate growth in the decades to come.

The remainder of our paper is structured as follows. In Section 2 we present our baseline model and derive our main results. In Section 3 we extend this framework by allowing for endogenous market power. In Sections 4 and 5 we calibrate our theory and quantify the effect of falling population growth for the process of firm dynamics and growth. Section 6 concludes. An Online Appendix contains the formal derivations of our theoretical results and details on our quantitative analysis.

2 The Baseline Model

We start with a baseline version of our model, where markups are constant and the efficiency of firms' existing products grows exogenously. This version of the model has an analytical solution and allows for a tight characterization how population growth affects entrants' and incumbents' incentives to engage in creative destruction and the creation of new products. Below we extend our analysis by explicitly allowing for endogenous markups and endogenous own-innovation.

[2] See also Jones (2020) for a recent analysis of the implications of negative population growth.

2.1 Environment

Time is continuous. There is a mass L_t of identical individuals, each supplying one unit of labour inelastically. This mass grows at rate η_t, such that $\dot{L}_t/L_t = \eta_t$. The rate of population growth η_t, which we take as exogenous, is the crucial parameter of this paper. Households have preferences over a final consumption good c_t, which are given by

$$U = \int_0^\infty e^{-(\rho-\eta)t} \ln(c_t)\, dt,$$

where $\rho > \eta$.

Production and Market Structure. The final consumption good is composed of a continuum of differentiated varieties, that (as in Klette and Kortum (2004)) may be produced by multiple firms. The production of the final good takes place in a competitive final sector, that combines the differentiated varieties according to

$$Y_t = \left(\int_0^{N_t} \left(\sum_{f \in S_{it}} y_{fit} \right)^{\frac{\sigma-1}{\sigma}} di \right)^{\frac{\sigma}{\sigma-1}}. \tag{1}$$

Here N_t is the number of active product lines, where these product lines are indexed by i. This number evolves endogenously with the creation and destruction of new products. S_{it} is the set of firms with the knowledge to produce product i, which likewise evolves endogenously.

Firms can be active in multiple product markets. Each firm f is characterized by a set of the products it produces, denoted by Θ_f, and an efficiency of producing these products, indexed by $\{q_{fi}\}_{i \in \Theta_f}$. We denote the number of products firm f produces by n_f. Production of each good uses only labor, and is given by

$$y_{fi} = q_{fi} l_{fi},$$

where l_{fi} is the amount of labor hired by firm f to produce product i, and q_{fi} denotes the efficiency of firm f in producing product i.

Because the output of firms producing the same product i is considered to be perfectly substitutable, each product is only produced by the most efficient firm. Suppose to begin with that the producing firm charges a constant markup over marginal cost $\mu = \frac{\sigma}{\sigma-1}$.[3] Below we explicitly allow for imperfect competition which gives rise to heterogenous markups. With constant markups, aggregate output Y_t and equilibrium wages w_t are given by

$$Y_t = Q_t N_t^{\frac{1}{\sigma-1}} L_t^P \quad \text{and} \quad w_t = \mu^{-1} Y_t / L_t^P, \tag{2}$$

where $Q_t \equiv \left(\int q_i^{\sigma-1} dF_t(q) \right)^{\frac{1}{\sigma-1}}$ is a measure of average efficiency, F_t is the distribution of product

[3]This can either be the case if the producer's relative efficiency advantage exceeds μ or if firms have to pay an infinitesimal sunk cost before producing, in which case the less productive firm will not enter (see Garcia-Macia et al. (2016)).

efficiency and L_t^P is the total amount of labor devoted to the production of goods. Equilibrium profits per product are given by

$$\pi_t(q) = (\mu - 1)\left(\frac{q}{Q_t}\right)^{\sigma-1}\frac{L_t^P}{N_t}w_t.$$

Hence profits are high if the product's efficiency q is large relative to average efficiency Q_t and if average employment per products, L_t^P/N_t, is large.

Entry, Innovation and Aggregate Growth. Both firms' productivities and the products they sell evolve endogenously. As in Garcia-Macia et al. (2016), our theory allows for three sources of firm dynamics. First we allow for *creative destruction* by incumbents and entrants (as in Klette and Kortum (2004)). Creative destruction occurs when either an existing firm or a new firm improves the efficiency of a product i, which is currently produced by another producer. Such efficiency improvements result in churning, whereby the old producer gets replaced. Second we allow for *own-innovation*, whereby firms improve the efficiency q of the products they are currently producing (see Atkeson and Burstein (2010) or Luttmer (2007)). Third, we allow for the endogenous creation of *new varieties*. This margin is the source of variety gains, whereby firms can generate product varieties which are entirely new to the economy. Allowing for variety creation is essential to ensure that the model has a stationary firm size distribution in the presence of population growth. It is this margin which implies that our model is a model of semi-endogenous growth, i.e. the growth rate depends on rate of population growth rather than the level of the population (Jones (1995)). Finally, we also assume that product lines die at an exogenous rate of δ. This can be interpreted as a taste shock in which consumers no longer value a product line for exogenous reasons. Doing so helps ensure stationarity at low or negatives levels of population growth.

We formalize these decisions in the following way. Existing firms increase the efficiency with which they produce their existing products deterministically at rate I:

$$\frac{\dot{q}_{it}}{q_{it}} = I.$$

To focus on the main economic mechanism how population growth affects firm dynamics we start by assuming that I is exogenous and constant over time. Below in Section 2.6 we show how to endogenize this rate.

Firms can also expand into new product lines. To do so, they choose the Poisson rate X at which the knowledge for how to produce a product new to them is created. Such expansion activities are costly, and we denote these costs (in units of labor) as

$$c_t^X(X, n) = \frac{1}{\varphi_x}X^{\zeta}n^{1-\zeta} = \frac{1}{\varphi_x}x^{\zeta}n, \tag{3}$$

where $\zeta > 1$, n denotes the number of products the firm is currently producing and $x = X/n$ is the firms' innovation intensity.[4]

[4] The particular functional form of the innovation cost function in (3) is not essential. All our results equally apply as

Conditional on successfully creating a new product, this product can either be a new variety to the aggregate economy, or it can improve upon an already existing product from another firm. We assume that innovation is "undirected", such that the firm cannot target new or existing varieties. With probability α the new product represents a technological advance over a (randomly selected) incumbent firm's product, increases the product's efficiency by a factor $\lambda > 1$ and forcing the current producer to exit ("incumbent creative destruction"). With the complementary probability $1-\alpha$, the product will be new to society as a whole, i.e. the mass of available products N_t grows ("incumbent new variety creation").

We assume that the efficiency of new varieties is given by $q = \omega Q_t$, where ω is drawn from a fixed distribution $\Gamma(\omega)$. Hence, as in Buera and Oberfield (2016), the efficiency of new varieties is determined both by the existing knowledge embedded in Q_t and by novel ideas. It is useful to define $\overline{\omega} \equiv \left(\int \omega^{\sigma-1} d\Gamma(\omega)\right)^{\frac{1}{\sigma-1}}$, which we also refer to as the mean efficiency of new products (appropriately scaled). As we show below, the equilibrium allocations only depend on $\overline{\omega}$ and not on $\Gamma(\omega)$.

Entrants have the same opportunities as incumbent firms. While they naturally do not own any products they could improve on, they also engage in creative destruction and new variety creation. As with incumbent firms, the share of innovations which result in creative destruction is exogenous and given by α. Entrants have access to a linear entry technology, where each worker they hire for research generates a flow of φ_E ideas.

Let Z_t denote the aggregate flow of entry and $z_t = Z_t/N_t$ the entry intensity per product. Similarly, let x_t denote the average expansion intensity by incumbent firms $x_t = \frac{1}{N_t}\int x_{it} di$. Letting ν_t denote the rate at which new varieties are created and τ_t the rate of creative destruction, i.e. the rate at which the producer of a given product is replaced by another producer, it follows that

$$\nu_t = (1-\alpha)(x_t + z_t) \quad \text{and} \quad \tau_t = \alpha (x_t + z_t).$$

The rate of variety growth is thus given by

$$g_t^N = \frac{\dot{N}_t}{N_t} = \nu_t - \delta = (1-\alpha)(x_t + z_t) - \delta = \frac{1-\alpha}{\alpha}\tau_t - \delta. \tag{4}$$

Note that creative destruction τ and variety creation are closely linked. Our formulation of undirected innovation makes this link particularly stark. However, as we show in Section 2.6, the optimal level of creative destruction and variety creation positively co-move even in a more general setting where α can be chosen directly by the firm and the direction of innovation is thus endogenous.

The rate of efficiency growth g^Q is given by

$$g_t^Q = \frac{\dot{Q}_t}{Q_t} = I + \frac{\lambda^{\sigma-1}-1}{\sigma-1}\tau_t + \frac{\overline{\omega}^{\sigma-1}-1}{\sigma-1}\nu_t.$$

long as $c_t^X(X,n)$ is homogeneous of degree one in both arguments.

Aggregate efficiency Q_t grows for three reasons. First, firms' own-innovation efforts raise the efficiency of production of individual products and hence also the aggregate efficiency index Q_t at rate I. Second, because $\lambda > 1$, creative destruction is a source of aggregate productivity growth. Finally, the creation of new varieties also affects the growth rate of average efficiency. If new products are on average as productive as existing products, i.e. $\overline{\omega} = 1$, the growth rate of average efficiency Q_t is independent of the rate of product creation g_N. If new products are an average worse, $\overline{\omega} < 1$, faster product creation is an adverse source of efficiency growth.

Using (4) to substitute for ν_t and τ_t, we can write g_t^Q as

$$g_t^Q = I + \frac{\overline{q}^{\sigma-1} - 1}{\sigma - 1} \left(x_t + z_t\right),$$

where

$$\overline{q} = \left(\alpha\lambda^{\sigma-1} + (1-\alpha)\,\overline{\omega}^{\sigma-1}\right)^{\frac{1}{\sigma-1}} \tag{5}$$

parametrizes the average efficiency gains of a product innovation and is simply a CES weighted average of the efficiency improvement of creative destruction λ and the relative efficiency of new varieties $\overline{\omega}$.The rate of quality growth g_t^Q is thus increasing in the aggregate rate of product innovation $x_t + z_t$ as long as $\overline{q} > 1$, i.e. as long as α and $\overline{\omega}$ are sufficiently large (recall that $\lambda > 1$).

Finally, the overall growth of labor productivity Y_t/L_t^P depends on both efficiency growth g_t^Q and variety growth g_t^N:[5]

$$\begin{aligned} g_t^{LP} = \frac{d}{dt}\ln\left(Q_t N_t^{\frac{1}{\sigma-1}}\right) &= g_t^Q + \frac{1}{\sigma-1} g_t^N = I + \frac{\lambda^{\sigma-1}-1}{\sigma-1}\tau_t + \frac{\overline{\omega}^{\sigma-1}}{\sigma-1}\nu_t - \frac{1}{\sigma-1}\delta \\ &= I + \frac{\overline{q}^{\sigma-1} - \alpha}{\sigma-1}\left(x_t + z_t\right) - \frac{1}{\sigma-1}\delta. \end{aligned}$$

Note that variety growth ν_t is always a source of aggregate growth, even if $\overline{\omega} < 1$. By contrast, the destruction of varieties δ has a negative effect on aggregate productivity though the loss of varieties. This also implies that g_t^{LP} is increasing in the aggregate rate of product innovation $x_t + z_t$ because $\overline{q}^{\sigma-1} > \alpha$ (recall that $\lambda > 1$ and $\overline{\omega} > 0$).

2.2 Optimal Product Creation and Entry

Firms' expansion decisions are forward-looking. The state variables at the firm-level are $\{q_{fi}\}_{i\in\Theta_f}$, which we for simplicity denote as $[q_i]$. The value function of a firm is given by the HJB equation

$$r_t V_t([q_i]) - \dot{V}_t([q_i]) = \underbrace{\sum_{k=1}^{n} \pi_t(q_k)}_{\text{Profits}} + \sum_{k=1}^{n} \underbrace{(\tau_t + \delta)\left[V_t\left([q_i]_{i\neq k}\right) - V([q_i])\right]}_{\text{Creative destruction and exit}} + \sum_{k=1}^{n} \underbrace{I\frac{\partial V_t([q_i])}{\partial q_k} q_k}_{\text{Own innovation}} + \Xi_t([q_i]), \tag{6}$$

[5] Along a BGP, where the share of production workers L_t^P/L_t is constant, income per capita also grows at rate g_t^{LP}.

where Ξ_t is the option value of product creation that is given by

$$\Xi_t\left([q_i]\right) = n \times \max_x \left\{ x \left(\underbrace{\alpha \int V_t\left([q_i], \lambda q\right) dF_t\left(q\right)}_{\text{Replacing an existing firm}} + \underbrace{\left(1-\alpha\right) \int V_t\left([q_i], \omega Q_t\right) d\Gamma\left(\omega\right)}_{\text{New variety}} - V_t\left([q_i]\right) \right) - \frac{1}{\varphi_x} x^{\zeta} w_t \right\}.$$

The value of a firm, $V_t\left([q_i]\right)$, consists of multiple additively separable parts. First, the value of the firm is increasing in the current flow profits. Second, the firm might lose any of its products to another firm, which happens at the endogenous rate of creative destruction τ and the exogenous rate of product loss δ. Third, own-innovation raises the efficiency q_i of each product, and hence profitability. Finally, the firm has the option to start producing a new product outside its current portfolio. With probability α it replaces a randomly selected product, with probability $1-\alpha$, the firm creates a new variety, whose efficiency is given by ωQ_t. The following Proposition summarizes the main properties of the value function:

Proposition 1. *Consider the value function $V_t\left([q_i]\right)$ given in (6). $V_t\left([q_i]\right)$ is given by ,*

$$V_t\left([q_i]\right) = \sum_{i=1}^{n} V_t\left(q_i\right) \quad \textit{where} \quad \left(r_t + \tau_t + \delta\right) V_t\left(q\right) - \dot{V}_t\left(q\right) = \pi_t\left(q\right) + I \frac{\partial V_t\left(q\right)}{\partial q} q + \frac{\zeta - 1}{\varphi_x} x^{\zeta} w_t,$$

and $\pi_t\left(q\right) = \left(\mu - 1\right)\left(\frac{q_i}{Q_t}\right)^{\sigma-1} \frac{L_t^P}{N_t} w_t$. The optimal expansion rate x is constant and given by

$$x = \left(\frac{\varphi_x}{\zeta}\right)^{\frac{1}{\zeta-1}} \left(\frac{\alpha V_t^{CD} + \left(1-\alpha\right) V_t^{NV}}{w_t}\right)^{\frac{1}{\zeta-1}}, \tag{7}$$

where

$$V_t^{CD} = \int V_t\left(\lambda q\right) dF_t\left(q\right) = V_t\left(\lambda Q_t\right) \text{ and } V_t^{NV} = \int V_t\left(\omega Q_t\right) d\Gamma\left(\omega\right) = V_t\left(\overline{\omega} Q_t\right).$$

Along a BGP, where the interest rate is constant and output grows at a constant rate g^Y, $V_t\left(q\right)$ is given by

$$V_t\left(q\right) = \left(\frac{\left(\mu - 1\right)\left(\frac{q_i}{Q_t}\right)^{\sigma-1} \frac{L_t^P}{N_t}}{\rho + \tau + \delta + \left(g^Q - I\right)\left(\sigma - 1\right)} + \frac{1}{\rho + \tau + \delta} \frac{\zeta - 1}{\varphi_x} x^{\zeta} \right) w_t \tag{8}$$

and L_t^P / N_t is constant. Moreover, the expected value of product creation is given by

$$\alpha V_t^{CD} + \left(1-\alpha\right) V_t^{NV} = V_t\left(\overline{q} Q_t\right),$$

where $\overline{q}$ is defined in (5).

Proof. See Section A-1.1 in the Appendix. □

Proposition 1 contains four results. First, the value function $V_t\left([q_i]\right)$ is additive, so we can focus on the value of a single product $V_t\left(q\right)$. Second, $V_t\left(q\right)$ is itself the sum of two components. The first part consists of the flow profits and the gains from own-innovation. The second is determined by the option value of expansion, which equals the inframarginal rents of the innovation technology. Third, the optimal expansion rate x is determined by the average of the creative destruction value V_t^{CD} and the value of new variety creation V_t^{NV} (both relative to the wage). Hence, the link between population growth η and firms' innovation rate x operates via V_t^{CD} and V_t^{NV}. Moreover, these values are in turn simply the value of a single product evaluated at the creative destruction entry point λQ_t or the initial efficiency of a new variety ωQ_t. All these properties do not hinge on the economy to be on a BGP and we use them below to compute the transitional dynamics.

Along a BGP, we can solve the value function $V_t(q)$ explicitly - see (8). It is the sum of the net present value of flow profits and the net present value of the expansion value. Note that flow profits are discounted at a different rate than the option value of innovation, reflecting the evolution of the relative competitiveness of the firm's product. Because the relative efficiency of a product $(q/Q_t)^{\sigma-1}$ changes at rate $\left(I - g^Q\right)(\sigma - 1)$, if Q_t grows faster (slower) than q, the product's profitability declines (increases) along its life-cycle. This difference in the rate of discounting between flow profits and innovation value turns out to be important to understand how population growth affects the equilibrium level of L_t^P/N_t. Finally, note that the expected value of product creation is simply given by $V_t(\bar{q})$, i.e. the value of entering with a product with efficiency $\bar{q}Q_t$. As we show below, this property makes the characterization of the BGP equilibrium very tractable.

Entry. Free entry requires that the expected value of a successfully created new product (which, like for incumbents, with probability α, stems from an existing firm and with probability $1-\alpha$ is entirely new to society) does not exceed the cost of entry, i.e.

$$V_t^{Entry} \equiv \alpha V_t^{CD} + (1-\alpha) V_t^{NV} \leq \frac{1}{\varphi_E} w_t. \tag{9}$$

For the remainder of this paper we focus on the empirically relevant case where the flow of entry is positive and equation (9) holds with equality.

The free entry condition in (9) is a crucial equation in our theory. Most importantly, it implies that the rate of product creation by incumbent firms is a function of technology only. Combining (9) with (7) yields

$$x = \left(\frac{1}{\zeta}\frac{\varphi_x}{\varphi_E}\right)^{\frac{1}{\zeta-1}}. \tag{10}$$

Hence, incumbent product creation is independent of *any* general equilibrium variables. In particular, it does not depend on the rate of population growth η. Note that equation (10) holds both on and off the BGP and only relies on the free entry condition to be binding.

This property plays an important role in our analysis and allows for a precise characterization of the role of population growth. It follows from the fact that incumbents' innovation technology has decreasing returns at the firm-level, while entry - that operates at the aggregate level - has constant returns.[6] Hence, the free entry condition pins down the value of product creation (relative to the wage) and incumbent firms optimally chose the rate of product creation to equalize the marginal cost and the marginal benefits. In Section 2.6 below we generalize our results to the case where the entry process has decreasing returns in the aggregate. In that case, x also depends on general equilibrium variables and is affected by population growth.

2.3 Balanced Growth Path Equilibrium

To characterize the BGP in this economy, define the two aggregate statistics

$$\mathcal{N}_t \equiv \frac{N_t}{L_t} \quad \text{and} \quad \ell_t^P \equiv \frac{L_t^P}{L_t}.$$

We will refer to $\mathcal{N}_t$ as the economy's *variety intensity* and to ℓ_t^P as the *production share*. These two aggregate statistics are sufficient to characterize the BGP.[7]

Note first that labor market clearing implies that

$$L_t = L_t^P + L_t^R = L_t^P + N_t \left(\frac{1}{\varphi_E} z_t + \frac{1}{\varphi_x} x^{\zeta} \right).$$

Using that $z_t = \frac{1}{1-\alpha} \nu_t - x$ and the optimal rate of incumbent expansion given in (10), labor market clearing requires that

$$\left(\frac{1-\ell_t^P}{\mathcal{N}_t} \right) = \frac{1}{\varphi_E} \left(\frac{\nu_t}{1-\alpha} - \frac{\zeta - 1}{\zeta} x \right). \tag{11}$$

Holding the variety intensity $\mathcal{N}_t$ constant, a higher production share ℓ_t^P reduces the creation of new varieties ν_t as less resources are allocated towards research. Conversely, for a given production share, variety creation ν_t is decreasing in the variety intensity $\mathcal{N}_t$ as the amount of research effort per existing variety is lower. Equation (11) is the first key equation to characterize the equilibrium.

The second key equation is the free entry condition. Along the BGP, Proposition 1 and the free entry condition in (9) implies that

$$\frac{1}{\varphi_E} = \frac{V_t(\bar{q} Q_t)}{w_t} = \frac{\bar{q}^{\sigma-1} (\mu - 1)}{\rho + \delta + \left(\frac{\bar{q}^{\sigma-1}}{1-\alpha} - 1 \right) \nu_t} \frac{\ell_t^P}{\mathcal{N}_t} + \frac{\frac{\zeta-1}{\varphi_x} x^{\zeta}}{\rho + \delta + \frac{\alpha}{1-\alpha} \nu_t}, \tag{12}$$

Equation (12) highlights that free entry determines the average number of production workers per product $\ell_t^P / \mathcal{N}_t = L_t^P / N_t$ as a function of the rate of variety creation ν_t. Whether $\ell_t^P / \mathcal{N}_t$ and ν_t are positively or negatively related depends on the sign of $\frac{\bar{q}^{\sigma-1}}{1-\alpha} - 1$. A sufficient condition for $\ell_t^P / \mathcal{N}_t$ to

[6] Note that incumbent product creation also has constant return in the aggregate: if the number of incumbent firms were to double, the anount of aggregate product creation performed by incumbents would also double.

[7] In Section A-1.1.4 in the Appendix we derive the full system of differential equations that characterize the equilibrium path during the transition.

increase in g_N is

$$\frac{\overline{q}^{\sigma-1}}{1-\alpha}-1>0, \tag{13}$$

which is satisfied if λ, α and $\overline{\omega}$ are sufficiently large.To understand the role of this restriction, recall that $\frac{\alpha}{1-\alpha}\nu_t = \tau_t$ and $\left(\frac{\overline{q}^{\sigma-1}}{1-\alpha}-1\right)\nu_t = (\sigma-1)(g_Q - I) + \tau_t$. A higher rate of variety growth always increases creative destruction τ This channel reduces the value of existing firms through an increase in the effective discount rate. At the same time, $g_Q - I$ decreases with a higher rate of variety growth if $\overline{q} < 1$. As long as (13) is satisfied (which is the case at our estimated parameters), the creative destruction effects always dominates the quality growth effect and a higher rate of variety growth increases creative destruction and hence reduces the value of existing firms through a higher rate of discounting.[8] Free entry therefore requires the *level* of flow profits to go up. This is achieved through an increase in the number of workers per product $\ell_t^P/\mathcal{N}_t$.

Along a BGP, the growth rate is constant. This implies that g_N grows at a constant rate. (11) and (12) therefore require that $\mathcal{N}_t$ and ℓ_t^P are constant. This has the important implication that the number of varieties N_t has to grow at the rate of population growth:

$$\eta = g_N = \nu_t - \delta = (1-\alpha)(z+x) - \delta$$

The aggregate quantity of innovation $z + x$ is thus directly tied to the growth rate of labor force. And given that equation (4) then also determines the rate of creative destruction we can analytically characterize the BGP allocations as a function of population growth.

Proposition 2. *On a BGP, the following holds:*

1. *The rate of creative destruction τ, the rate of incumbent product creation x and the rate of entry z are given by*

$$\nu = \eta + \delta \qquad \tau = \frac{\alpha}{1-\alpha}(\eta+\delta) \qquad x = \left(\frac{1}{\zeta}\frac{\varphi_x}{\varphi_E}\right)^{\frac{1}{\zeta-1}} \qquad z = \frac{\eta+\delta}{1-\alpha} - x. \tag{14}$$

2. *Aggregate growth g^y is given by*

$$g^y = I + \left(\frac{\overline{q}^{\sigma-1}-\alpha}{\sigma-1}\right)\frac{\eta}{1-\alpha} + \left(\frac{\overline{q}^{\sigma-1}-1}{\sigma-1}\right)\frac{\delta}{1-\alpha}, \tag{15}$$

where $\overline{q}^{\sigma-1} = \alpha\lambda^{\sigma-1} + (1-\alpha)\overline{\omega}^{\sigma-1} > \alpha$ (see (5)).

[8] Note that (13) is weaker than $\overline{q} > 1$, i.e. average quality growth can be declining in research expenditure. (13) requires that it cannot decline fast enough to outweigh the effect on creative destruction.

3. The production share ℓ^P and the variety intensity $\mathcal{N}$ are uniquely determined by the two equations

$$\left(\frac{1-\ell^P}{\mathcal{N}}\right) = \frac{1}{\varphi_E}\left(\frac{\eta+\delta}{1-\alpha} - \frac{\zeta-1}{\zeta}x\right) \tag{16}$$

$$1 = \frac{\varphi_E \overline{q}\,(\mu-1)}{\rho + \frac{\overline{q}^{\sigma-1}}{1-\alpha}\delta + \left(\frac{\overline{q}^{\sigma-1}}{1-\alpha}-1\right)\eta}\frac{\ell^P}{\mathcal{N}} + \frac{\zeta-1}{\zeta}\frac{x}{\rho+\frac{1}{1-\alpha}\delta+\frac{\alpha}{1-\alpha}\eta}. \tag{17}$$

Proof. See Section A-1.1.2 in the Appendix. □

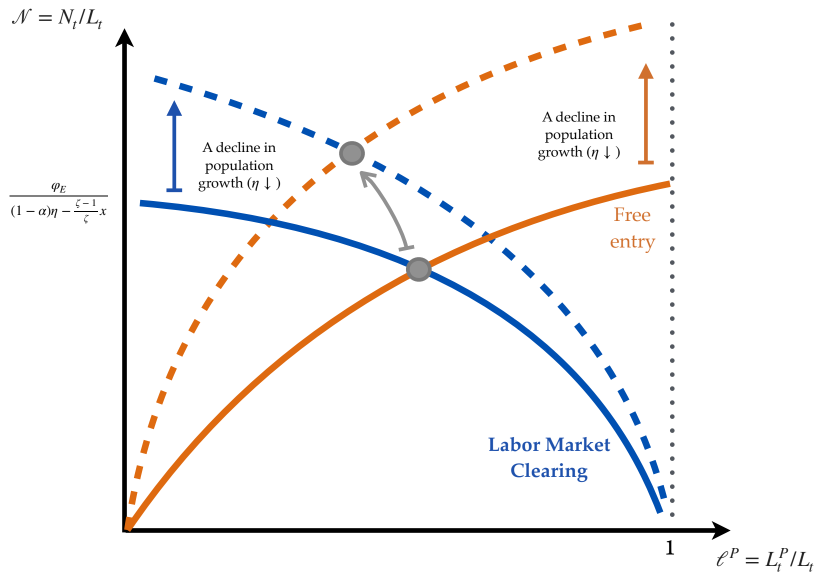
Proposition 2 contains three key theoretical results of this paper. First, a decline in population growth reduces creative destruction. Moreover, the entirety of the decline is absorbed by the economy's extensive margin - entrants do all the work. Hence, even though our model allows for incumbents' incentives to engage in product creation to respond, in equilibrium free entry implies that incumbents' rate of product creation is insulated from demographics and the economy lowers its aggregate innovative effort $z+x$ entirely through a quantity response: the flow of entrants declines.

Second, the rate of population growth directly affects the rate of growth. It does so in two ways. First, population growth determines variety creation, which is in itself a form of growth. Second, population growth also affects creative destruction and hence the rate of efficiency growth g^Q. While the effect of population growth on variety growth is always positive, its affect on efficiency growth depends on the average efficiency of newly created products $\overline{\omega}$ and the increment of creative destruction λ. The overall effect on income growth, however, is unambiguous: falling population growth reduces long-run income growth as is typical in models of semi-endogenous growth (Jones, 2021).

Third, the *level* of varieties N_t (relative to the population) is determined in equilibrium and is a function of population growth η. This is seen in Figure 2, where we depict the free entry condition (shown in orange) and the labor market clearing condition (shown in blue) from Proposition 2. The labor market clearing schedule always shifts upwards as population growth declines. Because a decline in population growth reduces the entry intensity z and keeps incumbent expansion x constant, the demand for research labor declines holding the number of varieties fixed. To satisfy the resource constraint, the variety $\mathcal{N}$ increases for a given sectoral allocation of labor. Similarly, the schedule describing the free entry condition shifts upwards if (13) is satisfied. Declining population growth raises the value of existing firms through the above-mentioned effect on firms' discount rates, and hence requires variety intensity to increase product market competition.

Note that this rise in the variety intensity is a countervailing force to the growth implications highlighted in Proposition 2. Because increases in N_t/L_t are a source of welfare gains, lower population growth has positive welfare consequences through a higher level of varieties (a "static" effect) but negative consequences via a decline in the growth rate (a "dynamic" effect). By contrast, the effect of population growth on the long-run share of production workers ℓ^P is theoretically ambiguous. In Figure 2, we show the case where a decline in population growth reduces the share of production workers ℓ^P and increases the share of workers employed in research $1-\ell^P$.

Figure 2: Declining Population Growth and Variety Creation along the BGP



Note: This figure shows the determination of $(\ell^P, \mathcal{N})$ along the BGP (see Proposition 2). It also depicts the consequences of a decline in population growth η.

2.4 Growth and Firm-Dynamics Without Scale Effects

One key implication of Proposition 2 is the absence of scale effects, both for the aggregate growth rate and the equilibrium firm size distribution. For the aggregate growth rate, this is immediately apparent from (15): as in the semi-endogenous growth model of Jones (1995), the rate of growth is fully determined from the rate of population growth and is independent of the level of the population.[9] Relatedly, our model also highlights the absence of scale effects for the equilibrium firm-size distribution. Firm-level employment is given by

$$l_{ft} = \sum_{i=1}^{n_f} \left(\frac{q_i}{Q_t}\right)^{\sigma-1} \times \frac{L_t^P}{N_t} = \sum_{i=1}^{n_f} \left(\frac{q_i}{Q_t}\right)^{\sigma-1} \times \frac{\ell^P}{\mathcal{N}} \tag{18}$$

and thus depends on the number of products the firm owns (n_f), the scaled efficiency of the firms' products (q_i/Q_t) and the mass of production workers relative to the number of varieties L_t^P/N_t. Along a BGP, both the distribution of the number of products and the distribution of scaled qualities are stationary and fully determined from the entry flow z, the rate of product innovation by incumbents x and the rate of own-innovation I, all of which are independent of the level of the population

[9] In contrast to Jones (1995) however, our model features the possibility of positive growth even if the population is stationary, i.e. $\eta = 0$. While there will not be any growth through variety creation, the "vertical" dimensions of own-innovation I and creative destruction τ are still available to achieve long-run growth if the population is constant (particularly if products die at the exogenous rate δ we assume in the quantitative section). In that sense, our model is similar to the model of Young (1998), which also features growth without scale effects.

L_t.[10] Similarly, the free entry equation (17) shows that $\ell^P/\mathcal{N}$ is also independent of the level of the population but only depends on the rate of population growth η. A larger population increases the number of firms but leaves the distribution of firm-size unchanged.

This symmetry whereby neither the firm size distribution nor the equilibrium growth rate depends on the level of the population is not a coincidence, because both are determined by firms' innovation incentives. Hence, firm-based models of growth that feature a stationary firm size distribution in the presence of population growth also point towards a world of semi-endogenous growth where population growth rather then the level of the population is the central determinant of rising living standards. As a case in point consider for example the baseline model of Klette and Kortum (2004), which features strong scale effects where the growth rate is increasing in the size of the population. At the same time, the firm size distribution is not stationary in the presence of population growth: a rising population will lead to ever increasing entry rates and smaller and smaller firms.

Our model does not have this implication because the *level* of varieties is endogenous. In fact, consider the case of a constant population, i.e. $\eta = 0$. Proposition 2 then directly shows that our model features growth without scale effects: the equilibrium then directly pins down ℓ^P and $\mathcal{N}$. Hence, a doubling of the level of population simply doubles the number of varieties N_t but leaves everything, including the process of firm-dynamics unchanged. In Klette and Kortum (2004), the produce space is exogenously fixed and free entry thus requires that a larger population, which comes with higher profits per potential entrant, leads to higher discounting via creative destruction. In our model, the adjustment operates through the level of flow profits. And while changes in creative destruction lead to changes in the growth rate and the firm size distribution , changes in the variety intensity $\mathcal{N}$ lead to changes in the level of productivity while keeping the rest of the economy stationary. In that sense, our model is a version of Young (1998) but augmented with a full endogenous process of firm dynamics.

We have also assumed that innovation costs scale with overall productivity Q_t. In particular, the entry cost is a fixed amount of labor, and hiring the same about of labor will generate a constant flow of useful ideas no matter how advanced the economy is. Doing so is crucial to generate a stationary firm size distribution. We can modify the theory to incorporate rising costs of entry as the economy grows (due, for example, to diminishing returns to research as ideas become harder to find, as in Bloom et al. (2020)). We analyze this case in Section A-1.2 in the Appendix. The main implication is that average firm size is no longer constant, but must rise forever on the balanced growth path. Intuitively, if the cost of entry rise faster than aggregate productivity, free entry requires that aggregate profits must also rise. This is achieved though a decline in competition and a secular increase in the size of firms.

[10]This result does not hinge on taking I to be exogenous, which we assumed for purely expositional purposes. Even if we treat I as endogenous (as we do below), it is still the case that the optimal rate of own-innovation I is independent of level of the population.

2.5 Population Growth and Firm Dynamics

The discussion above highlights the pivotal role of population growth for the process of firm-dynamics and the resulting equilibrium firm size distribution. In this section we leverage the analytical results of Proposition 2 to provide a characterization of the consequences of declining population growth for firm survival and the age distribution, the size distribution and market concentration and the equilibrium entry rate.

Population Growth and Firm Survival Consider first the impact of population growth on firms' chances to survive. To do so, define firms' *net* rate of product accumulation $\psi = x - (\tau + \delta)$, which is exactly the difference between the rate of product loss $\tau + \delta$ and the accumulation of products x. Using (14) to express τ in terms of the rate of population growth η yields

$$\psi = x - \frac{\alpha\eta + \delta}{1 - \alpha},$$

i.e. a decline in the rate of population growth increases the net rate of product accumulation at the firm-level as firms' face less of a threat of creative destruction.

This net accumulation rate ψ emerges as one key determinant for the process of firm dynamics. Let $S(a)$ denote the survival function, i.e. the probability that a given firm survives until age a. As we show in Section A-1.1.6 in the Appendix, this survival function is given by

$$S(a) = \frac{\psi e^{\psi a}}{\psi - x\left(1 - e^{\psi a}\right)}. \tag{19}$$

In the left panel of Figure 3 we display $S(a)$ graphically. Naturally, $S(a)$ is declining in a and satisfies $\lim_{a\to\infty} S(a) = 0$ because all firms exit eventually. More importantly, lower population growth *increases* firms' survival rates through an increase in the accumulation rate ψ. Hence, firms exit at a lower rate and become older on average. In fact, one can show that the average age of firms is given by $\mathbb{E}[\text{Age}] = \frac{1}{x}\ln\left(\frac{\alpha\eta+\delta}{\alpha\eta+\delta-(1-\alpha)x}\right)$, which is decreasing in η.

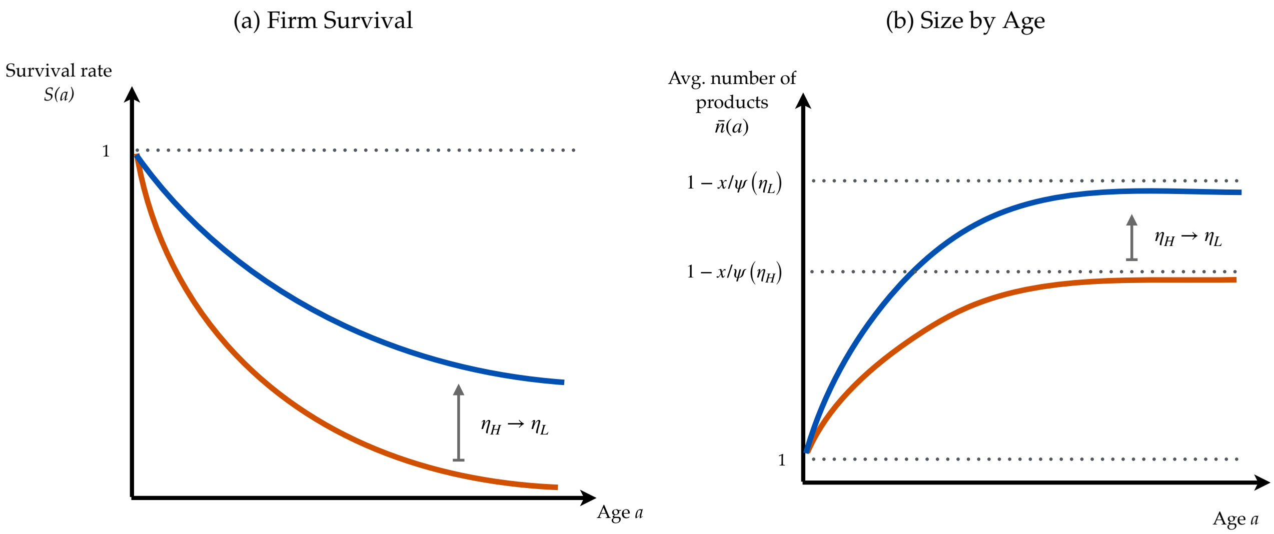
Population Growth, Concentration and Firm Size. Because firms on average grow as they age conditional on survival, lower population growth increases firm size and concentration by shifting the age distribution towards older firms. In addition, by increasing the net accumulation rate ψ, lower population growth also increases the whole profile of life-cycle growth, i.e. firms are becoming bigger *conditional* on age since their expansion incentives do not change but they lose products less often. In particular, let $\overline{n}(a)$ denote the average number of products of a firm of age a. Then it can be shown that

$$\overline{n}(a) = 1 - \frac{x}{\psi}\left(1 - e^{\psi a}\right), \tag{20}$$

which we display in the right panel of Figure 3. Not only is $\overline{n}(a)$ increasing in a, but it is also declining in η. These two forces imply that market concentration rises as population growth declines.

Interestingly, these increases in concentration and firm size go hand in hand with an *increase* in the

Figure 3: Falling Population Growth and Rising Concentration



Note: The figure shows the relationship between population growth η and firms' survival probabilities $S(a)$ (see (19)) in the left panel and the relationship between population growth η and the average number of products $\overline{n}(a)$ (see (20)) in the right panel.

aggregate variety intensity $\mathscr{N}_t = N_t/L_t$. This is due to the multi-product nature of our theory: while population growth reduces the number of firms per worker, it increases the number of products per worker because each existing firm offers a larger product portfolio. Hence, higher concentration can coexist with an expansion of product variety. This potential positive welfare effect is absent in theories without multi-product firms.

Another way to analyze the effect of population growth on concentration is by considering the right tail of the size distribution. As highlighted in equation (18), both the distribution of the number of products n and of scaled efficiency q/Q determine the distribution of firm size. In our model, both of these distributions have a pareto tail. The right tail of the employment distribution is thus given by

$$\varrho_l = \min\left\{\varrho_n, \frac{1}{\sigma - 1}\varrho_q\right\},$$

where ϱ_n is the tail of the product distribution and ϱ_q is the tail of the scaled efficiency distribution. Intuitively, there are two ways for firms to be very large: through having many products, or by having an extraordinarily good product.

As we show in the Section A-1.1.6 in the Appendix, as long as $\eta > \psi > 0$, the results of of Luttmer (2011) imply that the distribution of the number of products n_f has a pareto tail ζ_n, which is given by

$$\varrho_n = \frac{\eta}{\psi} = \frac{(1-\alpha)\,\eta}{x\,(1-\alpha) - \delta - \alpha\eta}. \tag{21}$$

Hence, the Pareto tail of the product distribution is a closed form expression of the rate of population growth η and a decline in population growth increases concentration, i.e lowers ϱ_n towards unity. Equation (21) also highlights that lower population growth affects the product distribution through two channels. Holding firms' net expansion rate ψ constant, lower population growth increases concentration because it reduces the rate at which new firms, which are on average small by virtue of being young, enter. In addition, lower population growth endogenously increases the net accumulation rate ψ by lowering creative destruction. This further increases market concentration and lowers the tail of the product distribution.[11]

The distribution of relative efficiency also has a Pareto tail. In particular, as long as the entrant efficiency distribution Γ has a thin tail, the tail parameter ϱ_q is implicitly defined by (see Section A-1.1.6 of the Appendix)

$$\varrho_q \left(\frac{\overline{q}^{\sigma-1} - 1}{\sigma - 1} \right) = -1 + \alpha \lambda^{\varrho_q}, \tag{22}$$

and hence depends on λ, α, σ and $\overline{q}$. Interestingly, and in stark contrast to (21), the tail of the efficiency distribution ϱ_q is independent of population growth η. As $\lambda \to 1$, ϱ_q approaches $\varrho_q = (\sigma - 1) / \left(1 - \overline{\omega}^{\sigma-1}\right)$. Hence, if creative destruction does not contribute to efficiency growth, the tail of the efficiency distribution will be thicker, the lower the relative efficiency of new varieties, i.e. the smaller $\bar{\omega}$.

To summarize, declining population growth always increases concentration because incumbent firms expand at a faster rate and survive longer. Whether this increase in concentration also shows up in the tail of the size distribution depends on the comparison between ϱ_n and ϱ_q . If $\varrho_n < \varrho_q$, lower population growth reduces the tail of the employment distribution. If $\varrho_q < \varrho_n$, the right tail of the employment distribution is unaffected by population growth. Which of these tail coefficients dominates is a quantitative question.

Population Growth and the Entry Rate. Finally, our theory highlights the implications of population growth for the equilibrium entry rate. Letting $\mathcal{F}_t$ denote the number of firms at time t, the entry rate is given by

$$\text{Entry rate}_t = \frac{Z_t}{\mathcal{F}_t} = z \times \frac{N_t}{\mathcal{F}_t}.$$

Holding the number of products per firm constant, a lower entry flow z reduces the rate of entry. Conversely, for a given entry intensity z, an increase in $N_t/\mathcal{F}_t$ increases the entry rate. Our theory reflects these two counteracting forces. A decline in population growth lowers z, which all else equal pushes the entry rate lower. At the same time, we argued above that $N_t/\mathcal{F}_t$ increases in response to a decline in population growth as both the variety intensity N_t/L_t increases and firms become

[11] Note that (21) can also be written as $\zeta_n = \frac{\eta}{\eta - z}$, i.e. concentration is large if the flow of new entrants z is small *relative* to population growth η. Our theory, in particular Proposition 2, implies that a decline in η will reduces both z and $\frac{\eta}{\eta - z}$. This also implies that the product distribution does not have a pareto tail in the absence of population growth (as is the case in the baseline model of Klette and Kortum (2004)).

larger, i.e. $L_t/\mathcal{F}_t$ rises.[12] Quantitatively, we find that the first effect decisively dominates: declining population growth lowers the rate of entry in equilibrium.

2.6 Discussion of Assumptions

Three assumptions make our theory particularly tractable. First, we assume that the economy has access to a linear entry technology. Second, product creation is undirected: a constant share α of innovation and entry results in creative destruction rather than new variety creation. Third, we assumed the rate of own-innovation I to be exogenous. In this section we show that our main results qualitatively do not hinge on these assumptions.

Decreasing Returns in the Entry Technology. Assume that the productivity of entrant labor hired to produce new ideas is given by

$$\varphi_E(z_t) = \tilde{\varphi}_E z_t^{-\chi} \text{ where } \chi \geq 0. \tag{23}$$

Here, z_t is the aggregate entry rate that each entrant takes as given. For $\chi = 0$, this specification yields the constant returns to case analyzed above. For $\chi > 0$, the cost of entry rises with the aggregate entry rate. We refer to χ as the strength of congestion.

Under (23), free entry requires that

$$\frac{V_t(\bar{q}Q_t)}{w_t} = \frac{1}{\varphi_E(z_t)} = \frac{1}{\tilde{\varphi}_E} z_t^{\chi}. \tag{24}$$

Hence, to the extent that there is congestion, i.e. $\chi > 0$, the average value of product creation (relative to the wage) is increasing in the aggregate entry rate. Alternatively, the aggregate entry supply curve is increasing in the value of entry with an elasticity $1/\chi$. For our baseline case of $\chi = 0$, entry is infinitely elastic.

Irrespective of the entry technology, it is still the case that the rate of variety growth is equal to the rate of population growth, i.e. $\nu = \eta + \delta$. This directly implies that two important results of Proposition 2 still apply: the rate of creative destruction is still given by $\tau = \frac{\alpha}{1-\alpha}(\eta + \delta)$ (see (14)) and the aggregate growth rate g_y is still given in (15).

By contrast, the composition of creative destruction into the entry flow z and incumbents' rate of product creation x, depends on the strength of congestion χ. Note that the policy function of incum-

[12] We have not found an analytic expression for $N_t/\mathcal{F}_t$. However, it is straightforward to calculate. Let $\nu(n) = \frac{\omega_t(n)}{N_t}$ denote the share of firms with n products. As we show in Section A-1.1.6 in the Appendix, $\nu(n)$ is given by

$$\nu(n+1) = \begin{cases} \left(2\frac{\alpha\eta}{1-\alpha}\right)^{-1}\left(\nu(1)\left(\frac{\eta}{1-\alpha} + x\right) - z\right) & \text{if } n = 1 \\ \left((n+1)\frac{\alpha\eta}{1-\alpha}\right)^{-1}\left(\nu(n)\,n\left(\frac{\alpha\eta}{1-\alpha} + x\right) + \nu(n)\,\eta - \nu(n-1)(n-1)\,x\right) & \text{if } n > 2 \end{cases}.$$

Together with the consistency condition $\sum_{n=1}^{\infty} \nu(n)\,n = 1$, these equations fully determine $\{\nu(n)\}_n$ as a function of parameters, in particular the rate of population growth η Then, $N_t/\mathcal{F}_t = \left(\sum_{n=1}^{\infty} \nu(n)\right)^{-1}$.

bents (7) and the congestion-adjusted free entry condition in (24) imply that

$$\tau = \alpha\left(z+x\right) = \alpha\left(z+\left(\frac{\varphi_x}{\zeta}\right)^{\frac{1}{\zeta-1}}\left(\frac{V_t\left(\bar{q}Q_t\right)}{w_t}\right)^{\frac{1}{\zeta-1}}\right) = \alpha\left(z+\left(\frac{\varphi_x}{\zeta\tilde{\varphi}_E}\right)^{\frac{1}{\zeta-1}} z^{\frac{\chi}{\zeta-1}}\right).$$

Using $\tau = \frac{\alpha}{1-\alpha}\left(\eta+\delta\right)$, this implies that the product entry flow z is uniquely determined from the equation

$$\frac{\eta+\delta}{1-\alpha} = z + \left(\frac{\varphi_x}{\zeta\tilde{\varphi}_E}\right)^{\frac{1}{\zeta-1}} z^{\frac{\chi}{\zeta-1}}.$$

It easy to see that z is declining in η, that is falling population growth still reduces the entry flow z. Given z, the rate of incumbent product creation x is given by

$$x = \left(\frac{\varphi_x}{\zeta\tilde{\varphi}_E}\right)^{\frac{1}{\zeta-1}} z^{\frac{\chi}{\zeta-1}}.$$

For the case of no congestion, $\chi = 0$, the solution is exactly as in Proposition 2 and x does not depend on population growth. If $\chi > 0$, x is increasing in z and hence also declining in population growth. Whether changes in population growth affect entrants or incumbents relatively more depends on the congestion elasticity χ relative to the convexity of the cost function ζ. In particular, $z/x \propto z^{\frac{\zeta-1-\chi}{\zeta-1}}$ so that entrants respond relatively more to changes in population growth if $\zeta - 1 > \chi$, i.e. if the entry cost elasticity χ is smaller than the incumbent cost elasticity $\zeta - 1$. Hence, qualitatively, all the results derived above hold true as long as $\chi < \zeta - 1$. The case of $\chi = 0$ makes the "entry dependence" particularly salient.

Endogenizing the Direction of Innovation α. Our second assumption concerns the direction of innovation α. In Section A-1.3 in the Appendix we present a detailed analysis of an extension of our model, where entrants and incumbents can directly chose the flow rate at which they want to creatively destroy products (x_{CD} and z_{CD}) and at which they want to create new varieties (x_{NV} and z_{NV}). Hence, $\tau = z_{CD} + x_{CD}$ and $g_N = x_{NV} + z_{NV}$.

For incumbent firms the value function takes exactly the same form as in (8) in Proposition 1: the value of having a product with quality q is given by

$$V_t\left(q\right) = \frac{\left(\mu-1\right)\left(\frac{q_i}{Q_t}\right)^{\sigma-1}\frac{L_t^P}{N_t}}{\rho+\tau+\delta+\left(g^Q - I\right)\left(\sigma-1\right)} w_t + \frac{1}{\rho+\tau+\delta}\Xi_t^*$$

where Ξ_t^* is the value of innovation is given by

$$\Xi_t^* = n \times \left(\max_{x_{CD}}\left\{x_{CD}\left(V_t^{CD} - \frac{1}{\varphi_{CD}} x_{CD}^{\zeta} w_t\right\} + \max_{x_{NV}}\left\{x_{NV}\left(V_t^{NV} - \frac{1}{\varphi_{NV}} x_{NV}^{\zeta} w_t\right\}\right), \tag{25}$$

where, as before, $V_t^{CD} = V_t\left(\lambda Q_t\right)$ and $V_t^{CD} = V_t\left(\overline{\omega} Q_t\right)$. For entrants we assume the following entry technology: as in the baseline model, each workers can generate φ_E new business ideas. To turn a business idea into a viable product, new firms have access to the same innovation technology as incumbent firms. This structure maintains both the symmetry between entrants and incumbents and the linear entry technology but endogenies the direction of innovation α.

As we show in Section A-1.3 in the Appendix, this extension of our model is still very tractable. First, the optimal rates of incumbent innovation are given by

$$x_{NV} = \left(\frac{\varphi_N}{\zeta} \frac{V_t\left(\overline{\omega} Q_t\right)}{w_t} \right)^{\frac{1}{\zeta-1}} \text{ and } x_{CD} = \left(\frac{\varphi_{CD}}{\zeta} \frac{V_t\left(\lambda Q_t\right)}{w_t} \right)^{\frac{1}{\zeta-1}}. \tag{26}$$

Second, because entering firms have the same innovation technology as incumbents, $z_{NV} = x_{VN} z$ and $z_{CD} = x_{CD} z$, where z is the aggregate flow of entry (per product N_t). Third, free entry requires that

$$\frac{1}{\varphi_E} = \frac{\Xi_t^*}{w_t} = \frac{\zeta - 1}{\varphi_{NV}} x_{NV}^{\zeta} + \frac{\zeta - 1}{\varphi_{CD}} x_{CD}^{\zeta}, \tag{27}$$

where the second equality stems from substituting (26) into (25).

Equation (26) highlights why variety creation and creative destruction are tightly linked: both depend on the same value function $V_t\left(q\right)/w_t$. In fact, in a special case, this model is exactly isomorphic to our simpler baseline model. Suppose what $\lambda = \overline{\omega}$, i.e. new varieties and creatively destroyed products have - on average - the same initial quality. (26) then implies that

$$\alpha = \frac{x_{CD}}{x_{CD} + x_{NV}} = \frac{\varphi_{CD}^{1/(\zeta-1)}}{\varphi_{CD}^{1/(\zeta-1)} + \varphi_N^{1/(\zeta-1)}},$$

i.e. the direction of innovation is constant. As in our baseline model we can thus write $x_{CD} = \alpha x$ and $x_{NV} = \left(1-\alpha\right) x$, where $x = x_{CD} + x_{NV}$ is the total quantity of incumbent innovation. The free entry condition (27) then implies that x is again fully determined from parameters and insulated from demographics. Finally, the rate of creative destruction and the amount of entry z are given by

$$\left(1-\alpha\right) x \left(1+z\right) = \eta + \delta \quad \text{and} \quad \tau = \alpha x \left(1+z\right) = \frac{\alpha}{1-\alpha}\left(\eta + \delta\right).$$

Hence, as in our baseline model, falling population reduces creative destruction and all the adjustment is achieved through a reduction in entry.

In Section A-1.3 in the Appendix we analyze the general case of $\lambda \neq \overline{\omega}$. This implies that α is no longer constant. However, we still show that falling population growth reduces both creative destruction and the relative importance of entrants z/x.

Endogenous Own-Innovation I**.** Finally, consider our choice of treating the rate of efficiency growth of incumbent firms I as exogenous. As we show in Section A-1.3 in the Appendix, all of our results extend to a case where I is endogenous in a straight-forward way. In particular the expressions for τ and g_N are exactly the same as in Proposition 2 and so is the expression for the equilibrium growth rate g_y, except that I is no longer a parameter but a choice variable. This endogenous rate of own-innovation is in turn implicitly defined by

$$I=\left(\frac{(\sigma-1)\,(\mu-1)\,L_t^P/N_t}{\rho+\tau+\delta+\left(g^Q-\frac{\zeta-1}{\zeta}I\right)(\sigma-1)}\frac{\varphi_I}{\zeta}\right)^{\frac{1}{\zeta-1}}. \tag{28}$$

Note that I depends on the rate of population growth both through the discount rate (i.e. τ and g^Q) and the level of flow of profits (i.e. L_t^P/N_t).[13] Because lower population growth reduces τ and g^Q and increases N_t/L_t^P, it seems that the effect of population on own-innovation is ambiguous. This, however, is not the case. Using the free entry condition, one can show that the optimal rate of own-innovation I is given by

$$I=\varsigma\left(1-\frac{\left(\frac{\zeta-1}{\zeta}\right)\left(\frac{1}{\zeta}\frac{\varphi_x}{\varphi_E}\right)^{\frac{1}{\zeta-1}}}{\rho+\tau+\delta}\right)^{\frac{1}{\zeta-1}}, \tag{29}$$

where ς is a collection of structural parameters. Note that this expresses I directly as a function of parameters and a single endogenous variable - the rate of creative destruction. And because I is increasing the rate of creative destruction, a decline in population growth *reduces* the rate of own innovation. This endogenous response of incumbents' own-innovation efforts this amplifies the negative growth consequences of falling population growth.

The fact that I is increasing in the rate of creative destruction might at first seem surprising. After all, a higher rate of creative destruction reduces the expected life-span, which should reduce firms' incentives to invest in productivity improvements. And as seen in (28), this intuition is indeed correct: holding the market size L_t^P/N_t fixed, a higher rate of creative destruction reduces the rate of own-innovation. However, once the change in L_t^P/N_t is taken into account, the general equilibrium effect of a higher rate of creative destruction becomes positive. The reason is the following: free entry requires the average production value *plus* the innovation value to be equal to the entry costs. A lower rate of population growth *increases* the innovation value because creative destruction declines. Hence, for the free entry condition to be satisfied, the production value has to *decrease.* And as the returns to own-innovation scale with the production value but not the innovation value, the returns to own-innovation are lower in an environment with lower population growth.

3 Extension for the Quantitative Analysis: Endogenous Market Power

So far we assumed that markups were constant and equal to the standard CES markup. We now generalize our model by assuming that firms compete *a la* Bertrand within product lines. Doing so makes the distribution of markups endogenous and allows us to study the effects of falling population growth on market power.

[13] Note also that I is independent of the level of the population, i.e. even when I is treated as endogenous, our model does not feature strong scale effects and the firm size distribution is stationary.

Given the CES structure of demand, each firm would like to charge a markup of $\frac{\sigma}{\sigma-1}$ over marginal cost. However, the presence of competing firms within their product line implies that the most efficient producer might have to resort to limit pricing. If they are unable to price at the optimal markup without inviting competition, they will set their price equal to the marginal cost of the next most efficient producer of that good, who is then indifferent between producing or not. The markup charged in product i, μ_i, is thus given by

$$\mu_i = \min\left\{\frac{\sigma}{\sigma-1}, \frac{q_i}{q_i^C}\right\} \equiv \min\left\{\frac{\sigma}{\sigma-1}, \Delta_i\right\}, \tag{30}$$

where q_i denotes the efficiency of current producer in product i, q_i^C is the efficiency of the next best competitor and $\Delta_i \equiv q_i/q_i^C > 1$ is the firm's efficiency advantage relative to it competitors (we also refer to this as the "gap"). Markups are rising in the gap Δ because higher efficiency shields the firm from competition.

The static equilibrium allocations generalize in a straight-forward way and aggregate output and equilibrium wages are now given by

$$Y_t = \mathcal{M}_t Q_t N_t^{\frac{1}{\sigma-1}} L_t^P \quad \text{and} \quad w_t = \Lambda_t Y_t / L_t^P = \Lambda_t \mathcal{M}_t Q_t N_t^{\frac{1}{\sigma-1}},$$

where

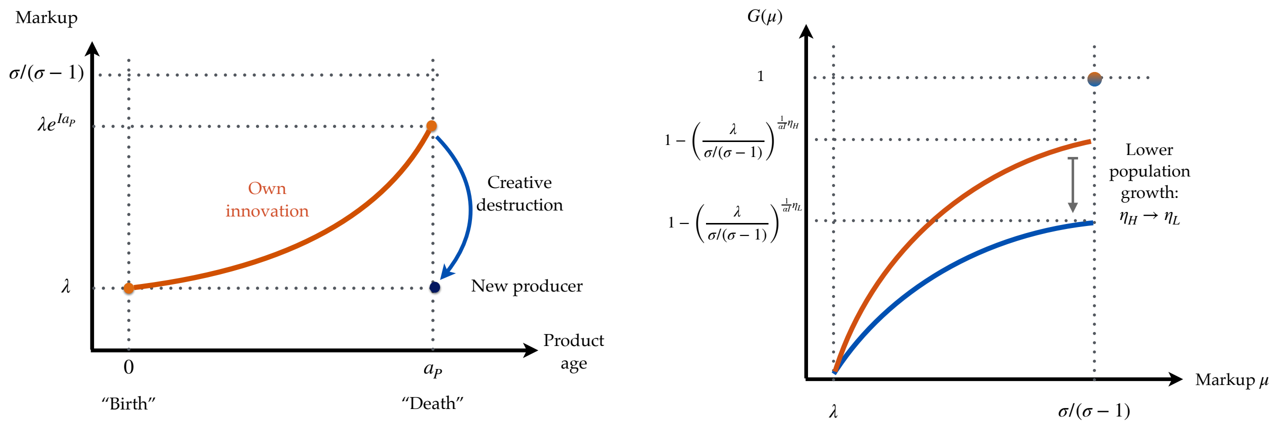
$$\mathcal{M}_t = \frac{\left(\int \mu^{1-\sigma} \left(q/Q_t\right)^{\sigma-1} dF_t\left(q,\mu\right)\right)^{\frac{\sigma}{\sigma-1}}}{\int \mu^{-\sigma} \left(q/Q_t\right)^{\sigma-1} dF_t\left(q,\mu\right)} \quad \text{and} \quad \Lambda_t = \frac{\int \mu^{-\sigma} \left(q/Q_t\right)^{\sigma-1} dF_t\left(q,\mu\right)}{\int \mu^{1-\sigma} \left(q/Q_t\right)^{\sigma-1} dF_t\left(q,\mu\right)}, \tag{31}$$

and $F_t(q,\mu)$ denotes the joint distribution of efficiency and markups. The two aggregate statistics $\mathcal{M}_t$ and Λ_t fully summarize the static macroeconomic consequences of monopoly power. Market power reduces both production efficiency (the misallocation term $\mathcal{M}_t$) and lowers factor prices relative to their social marginal product (the labor share Λ_t). In particular, a common increase in markups reduces the labor wedge Λ_t but keeps the allocation efficiency $\mathcal{M}_t$ unchanged. The latter is affected by the dispersion of markups. Because our model generates the joint distribution distribution of markups and efficiency $F_t\left(q,\mu\right)$ endogenously and this distribution is a function of the rate of population growth, a decline in the rate of population growth affects allocative efficiency via $\mathcal{M}_t$ and has distributional consequences through Λ_t.

Perhaps more surprisingly, the dynamic implications are very similar to our baseline model. While the value function is more involved, we show in Section A-1.4 in the Appendix that we can still derive an analytic expression which has a similar form to the one derived in the constant markup case. More importantly, all the results of Proposition 2 *exactly* hold in the model with Bertrand competition, i.e. the equilibrium rate of creative destruction τ, the entry rate z and the rate of incumbent expansion x are still given by (14). Hence, our findings that lower population growth increases concentration and shifts the age distribution towards older firms directly carries over to the environment with Bertrand

Figure 4: Falling Population Growth and Rising Market Power

(a) The Life-Cycle of Product Markups (b) The Distribution of Markups



Notes: This left panel shows a stylized example of how markups evolve at the product level. When a firm takes over a product, markups increase through own-innovation. Once the product is lost to another firm, markups are reset to the baseline level of λ. The right panel shows what happens to the distribution of markups when population growth falls.

competition.

Allowing for imperfect competition, however, yields additional insights. Our model features a crucial asymmetry between productivity growth due to creative destruction and own-innovation. Suppose the current producer of product i has an efficiency gap of Δ_i. If this firm is replaced by another producer, the efficiency gaps *reduces* to λ as the new firm's efficiency exceed the one of the previous producer by the creative destruction step size λ. Hence, churning through creative destruction reduces markups. By contrast, if the existing firm successfully increases its efficiency through own-innovation, the efficiency gap and hence the markup increase at rate I (as long as $\Delta_i \leq \frac{\sigma}{\sigma-1}$). Hence, own-innovation is akin to a positive drift for the evolution of markups, while creative destruction is similar to a "reset" shock, which lowers markups and keeps the accumulation of market power in check.

This process is displayed in the left panel of Figure 4. When a firm adds a product to its portfolio, the initial markup is λ. Conditional on survival, markups increase at rate I. A faster rate of creative destruction lowers the expected time a given firm produces a particular product and limits the opportunities for incumbent firms to accumulate market power.

The stochastic process shown in the left panel ofFigure 4 gives rise a stationary distribution of efficiency gaps Δ and hence markups. Newly created varieties do not face any competitor and hence charge a markup of $\frac{\sigma}{\sigma-1}$. Products that have been creatively destroyed at some point in the past are subject to Bertrand competition and the markup depends on Δ. Let N_t^{NC} denote the mass of products without any competitor and N_t^C be the mass of products that are subject to competition. Consistency requires that $N_t = N_t^{NC} + N_t^C$. In Section A-1.4.5 in the Appendix we prove two results. First, we show that, along a BGP, the share of product without any competitor is given by

$$N_t^{NC}/N_t = 1-\alpha,$$

i.e. it is simply given by the share of product creation that results in new varieties (rather than creative destruction).[14] Second, the distribution of efficiency gaps among products with a competitor is given by

$$F^C(\Delta) = 1-\left(\frac{\lambda}{\Delta}\right)^{\frac{\tau+\eta}{I}} = 1-\left(\frac{\lambda}{\Delta}\right)^{\frac{1}{1-\alpha}\frac{\eta}{I}}, \tag{32}$$

i.e. the marginal distribution of efficiency gaps is a Pareto distribution with tail parameter of $\frac{\tau+\eta}{I}$. As such, slower population growth increases the equilibrium distribution of efficiency gaps in a first-order stochastic dominance sense. First of all, slower population (and hence product) growth shifts the product distribution towards old products, which on average have higher markups. In addition, because slower population growth also reduces creative destruction, this effect is amplified, i.e. the average product age is increasing even for a given cohort of firms.[15]

To translate the distribution of efficiency gaps into the distribution of markups, recall from (30) that $\mu(\Delta) = \min\left\{\frac{\sigma}{\sigma-1}, \Delta\right\}$. Hence, for the case where markups are below the "unconstrained", monopolistically competitive markup $\frac{\sigma}{\sigma-1}$, the distribution of markups is a truncated Pareto. Among products without a competitor, the markup is given by $\frac{\sigma}{\sigma-1}$. Hence, the cross-sectional distribution of markups across products is given by

$$G(\mu) = \begin{cases} \alpha F^C(\mu) & \mu < \frac{\sigma}{\sigma-1} \\ 1 & \mu = \frac{\sigma}{\sigma-1} \end{cases}.$$

A reduction in population growth therefore increases markups along the whole distribution and shifts more mass towards the maximum CES markup. In the right panel of Figure 4 we depict how the distribution of markups changes in response to a decline in population growth from η_H to η_L.

The macroeconomic consequences of misallocation are summarized by $\mathcal{M}$ and Λ, which depend on the joint distribution between efficiency gaps Δ and efficiency q. To derive this distribution, define relative efficiency $\hat{q} = \ln(q/Q_t)^{\sigma-1}$ and let $\hat{\lambda} = \ln\lambda^{\sigma-1}$. Denote $F_t^C(\Delta,\hat{q})$ as the joint distribution

[14] Let $N_t^{NC}(a)$ the number of products without a competitor that have been around for a years at time t. Because $(1-\alpha)(z+x)N_t e^{-\eta a}$ such products entered at time $t-a$ and receive a competitor at the rate of creative destruction τ, $N_t^{NC}(a) = (1-\alpha)(z+x)N_t e^{-(\eta+\tau)a}$. Hence,

$$N_t^{NC} = \int_{a=0}^{\infty} N_t^{NC}(a)\,da = N_t\left(\frac{1-\alpha}{\alpha}\right)\left(\frac{\tau}{\tau+\eta}\right) = (1-\alpha)N_t,$$

because $\tau = \frac{1-\alpha}{\alpha}\eta$.

[15] To see intuitively, why our model gives rise to a Pareto distribution, let a_P denote the time a product has been produced by the same firm. Because the efficiency gap Δ increases at rate I as long as the firm is not replaced, $\Delta(a_P) = e^{Ia_P}$. And because the current producer gets replaced at rate τ and the number of products increases at rate η, the distribution of a_P is $P(a_P \le a) = 1-e^{-(\eta+\tau)a}$. Hence, $P\left(e^{Ia_P} < \Delta\right) = 1-e^{-\left(\frac{\eta+\tau}{I}\right)\ln\Delta}$, which is (32).

of efficiency gaps and relative efficiency for products which have a next best competitor. Similarly, denote $F_t^{NC}(\hat{q})$ as the distribution of relative efficiency for products that do not have a competitor. We show in Appendix A-1.4.4 that these objects evolve according to laws of motion given by

$$
\begin{aligned}
\frac{\partial F_t^C(\Delta,\hat{q})}{\partial t} &= \underbrace{-\frac{\partial F_t^C(\Delta,\hat{q})}{\partial \Delta} I\Delta - (\sigma-1)(I-g_t^Q)\frac{\partial F_t^C(\Delta,\hat{q})}{\partial \hat{q}}}_{\text{drift from own innovation}} - \underbrace{(\tau_t+\delta+\eta)\, F_t\left(\Delta,\hat{q}\right)}_{\text{product loss}} \\
&+ \underbrace{\lim_{s\to\infty} \tau_t F_t^C\left(s,\hat{q}-\hat{\lambda}\right)}_{\text{creative destruction of } C \text{ products}} + \underbrace{\tau_t \frac{N_t^{NC}}{N_t^C} F_t^{NC}(\hat{q}-\hat{\lambda})}_{\text{creative destruction of } NC \text{ products}}, \\
\frac{\partial F_t^{NC}(\hat{q})}{\partial t} &= \underbrace{-\frac{\partial F_t^{NC}(\hat{q})}{\partial \hat{q}} (\sigma-1)\left(I-g_t^Q\right)}_{\text{drift from own innovation}} - \underbrace{(\tau_t+\delta+\eta)\, F_t^{NC}(\hat{q})}_{\text{product loss}} + \underbrace{\frac{(1-\alpha)}{\alpha}\tau_t \Gamma\left(\frac{\exp(\hat{q})}{\sigma-1}\right)}_{\text{new products}}.
\end{aligned}
$$

These expressions highlight the separate roles of own innovation and creative destruction in influencing the evolution of efficiency and markups. Own innovation causes both the production efficiency and the gap to drift upwards at the deterministic rate I, while creative destruction "resets" the mass in the distribution above Δ to have a gap of λ. Also note that there is a one-way flow of products from the non-competitive mass to the competitive through creative destruction events, while the entrant distribution Γ only directly affects the non-competitive mass.

Though these distributions do not have a closed form solution on the BGP, they can easily be computed. And given $F^C(\Delta,\hat{q})$ and $F^{NC}(\Delta,\hat{q})$, the economy-wide joint distribution is given by

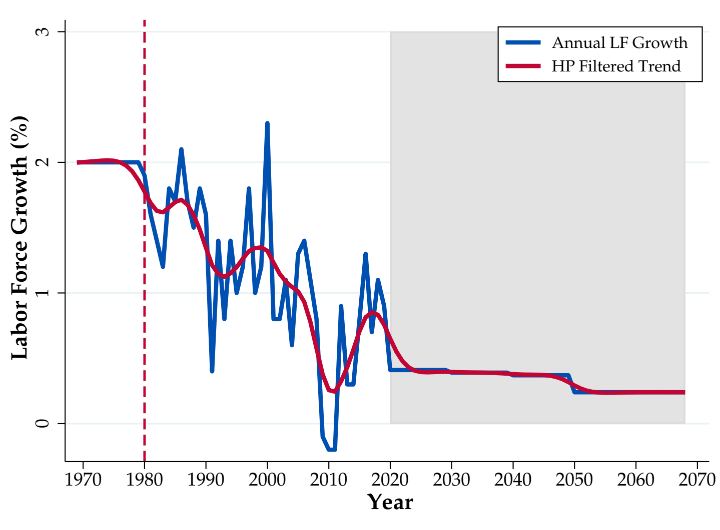
$$
F(\Delta,\hat{q}) = (1-\alpha)\, F^C(\Delta,\hat{q}) + \alpha F^{NC}(\Delta,\hat{q}),
$$

because α is exactly the steady-state fraction of products that have a competitor. Given $F(\Delta,\hat{q})$ we can then quantify the aggregate consequences of market power. Because higher markups reduce the labor share Λ and more dispersed markups reduce allocative efficiency $\mathcal{M}$, lower population growth tends to increase profits relative factor payments and has adverse effects on static allocation efficiency. Below we quantify the strength of these forces and solve for the joint distribution $F_t^C(\Delta,\hat{q})$ computationally, both along the BGP and during the transition.

4 Quantitative Analysis: Calibration

To quantify the importance of declining population growth we now calibrate our model to data from the US. In Figure 5 we display the historical growth rate of the labor force since 1980 and the official projections of the BLS. Our exercise to quantify the aggregate impact of this actual and projected decline is conceptually simple. We parametrize the model to a balanced growth path matching key

Figure 5: Labor Force Growth in the US



Notes: The figure shows the growth rate of the labor force in the U.S., with the raw series in blue and a HP-filtered trend component in red. The data is sourced from the BLS, accessed through FRED. Grey shading indicates projections.

moments of the data in 1980, when labor force growth was approximately 2%. We then study the aggregate impact of the path of labor force growth shown in Figure 5 by computing the dynamic response in our model. To do so, we treat the projections of the BLS as agents' rational expectations in our model, and also assume that the projected labor force growth rate after 2050 persists in the long-run..

4.1 Data

Our main dataset is the U.S. Census Longitudinal Business Database (LBD). The LBD is an administrative dataset containing information on the universe of employer establishments since 1978. It contains information on the age, industry, employment and payroll of each establishment, along with identifiers at the firm level that allow us to track the ownership of each establishment over time. We define the age of the firm in the LBD as the age of the oldest establishment that the firm owns. The birth of a new firm requires both a new firm ID in the Census and a new establishment record. We also modify the Census firm ID's to deal with some issues involving multi-establishment firms in the same way as developed in Walsh (2019).

To measure firms' markups, we require information on sales. We therefore augment the LBD data with information on firm revenue from administrative data contained in the Census' Business Register, following Moreira (2015) and Haltiwanger et al. (2016). The Business Register is the master list of establishments and firms for the U.S. Census and we are able to match approximately 70% of the records to the LBD.

In Table A-1 we provide some basic summary statistics on the firms in our data set. In total our data comprises about 3.61m firms in 1980 and 4.95m firms in 2010. During that time period, average firm employment increased by around 10% from 20 to 22 employees. The aggregate employment share of firms with less than 20 employees declined from 21.5% to 18.8% and the one of very large firms (with more than 10,000 employees) increased from 25.7% to 27%. Furthermore, firms became substantially older: the employment share of firms less than five years old declined from 38% to 30%. Qualitatively, all these are implications of our theory. Below we show the observed decline in population growth goes a long way to also replicate these patterns quantitatively.

4.2 Calibration

Our model is parsimoniously parametrized and rests on 11 parameters:

$$\Psi = \left\{ \underbrace{\alpha, \zeta, \varphi_E, \varphi_x, I, \bar{\omega}, \lambda}_{\text{Innovation \& Entry technology}}, \underbrace{\delta}_{\text{Exog. exit}}, \underbrace{\eta}_{\text{Pop. growth}}, \underbrace{\rho, \sigma}_{\text{Preferences}} \right\}.$$

Three of them - the demand elasticity σ, the discount rate ρ, and the convexity of the innovation cost function ζ - we set exogenously. We fix the elasticity of substitution between products σ at 4, following Garcia-Macia et al. (2016), set the discount rate ρ to 0.95 and assume a quadratic innovation cost function (i.e. $\zeta = 2$) as in Acemoglu et al. (2012).

The rate of labor force growth η is directly observed in the data (see Figure 5) and is our key parameter for the comparative statics. The remaining seven parameters are calibrated internally. We target key moments from the cross-sectional firm-size distribution in 1980 and observed life-cycle dynamics of markups and sales.[16] We are able to match these moments with arbitrary precision. Building a quantitatively accurate picture of the dynamic evolution of sales, employment and markups at the firm-level is crucial to credibly quantify the consequences of declining population growth. In Table 1 we report the parameters and the main moments we target.

While all moments are targeted simultaneously, there is nevertheless a tight mapping between particular moments and particular parameters which highlights how the different parameters are identified.

Innovation efficiency of incumbent firms: I and φ_x We identify the relative efficiency of the different sources of innovation from two dynamic moments: the life-cycle profile of sales and the life-cycle of markups. Because markup growth is driven by incumbents' own-innovation activities (see Figure

[16]The LBD data does not contain direct information on products. Argente et al. (2019) use data from Nielsen to provide direct evidence on the process of life-cycle growth at the product-level. Akcigit et al. (2021) analyze a related model and show that their model, calibrated to employment data, replicates the product-level distribution well. Cao et al. (2017) identify products (in the theory) with plants (in the data). For an early analysis of product-level data, see Bernard et al. (2011).

Table 1: Model Parameters

	Structural Parameters Description	Value	Moments	Data	Model
η	Labor force growth in 1980	0.02	Data from BLS	2%	2%
λ	Step size on quality ladder	1.11	Aggregate poductivity growth	2%	2%
I	Rate of own innovation	0.023	Markup growth by age 10 (RevLBD)	10.2%	10.2%
φ_X	Cost of inc. product creation	0.04	Sales growth by age 10 (RevLBD)	58%	58%
φ_E	Cost of entry	0.12	Avg. firm size (BDS)	20.7	20.7
δ	Destruction rate of products	0.06	Entry Rate in 1980 (BDS)	11.6 %	11.6 %
α	Share of creative destruction	0.59	Markup of entrants	-	18 %
$\bar{\omega}$	Relative efficiency of new products	0.45	Pareto tail of LBD employment distribution in 1980	1.1	1.1
ζ	Curvature of innovation cost	2	Set exogenously		
σ	Demand elasticity	4	Set exogenously		
ρ	Discount rate	0.05	Set exogenously		

Note: This table reports the calibrated parameters for the full model. Data for the firm lifecycle comes from the U.S. Census Longitudinal Database, augmented with revenues from tax-information using the Census' Business Register. Data for average firm size and the firm entry rate in 1980 are taken from the public use Business Dynamics Statistics.

4), this moment is informative about the rate of efficiency improvement I. Sales growth is in addition also affected by the rate of incumbent product creation, which depends directly on the cost of product expansion φ_X.

As we show in detail in Section A-2.3.3 in the Appendix, we can derive the life-cycle profiles of sales and markups (essentially) explicitly. This is not only convenient from a quantitative standpoint but also clarifies our identification strategy. The main insight to derive these moments is to first express markups and sales of a given product as a function of the product age a_P. Average relative sales as a function of a product age a_P are given by

$$s_P(a_P) \equiv E\left[\left.\frac{p_i y_i}{Y}\right| a_p\right] = E\left[\left.\mu_i^{1-\sigma}\left(\frac{q_i}{Q_t}\right)^{\sigma-1}\right| a_p\right] = \mu(a_p)^{1-\sigma} e^{(\sigma-1)\left(I-g^Q\right)a_p}\bar{q}^{\sigma-1},$$

where $\mu(a_P) = \min\left\{\frac{\sigma}{\sigma-1}, \Delta(a_P)\right\} = \min\left\{\frac{\sigma}{\sigma-1}, \lambda e^{Ia_P}\right\}$ and the remaining terms are the average relative quality . Because own-quality q increases at rate I while average quality Q increases at rate g^Q, $e^{(\sigma-1)\left(I-g^Q\right)a_p}$ is the relative drift of these random variables. The last term reflects that the initial average quality when the firm adds the product to its portfolio (see (5)).

With this expression for relative product sales $s(a_P)$ in hand, we can calculate the life-cycle of sales and markups at the firm-level. In particular, average sales and markups as a function of firm age a_f

are given by[17]

$$
\begin{aligned}
s_f\left(a_f\right) &= E\left[\left.\sum_{n=1}^{N_f} s_P\left(a_P\right)\right| a_f\right] \\
\mu_f\left(a_f\right) &= E\left[\left.\left(\sum_{i=1}^{N_f} \mu\left(a_p\right)^{-1} \frac{s_P\left(a_P\right)}{\sum_{i=1}^{N_f} s_P\left(a_P\right)}\right)^{-1}\right| a_f\right],
\end{aligned}
$$

where the expectations are taken with respect to the conditional distribution of N_f and a_P, conditional on a_f. Note that the conditional distribution of product age will in general depend on the age of the firm a_f, and will the conditional distribution of the number of products N. As we show in Section A-2.3.3 in the Appendix, we can calculate these conditional distributions of product age a_P and the number of products N_f given firm age a_f essentially explicitly. We can therefore calculate $s_f\left(a_f\right)$ and $\mu_f\left(a_f\right)$ without having to simulate the model.

Empirically, we measure markups at the firm level by the inverse labor share:

$$
\mu_f = \frac{py_f}{wl_f}, \tag{33}
$$

where py_f is the total revenue of the firm, and wl_f is the total wage bill. We calculate the total wage bill by aggregating establishment payroll. While this allows us in principle to measure markups for the population of U.S. firms, we only use firms' markup *growth* to calibrate our model. More specifically, letting $\mu_{f,t}$ be the mark-up of firm f at time t, we run a regression of the form

$$
\ln \mu_{f,t} = \sum_{a=0}^{20} \gamma_a^{\mu} \mathbb{I}_{Age_{ft}=a} + \theta_f + \theta_t + \epsilon_{f,t}, \tag{34}
$$

where $\mathbb{I}_{Age_{ft}=a}$ is an indicator for whether the firm is of age a and θ_f and θ_t are firm and time fixed effects respectively. Hence, γ_a^{μ} provides a non-parametric estimate of the rate of markup growth. We calibrate our model to the growth rate at the 10-year horizon, γ_{10}^{μ}.

Because we explicitly control for a firm fixed effect when estimating (34), we do not have to take a stand on firms' output elasticities as long as they are constant with age.[18] We follow the same approach when we estimate the life-cycle of sales, i.e. we also estimate (34) using log sales as the dependent variable and target γ_{10}^{py} in our quantitative model. In the LBD, firms increase their average

[17] Equivalently, the firm-level markup μ_f can also be expressed as a cost-weighted average of product-level markups, $\mu_f = \sum_{i=1}^{N_f} \mu_i \frac{wl_i}{\sum_{i=1}^{N_f} wl_i}$, as in Edmond et al. (2018).

[18] If, for example, firms within sectors had different production functions with different output elasticities, neither the level nor the dispersion of markups as measured from (33) could be distinguished from such differences in technology (see De Loecker and Warzynski (2012) and Peters (2020)). Also, by targeting markup growth, we avoid estimating output elasticities for labor, which is not feasible with the data we have as it does not contain data on capital or material inputs. Doing so would also complicate the mapping from model to data, since in our model labor is the only factor of production.

markup by roughly ten percentage points and grow in size by about 80% by age 10.

Entry Costs and product loss: φ_E and δ. We choose φ_E and δ to jointly match the entry rate and average firm size. The free condition determines market size L_t^P/N_t as a function of entry efficiency φ_E. This in turn is a key component of average firm employment. We thus choose φ_E to match an average firm employment of 20.76 in 1980 from the BDS. The higher the entry efficiency, the lower market size and the smaller the average size of firms. The exogenous rate of product loss δ directly influences the exit and hence - in a BGP - the entry rate of firms. We target the entry rate in 1980 of 11.6%.

Productivity growth through innovation: λ and $\overline{\omega}$. The parameters λ and $\overline{\omega}$ determine the relative quality of creatively destroyed products (λ) and newly generated varieties ($\overline{\omega}$). We infer these parameters from the aggregate rate of growth and the tail of the firm size distribution. That λ and $\overline{\omega}$ directly affect the growth rate is apparent from Proposition 2. For the tail of the firm size distribution, we find in our calibration that $\varsigma_n > \frac{1}{\sigma-1}\varsigma_q$, i.e. the tail of the employment distribution is given by $\varsigma_l = \frac{1}{\sigma-1}\varsigma_q$, where ς_q is given in (22). Given α and σ, this tail only depends on λ and $\overline{\omega}$. For our calibration we chose λ and $\bar{\omega}$ to target a rate of productivity growth of 2% and a tail parameter of the firm size distribution of 1.1 (close to Zipf's law). See Section A-2.2 in the Appendix for the details how we estimate the tail of the size distribution from our data.

New varieties vs. creative destruction: α. The share of new products in innovation, $1-\alpha$, plays an important role for the level of markups in the economy. The higher α, the lower the economy-wide markup because the higher the share of products that are subject to Bertrand competition. We target an economy-wide profit share of 25%.

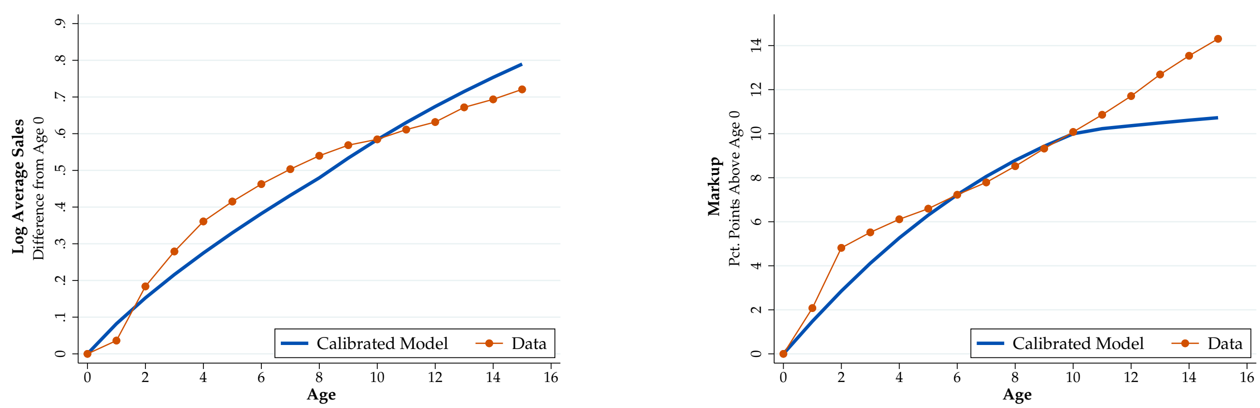
4.3 Estimates and Model Fit

As seen in Table 1, our model is able to match the targeted moments perfectly. To match the fact that firm-level markups grow by around ten percentage points at the ten year horizon, our model implies a rate of own-innovation of around 2.3%. For a creative destruction event, we estimate a productivity increase of 11%. This is required to match an annual aggregate growth rate of 2%. The initial production efficiency of new products is estimated to be low, about 10% of the average product in the economy. This relative low value is required to match the thickness of the tail of the employment distribution.

In addition to the targeted moments, our model, despite its parsimonious parametrization, also matches a variety of additional non-targeted moments. Consider first the sales and markup life cycle. In Figure 6 we show the model's performance by plotting the estimated coefficients γ_a^{μ} and γ_a^{py} from specification (34) estimated in the model and in the data. As highlighted in Table 1, we calibrate our model to match γ_{10}^{py} and γ_{10}^{μ}. Figure 6 shows that the model's implication for the whole age profile of sales (in the left panel) and markups (in the right panel) is quite close to what is observed in the data.

Figure 6: Lifecycle Growth in Firm Sales and Markups

(a) Sales Growth By Age (b) Markup Growth By Age



Note: Panel (a) in this Figure compares the lifecycle of firm sales in the model to the estimated lifecycle in the data. The data lifecycle plots the age coefficients from estimating equation (34) in the LBD. $N = \{35,300,000\}$, where this number has been rounded to accord with Census Bureau disclosure rules. The lifecycle of sales in the calibrated model is computed by simulating a panel of 10^6 firms , and averaging sales within age groups. Panel (b) does the same for relative markups.

For the case of sales, the model replicates the slight concavity of log sales well. In the model, this shape reflects survivorship bias; small firms either grow or are destroyed, while large firms can have products stolen and shrink without exiting. As such, average growth conditional on survival is declining with age for young firms before, eventually, becoming log-linear for large old firms, matching Gibrat's law. Quantitatively, firms in the US grow their sales by about 60 log points during their first 10 years. The fit for markups in Panel (b) is also relatively good, even though in the data markups appear more linear with age than emerge from the model. Empirically, markups are increasing almost linearly by 1% each year. In the model, the rate of markup growth is much more concave, reflecting the fact that markups are bounded from above by $\frac{\sigma}{\sigma-1}$.[19]

In Figure 7 we confront our model's predictions for the size distribution with the data. While we have explicitly targeted average size and the Pareto tail, our model matches the full non-parametric firm size and employment distribution very well. We plot the distribution of employment (left panel) and the number of firms (right panel) for both the model and the data in 1980.[20] Our model successfully matches both of these margins. Note in particular that it replicates the aggregate importance of very large firms with more than 1000 employees, that account for 25% of aggregate employment.

A central reason why our model successfully replicates the firm-size distribution is that it provides a good fit for the empirically observed exit hazards. In the left panel of Figure 8 we depict the exit rate by age from the micro-data in the LBD for the 1980 cohort of firms.[21] Our model is remarkably

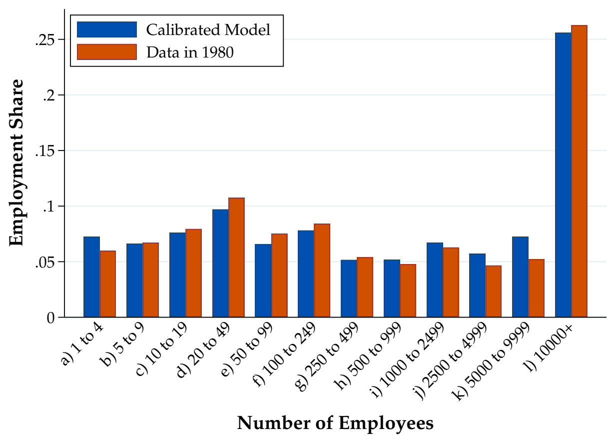
[19]In Figure A-5 in the Appendix we show the joint density of markups and efficiency at the product level, illustrating the positive correlation between markups and efficiency induced by survival and own-innovation.

[20]For replicability we chose size bins that are also available in the publicly available data from the BDS.

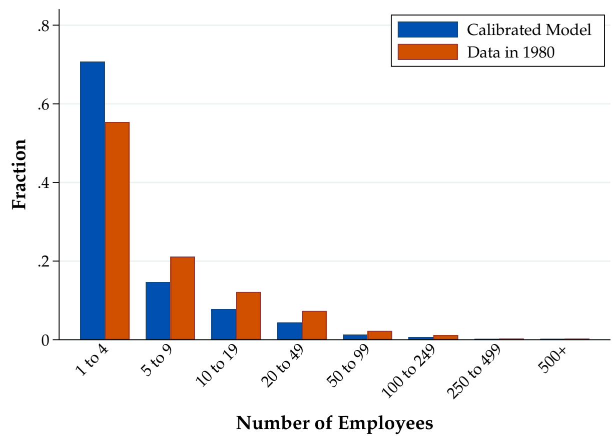
[21]To construct exit rates by age, we estimate a non-parametric Kaplan-Meier survival function by age for firms in the LBD. We select the cohort of firms born between 1980 and 1990, and follow them until 2015. We then take the exit rates to be the increments of the estimated survival functions. Each estimate is essentially the fraction of the sample that exits at

Figure 7: Size Distribution in Model and Data

(a) Employment Size Distribution



(b) Firm Counts in Size Bins



Notes: Panel (a) of this figure plots the employment shares by firm size in the calibrated model (blue bars) and the data (orange bars). Panel (b) shows the shares of the firm counts in model and data. The data is from the BDS release of 1980.

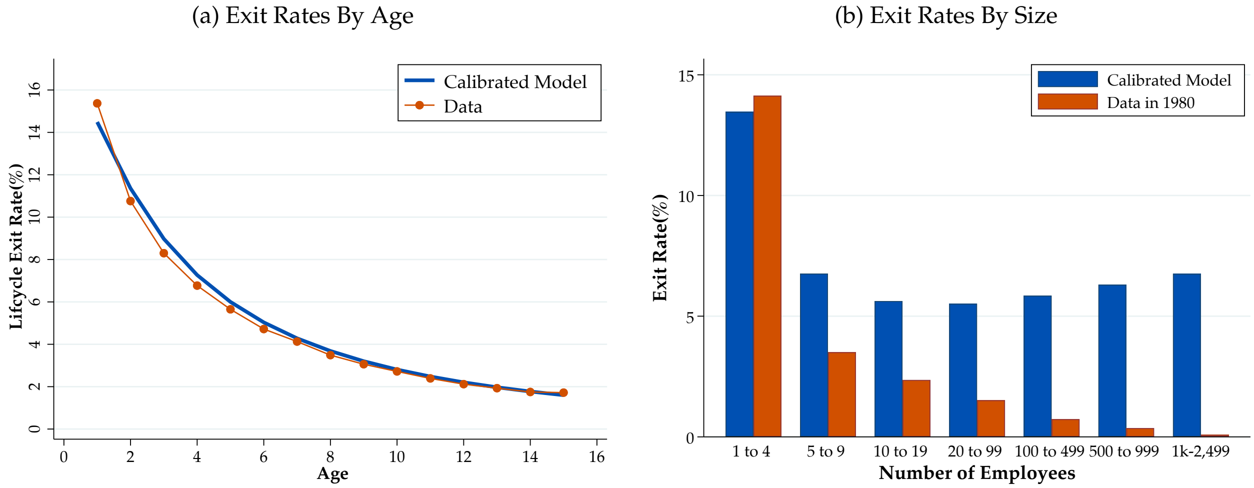
successful in replicating theses exit rates, despite the fact that we do not target them in the estimation. In our theory, exit rates are declining in age because older firms have more product lines and owning more products makes it progressively less likely that they will all be destroyed within a particular year.[22]

In the left panel we depict the exit rate for different size categories. Empirically, these exit rates are declining. Our model implies that this exit rate is initially declining but essentially independent of size for firms with more than 10 employees. The reason why our model has this counterfactual prediction is that (in our calibration) the thick tail of the employment distribution is driven by the distribution of product quality q and not the extensive margin of product creation. Hence, large firms are firms with a few superstar products, not those with many products. And because creative destruction is independent of product quality, such firms are as likely to exit as other firms. However, because - as seen in Figure 7 - the mass of large firms is relatively small, this prediction does not interfere greatly with our models' ability to provide a good fit to the firm size distribution. In Section A-2.6 of the Appendix, we extend our model to allow for type heterogeneity, whereby some young firms (sometimes flamboyantly described as "rockets" or "gazelles", see Pugsley et al. (2019)) grow at a faster rate for a time through product expansion. This allows a subset of firms to become large by adding a large number of products, an outcome which is unlikely in the baseline model. This extension improves the model's fit along this dimension substantially, since some large firms have many products and are thus unlikely to exit, but changes little else in the theoretical analysis.

age a (though the estimator accounts for the truncation from ceasing to observe firms after 2015).

[22] Data on the age distribution in the target year of our calibration is unavailable both in the public BDS and the administrative LBD, since the Census only begins tracking age for new establishments in 1978 (the first year the data is available).

Figure 8: Firm Exit Rates: Model and Data



Note: This figure presents a comparison of lifecycle exit rates between model and data. The exit rates in the data are taken from the increments in a Kaplan-Meier survival function estimated on all firms in the LBD born between 1980 and 1990. The model exit rates come from simulating a panel of 10^6 firms and calculating the fraction of the panel that exit at yearly frequencies. Age of a on the horizontal axis indicates that the firm exited between age $a-1$ and age a.

5 The Aggregate Impact of Falling Population Growth

We now use our calibrated model to quantify the effects the implications of the observed and projected decline in labor force growth shown in Figure 5. To do so, we start with the calibrated BGP in 1980 and then feed the path displayed in Figure 5 into the model. In the context of our theory, we assume that all agents have rational expectations about this path. All other parameters are held constant.

We focus both on the positive and normative aspects of our theory. On the positive side we focus on changes in the process of firm-dynamics, in particular the entry rate, average firm size, measures of concentration, the distribution of markups and firms' lifecycle growth. On the normative side we quantify the effect of the observed population growth decline on the economy-wide growth g^y, the static increase in the variety intensity $\mathcal{N}_t$ and overall welfare.

5.1 Declining Population Growth and Changing Firm Dynamics

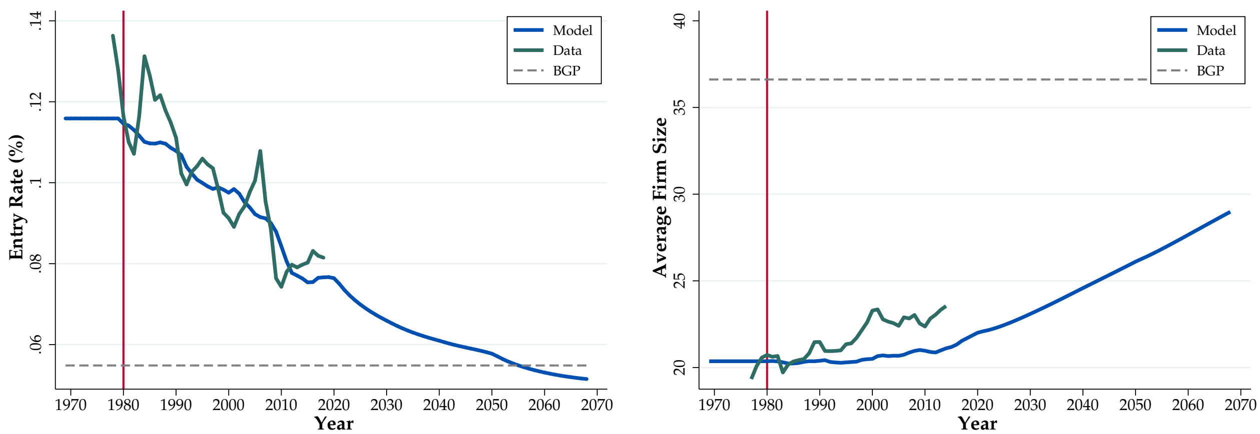
We start by considering the impact on firm dynamics. We focus first on the entry rate and average firm size. In Figure 9 we plot both the data and the implications of our theory.

Consider first the data, shown in green. The entry rate (shown in the left panel) declined markedly in the last 30 years from around 12% in the 1980s to around 8% in the mid 2000s. Note that this series of the entry rate tracks the evolution of population growth shown in Figure 5 very closely, and indeed the contemporaneous correlation is 0.74.[23] At the same time, average firm size (shown in the

[23] Karahan et al. (2016) and Hathaway and Litan (2014) study this link directly in the geographic cross-section, showing

Figure 9: Declining Population Growth and Changing Firm Dynamics

(a) Entry Rate | (b) Average Firm Size



Note: The figure displays the dynamic response of the entry rate (left panel) and average firm size (right panel) to the path of population growth shown in Figure 5.

right panel) rose from 20 to 23 employees, i.e. increased by around 15%. In blue, we superimpose the predictions of our theory. Recall that we used both the entry rate and average size in 1980 as a calibration target and hence match these numbers by construction. The subsequent fall in the entry rate and the rise in average size are then the sole consequence of the observed and projected decline in population growth.

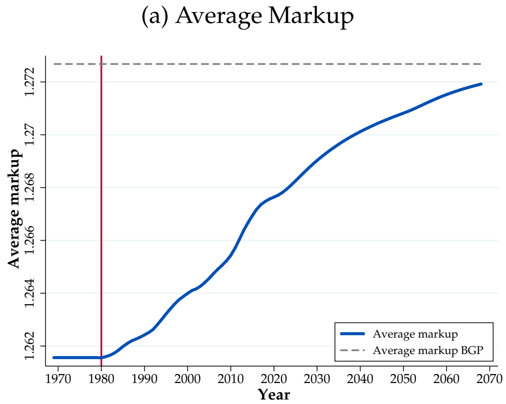
Figure 9 shows that the decline in population growth goes a long way to explain the observed changes in the entry rate and average size. For the entry rate, our model matches the US experience almost perfectly. For average size, our model also predicts an increase in average employment. However, our model implies a somewhat slower increase compared to what is observed in the data, and that the long-run increase will take many decades to settle at a higher value once labor force growth stabilizes. The increase in concentration is also similar to what is observed in the data, with the employment share of large firms (defined by the BDS to be 10,000 employees or more) increasing by 1% by 2015, roughly in line with the data (see Table A-1).

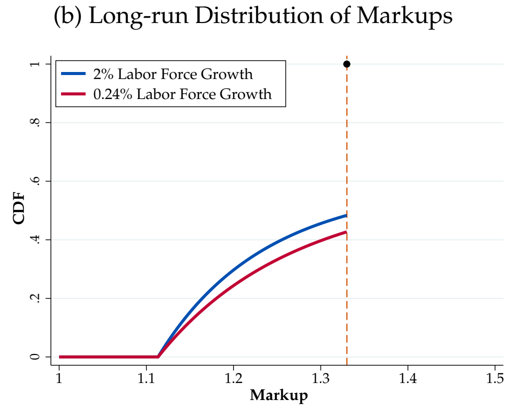
In Figure 9 we only display the implications of our theory until 2070. Given the population growth path shown in Figure 5, our model has not reached a new BGP at this point. Hence, we also plot the long-run implications for the entry rate and average size as dashed line. The entry rate adjusts relatively quickly and is already quite close to its long-run BGP value by 2070. By contrast, our model predicts that average size has some way left to run due to the slow-moving firm size distribution and in the long-run will increase substantially.

As highlighted in Section 2.5, average size is increasing both because of a shift in age distribution towards older firms and because lower population growth increases firm size conditional age. Quanti-

that states with slower labor force growth, as predicted by lagged birth rates in previous decades, see lower rates of firm entry.

Figure 10: Declining Population Growth and Rising Market Power





Note: The left panel shows the transition path of the average product markup as labor force growth changes according to the path in Figure 5. The right panel shows the markup distribution in the BGP before and after the transition.

tatively, however, much of the increase depicted in Figure 9 comes from shifts in the age distribution. In Figure A-6 in the Appendix we show the change in our targeted life-cycle moments of exit by age and sales growth by age. These objects do change as population growth declines, but only modestly. This dominant role of the age distribution is consistent with the data, where size or exit rates by age also changed little (see Karahan et al. (2016) and Hopenhayn et al. (2018)).

In Figure 10 we report the implied changes in product-level markups. We display both the evolution of the cost-weighted average markup (left panel) and the change in the distribution of markups in the BGP (right panel). As implied by our theoretical results, the decline in population growth increases markups in a first-order stochastic dominance sense. Quantitatively, the increase in market power is modest: the average product markup increases by about 1%. The markup distribution in the right panel highlights where this increase stems from. Declining population growth lowers creative destruction and hence increases markups among products that have a competitor. By contrast, the markup for products without a competitor is - by construction - $\sigma/(\sigma-1)$ and hence independent of η. Moreover, the share of non-competitive products is given by $1-\alpha$ and hence also independent of η.

As for average firm size, our model implies that the increase in average markups shown in Figure 10 occurs mostly across firms and is a reflection of the fact that firms become older. Within firms, products tend to become older since products are destroyed less frequently. On its own, this would tend to raise average markups. However, firms also tend to accumulate more products, which are on average younger and hence have lower markups (see Figure A-7). Quantitatively, these two forces almost exactly offset one another, so that the rise in markups reflects compositional changes whereby large and old firms with high markups increase their market share. This pattern is qualitatively consistent with the findings reported in Kehrig and Vincent (2017) and Autor et al. (2017).

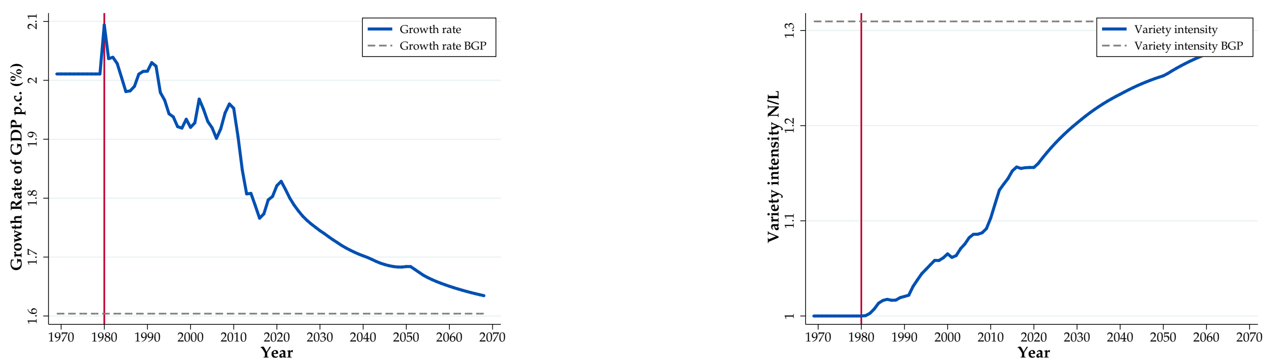
Table 2: Population Growth and Firm-level Moments

	Avg. Firm Size Emp. per Firm	Entry Rate %	Avg. Markup %	Large Emp Share >10000 Emp, %	Small Emp Share <20 Emp, %
Calibrated Baseline	20.04	11.60	26.4	25.6	37.8%
1.76% Decline in η	40.91	5.42	27.5	27.1	37.6%

Note: The table reports several firm level moments computed in the model. The first row refers to the BGP of the calibrated model. The second row is the counterfactual BGP where we reduce population growth by 1.76 to 0.24%. For ease of comparison we normalize N_t/L_t to 1 in the calibrated baseline.

Figure 11: Declining Population Growth and Income per Capita

(a) Income per Capita Growth g_t^y (b) Variety Intensity $\mathscr{N}_t$



Note: The figure displays the dynamic response of the aggregate growth rate (left panel) and the variety intensity $\mathscr{N}_t = N_t/L_t$to the path of population growth shown in Figure 5.

5.2 Declining Population Growth, Aggregate Growth and Welfare

We now turn to the normative implications of the decline in population growth. In Figure 11 we depict the growth rate of income per capita (left panel) and the change in the variety intensity $\mathscr{N}_t$ (right panel). As in Figure 9 above we trace out the model's implication until 2070 and indicate the long-run levels of the respective variables in the new BGP as dashed lines.

Interestingly, the effect of population growth on output growth is not monotone. On impact, a population growth decline *increases* output growth for about one decade. This is due to an increase in the variety intensity $\mathscr{N}_t$, which is a source of variety gains. These variety gains at the aggregate level coexist with rising average firm size because firms produce multiple products and the number of products per firm increases. Hence, rising concentration does not necessarily go hand in hand with falling variety. The mass of products available to consumers actually goes up in response to falling population growth. Because the increase in variety is only a transitory phenomenon, eventually, output growth declines and stabilizes at a lower level. In the long-run, declining population growth will reduce the growth rate of income per capita, as in most models of semi-endogenous growth (Jones,

Table 3: Population Growth and Economic Growth

	Welfare	Growth			Production labor	Variety intensity
		g^y	Variety (N_t)	Efficiency (Q_t)	L_t^P/L_t	N_t/L_t
Calibrated Baseline	100%	0.02	0.007	0.013	0.86	1
1.76% Decline in η	62%	0.016	0.001	0.015	0.88	1.21

Note: The table reports the aggregate growth rate (g^y), the growth stemming from variety gains $\left(\frac{1}{\sigma-1}g_N\right)$ and efficiency growth $\left(g^Q\right)$, the share of workers employed in researcher $\left(L^P/L\right)$ and the variety intensity (N_t/L_t). The first row refers to the calibrated model. The second row is the counterfactual where we reduce population growth by 1.76 to 0.24%. For ease of comparison we normalize N_t/L_t to 1 in the calibrated baseline.

2021). In our calibration, the long-run growth rate declines from around 2% to 1.6%.

In Table 3 we report the welfare implications of declining population growth and use the model to decompose the long-run growth rate into its different components. We measure welfare in consumption equivalent terms, that is by how much would we need to change the level of consumption per capita in the old BGP to achieve the same level of welfare for a member of the representative household. Welfare is 38% lower in the new BGP when population growth decreases from 2% to 0.24%. Around three quarters of this effect is due to slower growth in income, while the remainder comes from an an increase in misallocation $\mathcal{M}_t$.

The remainder of the table reports the aggregate growth rate and its composition between variety gains and efficiency growth. A decline in population growth reduces the long-run equilibrium growth rate from 2% to 1.6%. Furthermore, this decline stems almost entirely from falling variety growth. In fact, efficiency growth rises slightly in response to the decline in population growth. The reason is that we estimate the efficiency of production of new varieties $\overline{\omega}$ to be relatively low. The declining rate of variety creation therefore impacts average efficiency growth positively.

In the remaining two columns we also report the equilibrium allocation of labor and the long-run variety intensity, which is constant along a BGP. Note that falling population growth increases the level of productivity by increasing the variety intensity of the economy.

6 Conclusion

Most countries have experienced declining rates of fertility and a slowdown in population growth in recent decades. There is little reason to think that this trend is going to reverse any time soon; a world of low and falling population growth looks like it is here to stay.

In this paper we have shown that this trend is likely to have important implications for both the process of firm dynamics and for aggregate productivity. We proposed a rich firm-based model of semi-endogenous growth, that features creative destruction, variety growth, productivity growth by incumbent firm and heterogeneous markups, but nevertheless lends itself to an analytical characterization of the effects of population growth. We derived two main results: First, declining population growth reduces creative destruction and entry and increases average firm size and market concentration. Second, lower population growth reduces economic growth in the long-run, but has positive effects on productivity in the short-run.

At the heart of our mechanism is a simple insight: along a balanced growth path, the number of available products has to grow at the rate of population growth. In equilibrium, this leads to lower innovative activity by entering firms, lower creative destruction and higher concentration and - eventually - lower growth. At the same time, corporate valuations increase because future profits are discounted at a lower rate. Free entry therefore requires an increase in competition, which is accommodated by a rise in the economy's number of products per capita and hence average productivity. The short-fun welfare consequences of lower population growth are a priori ambiguous.

To quantify the importance of falling population growth, we calibrate our model to firm-level data of the US. The model, despite being parsimoniously parametrized, matches a variety of salient firm-level moments, and provides a laboratory to understand the quantitive impact changes in demographics. . We draw three main conclusions. First, the population growth channel can account for a large share of the change in entry rates and firm size since the 1980s. Hence, changes in population growth are likely to be an important contributor for the decline in dynamism in the US and the rest of the developed world. Second, even though the decline in population growth is predicted to lower economic growth in the long-run, economic growth remains higher for almost two decades. Finally, even though lower population growth increases market power and markups, we estimate this effect to be quantitatively small. Hence, the rise in markups and the fall in the labor share are unlikely to be driven by falling fertility, but rather due to technological or institutional changes.

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Appendix for "Population Growth and Firm Dynamics"

[NOT FOR PUBLICATION]

A-1 Theory

A-1.1 Characterization of the Baseline Model

This section contains the derivation of all results for the baseline model characterized in Section 2. Note that the household side is characterized by usual Euler equation $\frac{\dot{c}_t}{c_t} = r_t - \rho$ and the transversality condition

$$\lim_{t\to\infty}\left[e^{-\int_0^t (r_s-\eta)ds}a_t\right] = 0,$$

where a_t denotes per-capita assets of the representative household. Our assumption $\rho > \eta$ implies that the transversality condition is satisfied along a BGP.

A-1.1.1 Static Equilibrium

Consider first the static equilibrium allocations, in particular (2). Letting μ_i denote the markup in product i, the equilibrium wage is given by

$$w_t = \left(\int_0^{N_t} \mu_i^{1-\sigma} q_i^{\sigma-1} di\right)^{\frac{1}{\sigma-1}} = N_t^{\frac{1}{\sigma-1}}\left(\int \mu^{1-\sigma} q^{\sigma-1} dF_t(q,\mu)\right)^{\frac{1}{\sigma-1}}. \tag{A-1}$$

Similarly, aggregate output Y_t is given by

$$Y_t = N_t^{\frac{1}{\sigma-1}}\frac{\left(\int \mu^{1-\sigma}q^{\sigma-1}dF_t(q,\mu)\right)^{\frac{\sigma}{\sigma-1}}}{\int \mu^{-\sigma}q^{\sigma-1}dF_t(q,\mu)}L_t^P. \tag{A-2}$$

Defining $Q_t = \left(\int q^{\sigma-1}dF_t(q)\right)^{\frac{1}{\sigma-1}} = \left(E\left[q^{\sigma-1}\right]\right)^{\frac{1}{\sigma-1}}$ we can write (A-2) as

$$Y_t = N_t^{\frac{1}{\sigma-1}}Q_t\mathcal{M}_t L_t^P \qquad \text{where} \qquad \mathcal{M}_t = \frac{\left(\int \mu^{1-\sigma}\left(q/Q_t\right)^{\sigma-1}dF_t(q,\mu)\right)^{\frac{\sigma}{\sigma-1}}}{\int \mu^{-\sigma}\left(q/Q_t\right)^{\sigma-1}dF_t(q,\mu)}. \tag{A-3}$$

Similarly,

$$w_t L_P = \Lambda_t Y_t \qquad \text{where} \qquad \Lambda_t = \frac{\int \mu^{-\sigma}\left(q/Q_t\right)^{\sigma-1}dF_t(q,\mu)}{\int \mu^{1-\sigma}\left(q/Q_t\right)^{\sigma-1}dF_t(q,\mu)}. \tag{A-4}$$

For the case of $\mu_i = \mu$, $\mathcal{M}_t$ and Λ_t reduce to $\mathcal{M}_t = 1$ and $\Lambda_t = 1/\mu$ as required in (2).

Product-level sales and profits are given by

$$
\begin{aligned}
py_i &= \mu_i^{1-\sigma}\left(\frac{q_i}{Q_t}\right)^{\sigma-1}\left(\frac{1}{\mathcal{M}_t\Lambda_t}\right)^{\sigma-1}\frac{Y_t}{N_t} && \text{(A-5)}\\
\pi_i &= \left(1-\frac{1}{\mu_i}\right)\times\mu_i^{1-\sigma}\left(\frac{q_i}{Q_t}\right)^{\sigma-1}\left(\frac{1}{\mathcal{M}_t\Lambda_t}\right)^{\sigma-1}\frac{Y_t}{N_t}. && \text{(A-6)}
\end{aligned}
$$

If markups are constant, (A-5) reduces to

$$
py_i = \left(\frac{q_i}{Q_t}\right)^{\sigma-1}\frac{Y_t}{N_t} \quad \text{and} \quad \pi_i = \left(\frac{\mu-1}{\mu}\right)\left(\frac{q_i}{Q_t}\right)^{\sigma-1}\frac{Y_t}{N_t}.
$$

A-1.1.2 Aggregate Growth Rate

Given τ_t and $\nu_t = g_{Nt} + \delta$, the rate of quality growth is given by

$$
g_Q = \frac{\dot{Q}_t}{Q_t} = \left(\frac{\lambda^{\sigma-1}-1}{\sigma-1}\right)\tau_t + \frac{\left(\overline{\omega}^{\sigma-1}-1\right)}{\sigma-1}\nu_t + I. \tag{A-7}
$$

The growth rate of labor productivity is given by

$$
\begin{aligned}
g_t^{LP} = \frac{d}{dt}\ln\left(Q_t N_t^{\frac{1}{\sigma-1}}\right) &= g_t^Q + \frac{1}{\sigma-1}g_t^N\\
&= I + \left(\frac{\lambda^{\sigma-1}-1}{\sigma-1}\right)\tau_t + \frac{\overline{\omega}^{\sigma-1}}{\sigma-1}\nu_t - \frac{1}{\sigma-1}\delta. && \text{(A-8)}
\end{aligned}
$$

A-1.1.3 Proof of Proposition 1

We first derive the value function stated in Proposition 1. Upon rewriting the innovation value $\Xi_t\left([q_i]\right)$ as

$$
\Xi_t\left([q_i]\right) = n\times\max_x\left\{x\left(\alpha\int V_t\left([q_i],\lambda q\right)dF_t\left(q\right) + (1-\alpha)\int V_t\left([q_i],\omega Q_t\right)d\Gamma\left(\omega\right) - V_t\left([q_i]\right)\right) - \frac{1}{\varphi_x}x^{\zeta}w_t\right\},
$$

it is immediate that the value function is additive, i.e. $V_t\left([q_i]\right) = \sum_{i=1}^{n} V_t\left(q_i\right)$. The HJB equation associated with $V_t\left(q_i\right)$ is given by

$$
r_t V_t\left(q\right) - \dot{V}_t\left(q\right) = \pi_t\left(q\right) + I\frac{\partial V_t\left(q\right)}{\partial q}q - \left(\tau+\delta\right)V_t\left(q\right) + \Xi_t, \tag{A-9}
$$

where $\Xi_t = \max_x\left\{x\left(\alpha V_t^{CD} + (1-\alpha)V_t^{NV}\right) - \frac{1}{\varphi_x}x^{\zeta}w_t\right\}$ with $V_t^{CD} = \int V_t\left(\lambda q\right)dF_t\left(q\right)$ and $V_t^{NV} = \int V_t\left(\omega Q_t\right)d\Gamma\left(\omega\right)$. Suppose the value function takes the following forms

$$
V_t\left(q\right) = q^{\sigma-1}U_t + M_t, \tag{A-10}
$$

where M_t and U_t grow at some rate g_M and g_U respectively. Then

$$I\frac{\partial V_t(q)}{\partial q}q = I(\sigma-1)q^{\sigma-1}U_t.$$

Using that

$$\pi_t(q) = (\mu-1)\mu^{-\sigma}q^{\sigma-1}\frac{Y_t}{w_t^{\sigma-1}} = (\mu-1)\left(\frac{q}{Q_t}\right)^{\sigma-1}\frac{L_t^P}{N_t}w_t$$

(A-9) can be written as

$$(r_t+\tau+\delta-g_U)q^{\sigma-1}U_t + (r+\tau+\delta-g_M)M_t = \left((\mu-1)\left(\frac{1}{Q_t}\right)^{\sigma-1}\frac{L_t^P}{N_t}w_t + I(\sigma-1)U_t\right)q^{\sigma-1} + \Xi_t. \tag{A-11}$$

It is easy to show that along a BGP this implies that

$$\begin{aligned} U_t &= \frac{(\mu-1)\left(\frac{1}{Q_t}\right)^{\sigma-1}\frac{L_t^P}{N_t}w_t}{\rho+\tau+\delta+(\sigma-1)(g_Q-I)} \\ M_t &= \frac{\Xi_t}{\rho+\tau+\delta}, \end{aligned}$$

as $\Xi_t \propto w_t$. To see this note that

$$\Xi_t = \max_x\left\{x\left(\alpha V_t^{CD} + (1-\alpha)V_t^{NV}\right) - \frac{1}{\varphi_x}x^\zeta w_t\right\} = \frac{\zeta-1}{\varphi_x}x^\zeta w_t,$$

where

$$x = \left(\frac{\varphi_x}{\zeta}\right)^{\frac{1}{\zeta-1}}\left(\alpha\frac{V_t^{CD}}{w_t} + (1-\alpha)\frac{V_t^{NV}}{w_t}\right)^{\frac{1}{\zeta-1}}. \tag{A-12}$$

The value function is therefore given by

$$\begin{aligned} V_t(q) &= \frac{(\mu-1)\left(\frac{q}{Q_t}\right)^{\sigma-1}\frac{L_t^P}{N_t}w_t}{\rho+\tau+\delta+(\sigma-1)(g_Q-I)} + \frac{\frac{\zeta-1}{\varphi_x}x^\zeta w_t}{\rho+\tau+\delta} \\ &= \frac{\pi_t(q)}{\rho+\tau+\delta+(\sigma-1)(g_Q-I)} + \frac{\frac{\zeta-1}{\varphi_x}x^\zeta w_t}{\rho+\tau+\delta}. \end{aligned}$$

Note also that

$$V_t^{CD} = \int V_t(\lambda q)\,dF_t(q) = V_t(\lambda Q_t) \text{ and } V_t^{NV} = \int V_t(\omega Q_t)\,d\Gamma(\omega) = V_t(\overline{\omega}Q_t).$$

This concludes the proof of Proposition 1.

A-1.1.4 Characterization of Equilibrium

In this section we characterize the full equilibrium of our economy. We maintain the assumption that the free entry condition is binding along the equilibrium path. The equilibrium is characterized by the following conditions:

1. The evolution of aggregate productivity is given by (see (A-7))

$$g_Q = \frac{\dot{Q}_t}{Q_t} = \frac{\lambda^{\sigma-1}-1}{\sigma-1}\tau_t + \frac{\overline{\omega}^{\sigma-1}-1}{\sigma-1}\nu_t + I$$

   where $\nu_t = g_{Nt} + \delta$

2. The rate of creative destruction is linked to the growth rate of N_t according to

$$\tau = \frac{\alpha}{1-\alpha}\nu_t, \tag{A-13}$$

   where $\nu_t = (1-\alpha)(z_t + x)$. Note that x is constant because of the binding free entry condition.

3. Labor market clearing requires $L_t = L_{Pt} + L_{Rt}$, where

$$L_{Rt} = N_t\left(\frac{1}{\varphi_E}z_t + \frac{1}{\varphi_x}x^{\zeta}\right) = N_t\frac{1}{\varphi_E}\left(z_t + \frac{1}{\zeta}x\right)$$

   Hence,

$$\frac{L_t}{N_t} = \frac{L_{Pt}}{N_t} + \frac{1}{\varphi_E}\left(z_t + \frac{1}{\zeta}x\right) \tag{A-14}$$

4. The Euler equation is given by

$$r = \rho + g_c \tag{A-15}$$

   where g_c is the growth rate of per capita consumption. Wages and output are given by $Y_t = N_t^{\frac{1}{\sigma-1}}Q_t L_t^P$ and $w_t = \frac{1}{\mu}Y_t/L_t^P$. Note that market clearing requires $C_t = Y_t$. Hence, the growth rate of per capita consumption is given by

$$g_c = g_Y - \eta = g_w + g_{L^P} - \eta, \tag{A-16}$$

   where $g_w = \frac{1}{\sigma-1}g_N + g_Q$ (see (A-8)). The Euler equation in (A-15) therefore implies that the real interest rate is given by

$$r = \rho + g_w + g_{L^P} - \eta.$$

5. To derive the implications for the free entry condition, note that (A-11) implies that

$$\frac{1}{\varphi_E} = \overline{q}^{\sigma-1}u_t + m_t$$

where

$$
\begin{aligned}
m_t &= \frac{\frac{\zeta-1}{\varphi_x} x^{\zeta}}{\rho + g_{L^P} - \eta + \tau + \delta - g_m} \\
u_t &= \frac{(\mu - 1)\frac{L_t^P}{N_t}}{\rho + g_{L^P} - \eta + \tau + \delta - g_u + (\sigma - 1)(g_Q - I)}.
\end{aligned}
$$

Now define

$$
\ell_t^P \equiv \frac{L_t^P}{L_t} \quad \text{and} \quad \mathcal{N}_t = \frac{N_t}{L_t}.
$$

Also note that $\tau = \alpha(z + x) = \frac{\alpha}{1-\alpha}\nu_t$. Then we can write the free entry condition as

$$
\frac{1}{\varphi_E} = \frac{\bar{q}^{\sigma-1}(\mu - 1)}{\rho + g_\ell + \delta - g_u + \left(\frac{\bar{q}^{\sigma-1}}{1-\alpha} - 1\right)\nu_t} \frac{\ell_t^P}{\mathcal{N}_t} + \frac{\frac{\zeta-1}{\varphi_x} x^{\zeta}}{\rho + g_\ell + \frac{\alpha}{1-\alpha}\nu_t + \delta - g_m}.
$$

Hence, the equilibrium is characterized by a path $\left\{\ell_t^P, \mathcal{N}_t\right\}_t$ that satisfies the the free entry condition and labor market clearing

$$
\begin{aligned}
\frac{1}{\varphi_E} &= \frac{\bar{q}^{\sigma-1}(\mu - 1)}{\rho + g_\ell + \delta - g_u + \left(\frac{\bar{q}^{\sigma-1}}{1-\alpha} - 1\right)\nu_t} \frac{\ell_t^P}{\mathcal{N}_t} + \frac{\frac{\zeta-1}{\varphi_x} x^{\zeta}}{\rho + g_\ell + \frac{\alpha}{1-\alpha}\nu_t + \delta - g_m} && \text{(A-17)} \\
\frac{1 - \ell_t^P}{\mathcal{N}_t} &= \frac{1}{\varphi_E}\left(\frac{\nu_t}{1-\alpha} - \frac{\zeta - 1}{\zeta} x\right), && \text{(A-18)}
\end{aligned}
$$

where g_u and g_m are the growth rates of u_t and m_t given in

$$
\begin{aligned}
m_t &= \frac{\frac{\zeta-1}{\varphi_x} x^{\zeta}}{\rho + g_\ell + \frac{\alpha}{1-\alpha}\nu_t + \delta - g_m} \\
u_t &= \frac{(\mu - 1)\,\ell_t^P / \mathcal{N}_t}{\rho + g_\ell + \delta - g_u + \left(\frac{\bar{q}^{\sigma-1}}{1-\alpha} - 1\right)\nu_t}.
\end{aligned}
$$

For a given initial condition $\mathcal{N}_0$ and the terminal condition that $\ell_t^P \to \bar{\ell}^P$ and $m_t \to m$ and $u_t \to u$ one can solve for the dynamic path $\left\{\ell_t^P, \mathcal{N}_t\right\}_t$.

A-1.1.5 Balanced Growth Path

Along a BGP, income per capita grows at a constant rate. (A-16) implies that

$$
g_c = g_w + g_{L^P} - \eta = \left(\frac{\bar{q}^{\sigma-1} - \alpha}{\sigma - 1}\right)\frac{\nu_t}{1-\alpha} - \frac{1}{\sigma - 1}\delta + I + g_{\ell^P}.
$$

Along the BGP it also has to be the case that $\ell^P = L_t^P / L_t$ is constant. Hence, g^N is constant along a BGP. (A-18) therefore implies that $\mathcal{N}_t$ has to be constant, i.e.

$$g_N = \nu - \delta = \eta.$$

Hence, along the BGP the mass of products N_t grows at the same rate as the population. With ℓ^P and $\mathcal{N}$ constant, $g_u = g_m = 0$ along the BGP. Hence, $\left(\mathcal{N}, \ell^P\right)$ are given by

$$\frac{1}{\varphi_E} = \frac{\overline{q}^{\sigma-1}(\mu-1)}{\rho + \left(\frac{\overline{q}^{\sigma-1}}{1-\alpha} - 1\right)\eta + \frac{\overline{q}^{\sigma-1}}{1-\alpha}\delta} \frac{\ell_t^P}{\mathcal{N}_t} + \frac{\frac{\zeta-1}{\varphi_x} x^{\zeta}}{\rho + \frac{\alpha\eta+\delta}{1-\alpha}} \tag{A-19}$$

$$\frac{1-\ell^P}{\mathcal{N}} = \frac{1}{\varphi_E}\left(\frac{\eta+\delta}{1-\alpha} - \frac{\zeta-1}{\zeta}x\right). \tag{A-20}$$

To characterize the solution, note that the free entry condition (A-19) defines a relationship $\mathcal{N}^{FE}\left(\ell^P\right)$ which is increasing and satisfies $\lim_{\ell^P \to 0} \mathcal{N}^{FE}\left(\ell^P\right) = 0$ and $\mathcal{N}^{FE}(1) = \left(\frac{1}{\varphi_E} - \frac{\frac{\zeta-1}{\varphi_x}x^{\zeta}}{\rho+\frac{\alpha\eta+\delta}{1-\alpha}}\right)^{-1} \frac{(\mu-1)\overline{q}^{\sigma-1}}{\rho+\left(\frac{\overline{q}^{\sigma-1}}{1-\alpha}-1\right)\eta+\frac{\overline{q}^{\sigma-1}}{1-\alpha}\delta}$. Similarly, the resource constraint (A-20) defines a schedule $\mathcal{N}^{RC}\left(\ell^P\right)$, which is decreasing and satisfies $\mathcal{N}^{RC}(0) = \frac{\varphi_E}{\frac{\eta+\delta}{1-\alpha} - \frac{\zeta-1}{\zeta}x}$ and $\lim_{\ell^P \to 1} \mathcal{N}^{FE}\left(\ell^P\right) = 0$. Hence, these equations have a unique solution for $\mathcal{N} > 0$ and $\ell^P \in (0,1)$. Equation (A-20) also implies that $\mathcal{N}^{RC}\left(\ell^P\right)$ is decreasing in η holding ℓ^P constant. It can also be shown that $\mathcal{N}^{FE}\left(\ell^P\right)$ is decreasing in η holding ℓ^P constant if α and $\overline{\omega}$ are sufficiently large. A sufficient condition for that to be the case is $\frac{\overline{q}^{\sigma-1}}{1-\alpha} > 1$. If $\frac{\overline{q}^{\sigma-1}}{1-\alpha} > 1$, future profits are discounted at a higher rate if η increases. This implies that - holding ℓ^P constant - $\mathcal{N}^{FE}\left(\ell^P\right)$ shifts down. In Figure 2 in the main text we depict these loci for the case of $\frac{\overline{q}^{\sigma-1}}{1-\alpha} > 1$, which is satisfied at our estimated parameters.

A-1.1.6 Population Growth and Firm Dynamics (Section 2.5)

In this section we derive the relationship between population growth η and the different moments of the process of firm dynamics. In particular, we derive

1. the survival function $S(a)$ in (19),
2. the average number of products by age $\overline{n}(a)$ in (20),
3. the pareto tail of the product distribution ζ_n in (21).

Firm survival $S(a)$ and the average number of products $\overline{n}(a)$ Let $p_n(a)$ be the probability that a firm has n products at age a. This evolves according to

$$\dot{p}_n(a) = (n-1)\, x p_{n-1}(a) + (n+1)(\tau+\delta)\, p_{n+1}(a) - n(x+\tau+\delta)\, p_n(a). \tag{A-21}$$

Because exit is an absorbing state, $\dot{p}_0(a) = (\tau+\delta)\, p_1(a)$. The solution to this set of differential equations is (see Klette and Kortum (2004))

$$\begin{aligned} p_0(a) &= \frac{\tau+\delta}{x}\gamma(a) && \text{(A-22)} \\ p_1(a) &= (1-p_0(a))(1-\gamma(a)) \\ p_n(a) &= p_{n-1}(a)\,\gamma(a) && \text{(A-23)} \end{aligned}$$

where

$$\gamma(a) = \frac{x\left(1-e^{-(\tau+\delta-x)a}\right)}{\tau+\delta-x\times e^{-(\tau+\delta-x)a}}. \tag{A-24}$$

Given that $\frac{1-\alpha}{\alpha}\tau = \delta+\eta$, the net rate of accumulation ψ is given by

$$\begin{aligned} \psi &\equiv x-\tau-\delta = x-\frac{\alpha}{1-\alpha}(\eta+\delta)-\delta && \text{(A-25)} \\ &= x-\frac{\alpha\eta+\delta}{1-\alpha}. \end{aligned}$$

Hence, ψ is decreasing in η. Also note that $\psi = \eta - z$.

To make the firm-size distribution stationary, we need that $\eta > x-\tau-\delta$. Using equation (A-25), this implies that $z > 0$, i.e. stationary requires the entry flow to be positive. From this solution for $p_n(a)$ we can calculate both the survival rate and the cross-sectional age distribution.

The survival function $S(a)$***.*** Let $S(a)$ denote share of firms that survive until age a. Then

$$S(a) = 1-p_0(a) = \frac{\psi e^{\psi a}}{\psi - x(1-e^{\psi a})}, \tag{A-26}$$

which is equation (19) in the main text. The average age is given by (again see Klette and Kortum (2004))

$$\mathbb{E}[\text{Age}] = \int_0^\infty (1-p_0(a))\,da = \frac{\ln\left(\frac{\tau+\delta}{\tau+\delta-x}\right)}{x} = \frac{\ln\left(\frac{\tau+\delta}{\tau+\delta-x}\right)}{x}$$

The expected number of products by age $\overline{n}(a)$***.*** To derive $\overline{n}(a)$ in (20), let $\overline{p}_n(a)$ denote the share of firms of age a with n production *conditional on survival. Then,*

$$\overline{p}_n(a) = \frac{p_n(a)}{1-p_0(a)} \text{ for } n \geq 1.$$

Using $p_n(a)$ in (A-22)-(A-23), this implies that

$$\overline{p}_n(a) = \gamma(a)^{n-1}(1-\gamma(a)). \tag{A-27}$$

Then,

$$\overline{n}(a) = E\left[N|\, A_f = a\right] \;=\; \sum_{n=1}^{\infty} n\overline{p}_n(a) = (1-\gamma(a)) \sum_{n=1}^{\infty} n\gamma(a)^{n-1} = \frac{1}{1-\gamma(a)}. \qquad \text{(A-28)}$$

Using (A-24), this implies $\overline{n}(a) = 1 - \frac{x}{\psi}\left(1 - e^{\psi a}\right)$, which is the expression in (20).

The pareto tail of the product distribution ϱ_n. To derive the tail of the product distribution, let $\omega_t(n)$ be the mass of firms with n products at time t. Consider $n \geq 2$. Then

$$\dot{\omega}_t(n) = \underbrace{\omega_t(n-1)(n-1)x}_{\text{From } n-1 \text{ to} n \text{ products}} + \underbrace{\omega_t(n+1)(n+1)(\tau+\delta)}_{\text{From } n+1 \text{ to} n \text{ products}} - \underbrace{\omega_t(n)n(\tau+x+\delta)}_{\text{From } n \text{ to } n-1 \text{ or } n+1 \text{ products}} .$$

For $n = 1$ we have

$$\dot{\omega}_t(1) = Z_t + \omega_t(2)\, 2(\tau+\delta) - \omega_t(1)(\tau+x+\delta).$$

Along the BGP the mass of firms grows at rate η. Intuitively: the distribution of firms across products is stationary and the number of products N_t is increasing at rate η. Hence, the mass of firms is increasing at rate η. Hence, along the BGP we have

$$\dot{\omega}_t(n) = \eta\omega_t(n).$$

Denote $\nu(n) = \frac{\omega_t(n)}{N_t}$ and $z = \frac{Z_t}{N_t}$. Along the BGP, $\{\nu(n)\}_{n=1}^{\infty}$ is determined by

$$\nu(2) = \frac{\nu(1)(\tau+x+\delta+\eta) - z}{2(\tau+\delta)} \qquad \text{(A-29)}$$

and

$$\nu(n+1) = \frac{\nu(n)n(\tau+x+\delta) + \nu(n)\eta - \nu(n-1)(n-1)x}{(n+1)(\tau+\delta)} \quad \text{for } n \geq 2 \qquad \text{(A-30)}$$

Given $\nu(1)$, these equations fully determine $[\nu(n)]_{n\geq 2}$ as a function of (x, z, τ). We can then pin down $\nu(1)$ from the consistency condition that

$$\sum_{n=1}^{\infty} \nu(n)n = \sum_{n=1}^{\infty} \frac{\omega_t(n)}{N_t} n = \frac{\sum_{n=1}^{\infty} \omega_t(n)n}{N_t} = 1. \qquad \text{(A-31)}$$

Hence, equations (A-29), (A-30) and (A-31) fully determine the firm-size distribution $[\nu(n)]_{n\geq 1}$.

In particular, the average number of products per firm are given by $\overline{n} = \frac{1}{\sum_{n=1}^{\infty} \nu_t(n)}$.

Importantly, the distribution described by (A-29), (A-30) and (A-31) has a pareto tail as long as

$$\eta > x - \tau - \delta > 0.$$

In particular, applying Proposition 3 in Luttmer (2011), the tail index of the product distribution is given by[24]

$$\varrho_n = \frac{\eta}{x - \tau - \delta}.$$

Using that $\tau = \frac{\alpha}{1-\alpha}(\eta + \delta)$ we get that

$$\varrho_n = \frac{(1-\alpha)\,\eta}{x\,(1-\alpha) - \delta - \alpha\eta} = \frac{\eta}{\eta - z},$$

where the second equality uses that $z = \frac{\eta+\delta}{1-\alpha} - x$. Also

$$\frac{\partial \varrho_n}{\partial \eta} = (1-\alpha)\,\frac{x\,(1-\alpha) - \delta}{(x\,(1-\alpha) - \delta - \alpha\eta)^2} > 0.$$

Note that the requirement that $x - \tau - \delta > 0$ ensures that $x\,(1-\alpha) - \delta > 0$.[25] Hence, a decline in population growth reduces the Pareto tail towards unity and increases concentration.

The pareto tail of the efficiency distribution ϱ_q. In this section we derive the marginal distribution of efficiency q. In particular we derive (22), which we use to calibrate $\overline{\omega}$.

Define $\hat{q}_t$ as the relative productivity of a product

$$\hat{q}_t \equiv \ln\left(q_t/Q_t\right)^{\sigma-1}. \tag{A-32}$$

The drift of $\hat{q}_t$ (conditional on survival) is given by

$$\frac{\partial \hat{q}_t}{\partial t} = (\sigma-1)\,I - (\sigma-1)\,d\ln Q_t = -\left(\frac{\alpha\left(\lambda^{\sigma-1}-1\right)}{1-\alpha} + \overline{\omega}^{\sigma-1} - 1\right)(\eta+\delta), \tag{A-33}$$

where the second equality uses (15).

Let $F_t(\hat{q})$ denote the share of products at time t with $\hat{q}_i \le \hat{q}$. This cdf evolves according to the

[24] To map the formulation of Luttmer (2011) to our model, note that he expresses the law of motion for the number of products as

$$DM_1 = \lambda 2 M_2 + \nu N - (\mu + \lambda)\,M_1$$

and

$$DM_n = \mu\,(n-1)\,M_{n-1} + \lambda\,(n+1)\,M_{n+1} - (\mu+\lambda)\,n M_n.$$

This is the same law of motion as ours once we chose $\nu = z$, $\mu = x$ and $\lambda = \tau + \delta$. He shows that the pareto tail is given by $\frac{\eta}{\mu-\lambda}$ or (using our notation) $\frac{\eta}{x-\tau-\delta}$.

[25] Using that $\tau = \frac{\alpha}{1-\alpha}(\eta+\delta)$, it follows that

$$x - \tau - \delta = \frac{1}{1-\alpha}\left(x\,(1-\alpha) - \alpha\eta - \delta\right).$$

Hence, $x - \tau - \delta > 0$ implies that $x\,(1-\alpha) - \delta > \alpha\eta > 0$.

differential equation

$$\frac{\partial F_t(\hat{q})}{\partial t} = -\underbrace{\frac{\partial F_t(\hat{q})}{\partial \hat{q}}\frac{\partial \hat{q}_t}{\partial t}}_{\text{Drift of } \hat{q}} + \underbrace{\tau\left(F_t(\hat{q}-\hat{\lambda}) - F_t(\hat{q})\right)}_{\text{Creative destruction}} \underbrace{- (\delta+\eta)\left(F_t(\hat{q}) - \Gamma\left(\exp\left(\frac{\hat{q}}{\sigma-1}\right)\right)\right)}_{\text{Product loss vs new product creation}},$$

where $\hat{\lambda} = \ln \lambda^{\sigma-1}$. In the steady state, $\frac{\partial F_t(\hat{q})}{\partial t} = 0$ so that

$$\frac{dF(\hat{q})}{dq}\frac{\partial \hat{q}_t}{\partial t} = \tau\left(F_t(\hat{q}-\hat{\lambda}) - F_t(\hat{q})\right) - (\delta+\eta)\left(F_t(\hat{q}) - \Gamma\left(\exp\left(\frac{\hat{q}}{\sigma-1}\right)\right)\right). \tag{A-34}$$

Guess that F is exponential in the tail with index ϱ_q, that is

$$\underset{\hat{q}\to\infty}{lim}\, e^{\varrho_q \hat{q}}(1 - F(\hat{q})) = a$$

for some a and ϱ_q. If we assume that Γ has a thin tail[26] then as $\hat{q} \to \infty$, (A-34) implies that

$$\lim_{\hat{q}\to\infty}\left(ae^{-\varrho_q\hat{q}}\varrho_q\frac{\partial \hat{q}_t}{\partial t}\right) = \lim_{\hat{q}\to\infty}\left[(\delta+\eta+\tau) - \tau e^{\varrho_q\hat{\lambda}}\right]ae^{-\varrho_q\hat{q}} - (\delta+\eta).$$

Hence, the tail coefficient ϱ_q solves the equation

$$-\varrho_q\frac{\partial \hat{q}_t}{\partial t} = -(\delta+\eta+\tau) + \tau e^{\varrho_q\hat{\lambda}}.$$

Substituting for (A-33) and noting that $\tau = \frac{\alpha}{1-\alpha}(\eta+\delta)$ yields

$$\varrho_q\left(\lambda^{\sigma-1} + \frac{1-\alpha}{\alpha}\overline{\omega}^{\sigma-1} - \frac{1}{\alpha}\right) = -\frac{1}{\alpha} + e^{\varrho_q\hat{\lambda}}. \tag{A-35}$$

Using that $\hat{\lambda} = \ln \lambda^{\sigma-1}$, (A-35) yields

$$\varrho_q\left(\alpha\lambda^{\sigma-1} + (1-\alpha)\overline{\omega}^{\sigma-1} - 1\right) = -1 + \alpha\lambda^{\varrho_q(\sigma-1)}.$$

This is equation (22) in the main text. For the special case where creative destruction does not lead to any productivity advancements, i.e. $\lambda = 1$, the tail coefficient is given by

$$\varrho_q = \frac{1}{1-\overline{\omega}^{\sigma-1}}.$$

[26]Formally, assume that for any κ, we have $\underset{\hat{q}\to\infty}{lim}\, e^{\kappa\hat{q}}(1 - \Gamma\left(\exp\left(\frac{\hat{q}}{\sigma-1}\right)\right)) = 0$.

A-1.2 Diminishing returns to research in entry and product creation

Now we suppose that there are diminishing returns to research labor as the general state of technology improves, as suggested by the evidence in Bloom et al. (2020). In particular, suppose that the entry cost in units of labor is $\frac{1}{\varphi_E} Q_t^{\varsigma}$ with $\varsigma > 0$. Moreover, suppose that the innovation cost function for expanding the portfolio of products the firm has is

$$c_t^X(x, n) = \frac{1}{\varphi_x} x^{\zeta} n Q_t^{\varsigma}$$

In this case, the HJB for a single product can still be written

$$r_t V_t(q) - \dot{V}_t(q) = \pi_t(q) + I \frac{\partial V_t(q)}{\partial q} q - \tau V_t(q) + \Xi_t,$$

where

$$\Xi_t = \max_x \left\{ x \left(\alpha V_t^{CD} + (1-\alpha) V_t^{NV} \right) - \frac{1}{\varphi_x} x^{\zeta} w_t Q_t^{\varsigma} \right\}$$

And the solution is given by

$$\begin{aligned} V_t(q) &= q^{\sigma-1} U_t + M_t \\ U_t &= \frac{(\mu - 1) \left(\frac{1}{Q_t} \right)^{\sigma-1} \frac{L_t^P}{N_t} w_t}{(\rho + \tau + (\sigma-1)(g_Q - I) - g_{l^P})} \\ M_t &= \frac{(\zeta - 1) x Q_t^{\varsigma} w_t}{(\rho + \tau - \varsigma g_Q)} \end{aligned}$$

Free entry then requires that

$$\frac{1}{\varphi_E} w_t Q_t^{\varsigma} = \bar{q} \frac{(\mu - 1) \frac{L_t^P}{N_t} w_t}{(\rho + \tau + (\sigma-1)(g_Q - I) - g_{l^P})} + \frac{(\zeta - 1) x Q_t^{\varsigma} w_t}{(\rho + \tau - \varsigma g_Q)}$$

So that

$$\frac{1}{\varphi_E} = \frac{(\mu - 1) \frac{L_t^P}{N_t} Q_t^{-\varsigma}}{(\rho + \tau + (\sigma-1)(g_Q - I) - g_{l^P})} + \frac{(\zeta - 1) x}{(\rho + \tau - \varsigma g_Q)} \tag{A-36}$$

For this to hold on a BGP, it must be that

$$g_{l^P} = \varsigma g_Q$$

So that the number of workers per product is rising at a constant rate on the BGP.

$$\eta = g_N + \varsigma g_Q$$

In this model specification it is still true that

$$g_t^Q = \frac{\dot{Q}_t}{Q_t} = I + \frac{\lambda^{\sigma-1} - 1}{\sigma - 1}\tau_t + \frac{\overline{\omega}^{\sigma-1} - 1}{\sigma - 1} g_t^N$$

$$g_t^N = \frac{\dot{N}_t}{N_t} = (1-\alpha)(x_t + z_t) = \frac{1-\alpha}{\alpha}\tau_t$$

so we can find that the rate of creative destruction is

$$\tau = \frac{\eta - \varsigma I}{\frac{1-\alpha}{\alpha} + \varsigma\left(\frac{\lambda^{\sigma-1}-1}{\sigma-1} + \frac{\overline{\omega}^{\sigma-1}-1}{\sigma-1}\frac{1-\alpha}{\alpha}\right)}$$

To fully characterize the equilibrium, the research share of the economy again comes from the labor market clearing condition, which now reads

$$L_t = L_t^P + L_t^R = L_t^P + Q_t^{\varsigma} N_t \left(\frac{1}{\varphi_E} z_t + \frac{1}{\varphi_x} x^{\zeta}\right).$$

So that

$$\frac{1}{\ell_t^P} = 1 + \frac{Q_t^{\varsigma} N_t}{L_t^P}\left(\frac{1}{\varphi_E} z + \frac{1}{\varphi_x} x^{\zeta}\right)$$

Given that the ratio $\frac{Q_t^{\varsigma} N_t}{L_t^P}$ is fully determined from free entry and constant on the BGP (see (A-36)), and z and x are constant on the BGP, the share of production labor ℓ_t^P is constant. The main difference between the baseline model and this modification is that average firm size should be increasing on the BGP at a constant rate. Higher profits per firm are needed to offset the constantly increasing entry cost.

A-1.3 Model Extensions (Section 2.6)

A-1.3.1 Endogenizing the Direction of Innovation α.

In the baseline model in we assume that innovation was undirected, i.e. the share of product innovation resulting in creative destruction (rather than new varieties) was constant and equal to α. In this section we show that we can extend our theory to a setting where the direction of innovation is a choice variable of the firm.

Incumbent Innovation and the Value Function Suppose that the firm can chose the flow of new varieties x_N and creative destruction x_{CD}. The value function is then given by

$$r_t V_t(q) - \dot{V}_t(q) = \pi_t(q) + I\frac{\partial V_t(q)}{\partial q} q - \tau_t V_t(q) + \Xi_t$$

where

$$\Xi_t \equiv \max_{x_N} \left\{ x_N V_t^N - \frac{1}{\varphi_N} x_N^{\zeta} w_t \right\} + \max_{x_{CD}} \left\{ x_{CD} V_t^{CD} - \frac{1}{\varphi_{CD}} x_{CD}^{\zeta} w_t \right\}, \tag{A-37}$$

where φ_{CD} and φ_N parametrize the efficiency of creative destruction and new variety creation and V_t^N and V_t^N denote the value of creative destruction and new variety creation respectively. Along the BGP, the solution of $V_t(q)$ is given by

$$V_t(q) = \frac{(\mu - 1)}{\rho + (g_N - \eta) + (g_Q - I)(\sigma - 1) + \tau} \left(\frac{q}{Q_t} \right)^{\sigma - 1} \frac{L_t^P}{N_t} w_t + \frac{\Xi_t}{r + \tau - g_{\Xi_t}}.$$

Optimal Innovation and the Value of Innovation The optimal innovation rates associated with (A-37) are given by

$$x_{NV} = \left(\frac{\varphi_N}{\zeta} \frac{V_t^{NV}}{w_t} \right)^{\frac{1}{\zeta - 1}} \text{ and } x_{CD} = \left(\frac{\varphi_{CD}}{\zeta} \frac{V_t^{CD}}{w_t} \right)^{\frac{1}{\zeta - 1}}. \tag{A-38}$$

Note that this implies that the endogenous share of product creation directed to creative destruction is given by

$$\tilde{\alpha} = \frac{\left(\varphi_{CD} \frac{V_t^{CD}}{w_t} \right)^{\frac{1}{\zeta - 1}}}{\left(\varphi_N \frac{V_t^N}{w_t} \right)^{\frac{1}{\zeta - 1}} + \left(\varphi_{CD} \frac{V_t^{CD}}{w_t} \right)^{\frac{1}{\zeta - 1}}},$$

i.e. the relative "bias" of innovation depends on the relative valuations. This also implies that

$$\Xi_t = \left(\frac{\zeta - 1}{\varphi_{NV}} x_{NV}^{\zeta} + \frac{\zeta - 1}{\varphi_{CD}} x_{CD}^{\zeta} \right) w_t, \tag{A-39}$$

where x_{NV} and x_{CD} are constant (see below). Hence, the value of product creation grows at rate w_t, i.e.

$$g_{\Xi_t} = g_w = r - \rho.$$

Similarly, along the BGP we have $g_N = \eta$. Hence,

$$V_t(q) = \frac{(\mu - 1)}{\rho + (g_Q - I)(\sigma - 1) + \tau} \left(\frac{q}{Q_t} \right)^{\sigma - 1} \frac{L_t^P}{N_t} w_t + \frac{\Xi_t}{\rho + \tau},$$

where Ξ_t is given in (A-39).

To solve for Ξ_t and x_{NV} and x_{CD}, we need V_t^N and V_t^{CD}. As before these are given by

$$\begin{aligned} V_t^{CD} = \int V(\lambda q)\, dF_t(q) &= \frac{(\mu - 1)\lambda^{\sigma - 1}}{\rho + (g_Q - I)(\sigma - 1) + \tau} \frac{L_t^P}{N_t} w_t + \frac{\Xi_t}{\rho + \tau} \\ &= \left(\frac{(\mu - 1)\lambda^{\sigma - 1}}{\rho + (g_Q - I)(\sigma - 1) + \tau} \frac{L_t^P}{N_t} + \frac{\frac{\zeta - 1}{\varphi_{NV}} x_{NV}^{\zeta} + \frac{\zeta - 1}{\varphi_{CD}} x_{CD}^{\zeta}}{\rho + \tau} \right) w_t. \end{aligned}$$

Similarly, the value of new variety creation is given by

$$V_t^{NV} = V\left(\omega Q_t\right) \quad = \quad \left(\frac{(\mu-1)\,\omega^{\sigma-1}}{\rho+(g_Q-I)\,(\sigma-1)+\tau}\frac{L_t^P}{N_t}+\frac{\frac{\zeta-1}{\varphi_{NV}}x_{NV}^{\zeta}+\frac{\zeta-1}{\varphi_{CD}}x_{CD}^{\zeta}}{\rho+\tau}\right)w_t \tag{A-40}$$

Entry We assume the following process of entry. As in the baseline model, the economy has access to a linear entry technology whereby each worker generates a flow of φ_E new firms. These firms then have access to the same innovation technology as incumbents to eventually start producing either a creatively destroyed product or a new variety. In the event that no product is discovered, the potential firm exits.

Because new firms have - after paying the entry costs $\frac{1}{\varphi_E}w_t$ - the same opportunity as incumbents, their direction of innovation (i.e. new varieties versus creative destruction) is exactly the same as the one of incumbent firms. Hence, if z new firms are created (per product N_t), the total amount of creative destruction and new variety creation by entrants is given by zx_{CD} and zx_{NV} respectively. It also implies that the free entry condition is given by

$$\frac{1}{\varphi_E}w_t = \Xi_t,$$

where Ξ_t is the value of innovation given in (A-39). Note that the value of entry is only the flow value of innovation Ξ_t, not the present discounted value.

BGP equilibrium The BGP equilibrium in this economy is fully characterized by innovation choices x_{NV} and x_{CD}, the entry flow z, value functions V^{NV}/w_t and V^{CD}/w_t, the rate of creative destruction τ and the mass of production labor per product L_t^P/N_t. These objects are determined from the following conditions:

1. Because $g_N = \eta$ along the BGP,

$$x_{NV} + zx_{NV} = x_{NV}\,(1+z) = \eta. \tag{A-41}$$

2. Creative destruction τ is given by

$$\tau = x_{CD} + zx_{CD} = x_{CD}\,(1+z) \tag{A-42}$$

3. The first order condition for x_{NV} and x_{CD} are given by (see (A-38))

$$x_N = \left(\frac{\varphi_N}{\zeta}\frac{V_t^N}{w_t}\right)^{\frac{1}{\zeta-1}} \text{ and } x_{CD} = \left(\frac{\varphi_{CD}}{\zeta}\frac{V_t^{CD}}{w_t}\right)^{\frac{1}{\zeta-1}}. \tag{A-43}$$

4. The free entry condition is (see (A-39))

$$\frac{1}{\varphi_E} = \frac{\Xi_t}{w_t} = \frac{\zeta - 1}{\varphi_{NV}} x_{NV}^{\zeta} + \frac{\zeta - 1}{\varphi_{CD}} x_{CD}^{\zeta}.$$

5. To solve for the value functions V^{NV}/w_t and V^{CD}/w_t note that

$$\begin{aligned}(g_Q - I)(\sigma - 1) + \tau &= \left(\lambda^{\sigma-1} - 1\right)\tau + \left(\omega^{\sigma-1} - 1\right) g_N + \tau \\ &= \lambda^{\sigma-1}\tau + \left(\omega^{\sigma-1} - 1\right)\eta.\end{aligned}$$

   Hence, V^{NV}/w_t and V^{CD}/w_t are given by

$$\frac{V_t^{NV}}{w_t} = \frac{(\mu - 1)\,\omega^{\sigma-1}}{\lambda^{\sigma-1}\tau + (\omega^{\sigma-1} - 1)\,\eta} \frac{L_t^P}{N_t} + \frac{1}{\rho + \tau} \frac{1}{\varphi_E} \tag{A-44}$$

   and

$$\frac{V_t^{CD}}{w_t} = \frac{(\mu - 1)\,\lambda^{\sigma-1}}{\lambda^{\sigma-1}\tau + (\omega^{\sigma-1} - 1)\,\eta} \frac{L_t^P}{N_t} + \frac{1}{\rho + \tau} \frac{1}{\varphi_E}. \tag{A-45}$$

These are 7 equations in 7 unknowns $(z, x_{NV}, x_{CD}, \frac{V_t^{CD}}{w_t}, \frac{V_t^{NV}}{w_t}, \tau, \frac{L_t^P}{N_t})$, which fully determine the BGP equilibrium.

We can simplify this system further and express the BGP equilibrium in terms of x_{NV} and x_{CD}. Using (A-41) and (A-42) we get that

$$\tau = \frac{x_{CD}}{x_{NV}}\eta.$$

From (A-43) we get that

$$x_N^{\zeta} \frac{\zeta - 1}{\varphi_N} = \frac{\zeta - 1}{\zeta} \frac{V_t^N}{w_t} x_N \quad \text{and} \quad x_{CD}^{\zeta} \frac{\zeta - 1}{\varphi_N} = \frac{\zeta - 1}{\zeta} \frac{V_t^{CD}}{w_t} x_{CD}. \tag{A-46}$$

Free entry therefore requires that

$$\frac{1}{\varphi_E} = \frac{\zeta - 1}{\zeta} \left(\frac{V_t^{CD}}{w_t} x_N + \frac{V_t^{NV}}{w_t} x_{CD} \right).$$

Using the expressions for $\frac{V_t^{CD}}{w_t}$ and $\frac{V_t^{NV}}{w_t}$ in (A-44) and (A-45), we can solve for $\frac{L_t^P}{N_t}$ as

$$\frac{(\mu - 1)}{\lambda^{\sigma-1} \frac{x_{CD}}{x_{NV}}\eta + (\omega^{\sigma-1} - 1)\,\eta} \frac{L_t^P}{N_t} = \frac{1}{\varphi_E} \left(\frac{\frac{\zeta}{\zeta-1} - \frac{1}{\rho + \frac{x_{CD}}{x_{NV}}\eta} (x_N + x_{CD})}{\lambda^{\sigma-1} x_{CD} + \omega^{\sigma-1} x_{NV}} \right). \tag{A-47}$$

This implies that $\frac{V_t^{CD}}{w_t}$ and $\frac{V_t^{NV}}{w_t}$ are given by

$$\begin{aligned}\frac{V_t^{CD}}{w_t} &= \frac{1}{\varphi_E}\left(\frac{\frac{\zeta}{\zeta-1}\lambda^{\sigma-1}}{\lambda^{\sigma-1}x_{CD}+\omega^{\sigma-1}x_{NV}} - \frac{1}{\rho+\frac{x_{CD}}{x_{NV}}\eta}\left(\frac{\left(\lambda^{\sigma-1}-\omega^{\sigma-1}\right)x_{NV}}{\lambda^{\sigma-1}x_{CD}+\omega^{\sigma-1}x_{NV}}\right)\right)\\ \frac{V_t^{NV}}{w_t} &= \frac{1}{\varphi_E}\left(\frac{\frac{\zeta}{\zeta-1}\omega^{\sigma-1}}{\lambda^{\sigma-1}x_{CD}+\omega^{\sigma-1}x_{NV}} - \frac{1}{\rho+\frac{x_{CD}}{x_{NV}}\eta}\left(\frac{\left(\omega^{\sigma-1}-\lambda^{\sigma-1}\right)x_{CD}}{\lambda^{\sigma-1}x_{CD}+\omega^{\sigma-1}x_{NV}}\right)\right).\end{aligned}$$

Hence, (A-46) implies that x_{NV} and x_{CD} are determined from the equations

$$x_N^{\zeta-1}\frac{\zeta\varphi_E}{\varphi_N} = \frac{\frac{\zeta}{\zeta-1}\omega^{\sigma-1}}{\lambda^{\sigma-1}x_{CD}+\omega^{\sigma-1}x_{NV}} - \frac{1}{\rho+\frac{x_{CD}}{x_{NV}}\eta}\frac{\left(\omega^{\sigma-1}-\lambda^{\sigma-1}\right)x_{CD}}{\lambda^{\sigma-1}x_{CD}+\omega^{\sigma-1}x_{NV}} \tag{A-48}$$

$$x_{CD}^{\zeta-1}\frac{\zeta\varphi_E}{\varphi_{CD}} = \frac{\frac{\zeta}{\zeta-1}\lambda^{\sigma-1}}{\lambda^{\sigma-1}x_{CD}+\omega^{\sigma-1}x_{NV}} - \frac{1}{\rho+\frac{x_{CD}}{x_{NV}}\eta}\frac{\left(\lambda^{\sigma-1}-\omega^{\sigma-1}\right)x_{NV}}{\lambda^{\sigma-1}x_{CD}+\omega^{\sigma-1}x_{NV}}. \tag{A-49}$$

Using that $\tau=\frac{x_{CD}}{x_{NV}}\eta$ we write (A-48) and (A-49) in terms of x_{NV} and τ. With some algebra we can solve for x_{NV} explicitly as a function of τ and parameters. In particular, one can show that

$$x_N = \left(\frac{\varphi_N}{(\zeta-1)\,\varphi_E}\right)^{1/\zeta}\left(1+\left(\frac{\tau}{\eta}\right)^{\zeta}\frac{\varphi_N}{\varphi_{CD}}\right)^{-1/\zeta}.$$

Substituting this expression for x_{NV} into (A-48), we arrive at the equation

$$\left(\left(\frac{\lambda}{\overline{\omega}}\right)^{\sigma-1}-1\right)\frac{1}{\eta}\frac{\tau}{\rho+\tau} = \frac{\zeta}{(\zeta-1)^{\frac{\zeta-1}{\zeta}}}\left(\frac{\varphi_E}{\varphi_N}\right)^{\frac{1}{\zeta}}\frac{\left(\frac{\lambda}{\overline{\omega}}\right)^{\sigma-1}\frac{\tau}{\eta}-\left(\frac{\tau}{\eta}\right)^{\zeta}\frac{\varphi_N}{\varphi_{CD}}}{\left(1+\left(\frac{\tau}{\eta}\right)^{\zeta}\frac{\varphi_N}{\varphi_{CD}}\right)^{\frac{\zeta-1}{\zeta}}}. \tag{A-50}$$

This equation determines τ as a function of parameters. In particular, τ depends directly on η.

A Special Case Consider first a special case of this setup which is exactly isomorphic to our baseline model where the direction of innovation α is exogenous. Assume that creative destruction and new variety creation leads to the same quality improvement, i.e.

$$\lambda = \overline{\omega}.$$

This implies that the value of creative destruction and new variety creation is equalized, i.e. $V_t^{CD} = V_t^{NV} = V_t$. This in turn directly yields

$$\alpha = \frac{x_{CD}}{x_{CD}+x_{NV}} = \frac{\left(\frac{\varphi_{CD}}{\zeta}\frac{V_t}{w_t}\right)^{\frac{1}{\zeta-1}}}{\left(\frac{\varphi_{CD}}{\zeta}\frac{V_t}{w_t}\right)^{\frac{1}{\zeta-1}}+\left(\frac{\varphi_{NV}}{\zeta}\frac{V_t}{w_t}\right)^{\frac{1}{\zeta-1}}} = \frac{(\varphi_{CD})^{\frac{1}{\zeta-1}}}{(\varphi_{CD})^{\frac{1}{\zeta-1}}+(\varphi_{NV})^{\frac{1}{\zeta-1}}}.$$

Hence, the share of innovation activity directed to creative destruction, α, is indeed endogenous and simply determined from the relative innovation efficiencies φ_{CD} and φ_{NV}. Hence, we can write $x_{NV} = (1-\alpha)\,x$ and $x_{CD} = \alpha x$ as in the baseline model. Along the BGP we still have

$$x_{NV}\,(1+z) = (1-\alpha)\,x\,(1+z) = \eta$$

Similarly, creative destruction is given by

$$\tau = \alpha x\,(1+z)\,.$$

Hence, as in the baseline model,

$$\tau = \frac{\alpha}{1-\alpha}\eta,$$

i.e. lower population growth reduces creative destruction.[27]

To solve for the level of x, note that free entry requires that

$$\begin{aligned} \frac{1}{\varphi_E} &= \frac{\Xi_t}{w_t} = \frac{\zeta-1}{\varphi_{NV}}x_{NV}^{\zeta} + \frac{\zeta-1}{\varphi_{CD}}x_{CD}^{\zeta} \\ &= (\zeta-1)\left(\frac{(1-\alpha)^{\zeta}}{\varphi_{NV}} + \frac{\alpha^{\zeta}}{\varphi_{CD}}\right)x^{\zeta}. \end{aligned}$$

Hence,

$$x = \left(\frac{1}{\zeta-1}\frac{1}{\varphi_E}\frac{1}{\left(\frac{(1-\alpha)^{\zeta}}{\varphi_{NV}} + \frac{\alpha^{\zeta}}{\varphi_{CD}}\right)}\right)^{1/\zeta}.$$

As in the baseline model, x is constant and fully determined from parameters governing the relative innovation technologies. And with x constant, we have

$$zx = \frac{\eta}{1-\alpha} - x,$$

i.e. the total entry flow per product, zx, is a decreasing function of population growth: in equilibrium entrants bear the brunt of declining population growth.

Finally, we can use (A-47) to solve for the level of market size $\frac{L_t^P}{N_t}$. For the case where $\lambda = \omega$, (A-47) implies that

$$\frac{L_t^P}{N_t} = \frac{1}{\varphi_E}\frac{\lambda^{\sigma-1}\frac{\alpha}{1-\alpha} + \left(\omega^{\sigma-1}-1\right)}{\mu-1}\left(\frac{\frac{\zeta}{\zeta-1}}{\left(\lambda^{\sigma-1}\alpha + \omega^{\sigma-1}\,(1-\alpha)\right)x} - \frac{1}{\rho + \frac{\alpha}{1-\alpha}\eta}\right)\eta.$$

[27]Note that this solution is also implied by (A-50). If $\lambda = \overline{\omega}$, (A-50) requires that $\frac{\tau}{\eta} - \left(\frac{\tau}{\eta}\right)^{\zeta}\frac{\varphi_N}{\varphi_{CD}} = 0$. This implies that $\tau = \left(\frac{\varphi_{CD}}{\varphi_N}\right)^{\frac{1}{\zeta-1}}\eta = \frac{\alpha}{1-\alpha}\eta$.

Hence, $\frac{L_t^P}{N_t}$ is increasing in η. Conversely, a fall in population growth reduces $\frac{L_t^P}{N_t}$ and increases the number of varieties per worker.

The General Case If $\lambda \neq \omega$, τ is determined from (A-50). We can rewrite (A-50) as

$$\left(\frac{\kappa-1}{\kappa}\right)\mathcal{A}\frac{1}{\rho+\tau}\left(1+\left(\frac{\tau}{\eta}\right)^{\zeta}\frac{\varphi_N}{\varphi_{CD}}\right)^{\frac{\zeta-1}{\zeta}}+\frac{1}{\kappa}\left(\frac{\tau}{\eta}\right)^{\zeta-1}\frac{\varphi_N}{\varphi_{CD}}=1,$$

where $\kappa=\left(\frac{\lambda}{\overline{\omega}}\right)^{\sigma-1}$ and $\mathcal{A}=\frac{(\zeta-1)^{\frac{\zeta-1}{\zeta}}}{\zeta\left(\frac{\varphi_E}{\varphi_N}\right)^{\frac{1}{\zeta}}}$. Define the function

$$h(\tau)=\left(\frac{\kappa-1}{\kappa}\right)\mathcal{A}\frac{1}{\rho+\tau}\left(1+\tau^{\zeta}\frac{1}{\eta^{\zeta}}\frac{\varphi_N}{\varphi_{CD}}\right)^{\frac{\zeta-1}{\zeta}}+\frac{1}{\kappa}\tau^{\zeta-1}\frac{1}{\eta^{\zeta-1}}\frac{\varphi_N}{\varphi_{CD}}.$$

Then, a solution τ^* is implicitly defined by $h(\tau^*)=1$. Note that $h(.)$ satisfies

$$h(0)=0 \quad \text{and} \quad \lim_{m\to\infty} h(m)=\infty.$$

Note also that h is continuous so that there is at least one solution $h(\tau^*)=1$. Moreover, at least one solution satisfies $h'(\tau^*)>0$. If there is a unique solution then $h'(\tau^*)>0$. Hence, focus on a solution where $h'(\tau^*)>0$. Note that an increase in η shifts the function $h(\tau)$ downwards. Hence, an increase in η will increase τ^*. As in our baseline model, falling population growth reduces creative destruction τ.

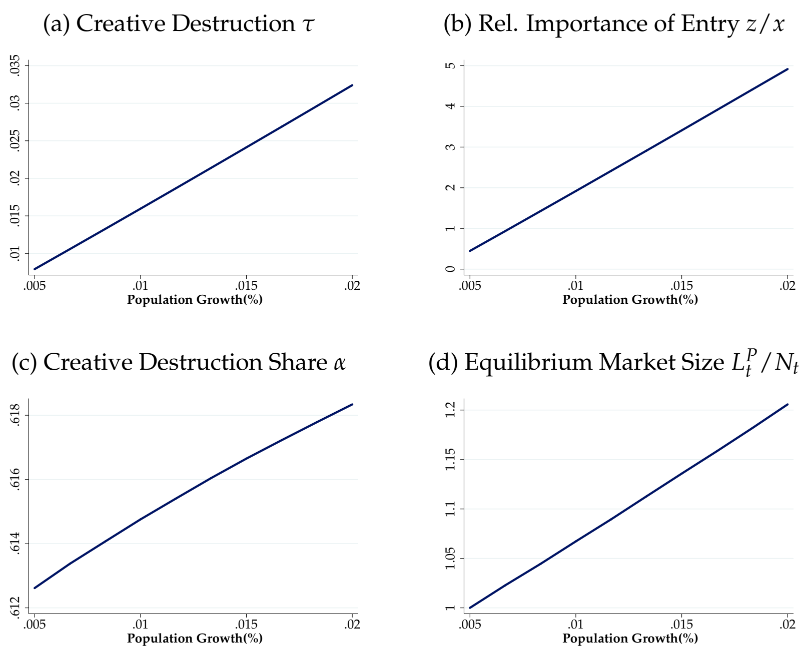
In Figure A-1 we show creative destruction τ, the relative importance of entry $z=\frac{z_{NV}}{x_{NV}}=\frac{z_{CD}}{x_{CD}}$, the share of innovation directed towards creative destruction $\alpha=\frac{x_{CD}}{x_{CD}+x_{NV}}$ and the size of the market L_t^P/N_t as a function of population growth.

The comparative static results shown in Figure A-1 accord well with our baseline model. First, shown on the upper left, creative destruction is an increasing function of population growth. Second, shown in the upper right, falling population reduces z, the flow of entrant product innovation relative to incumbents. Third, shown in the lower left, the creative destruction share α is increasing in η. Hence, falling population growth reduces creative destruction more than the creation of new varieties. Finally, shown in the lower right, falling population growth reduces the market size and flows profits through a decline in L_t^P/N_t.

The result that α is increasing in η is driven by our estimates that $\overline{\omega}<\lambda$. To see why, consider the special case where new entrants are extremely unproductive, i.e. $\omega\approx 0$. In that case, equations (A-48) and (A-49) can be solved analytically as

$$x_{NV}=\left(\frac{\varphi_N}{\zeta\varphi_E}\frac{1}{\rho+\frac{x_{CD}}{x_{NV}}\eta}\right)^{\frac{1}{\zeta-1}} \quad \text{and} \quad x_{CD}^{\zeta}=\frac{\varphi_{CD}}{\varphi_E}\frac{1}{\zeta-1}-\frac{\varphi_{CD}}{\varphi_N}x_{NV}^{\zeta}.$$

Figure A-1: The Effects of Population Growth (Endogenous α)



Note: This figure plots model outcomes in a calibrated version of the extended model with endogenous innovation direction as a function of the rate of population growth.

Hence, x_{NV} is decreasing in η and x_{CD} is decreasing in x_{NV} but does not depend on η conditional on x_{NV}. Falling population growth therefore increases x_{NV} but decreases x_{CD}. Hence, falling population growth reduces α as shown in Figure A-1. To see why, define the function

$$v(m) = \frac{(\mu - 1)\, m^{\sigma-1}}{\lambda^{\sigma-1}\tau + (\omega^{\sigma-1} - 1)\,\eta} \frac{L_t^P}{N_t} + \frac{1}{\rho+\tau}\frac{1}{\varphi_E}.$$

Then $V_t^{NV}/w_t = v(\overline{\omega})$ and $V_t^{CD}/w_t = v(\lambda)$. Now recall that a decline in population growth reduces both τ and L_t^P/N_t. The reduction in τ increases $v(m)$ due to a decline in the discount rate. The reduction in L_t^P/N_t obviously lowers $v(m)$. The initial quality term m thus governs the weight on the present discount value of profits (the first term) as opposed to the value of innovation (the second term). If m is small, $v(m)$ is not negatively affected by the decline in L_t^P/N_t but mostly benefits from the *increase* in the innovation value. A decline in population growth therefore raises the value of new variety creation relative to the value of creative destruction, leading to a decline in the creative destruction share α.

A-1.3.2 Endogenous own-innovation I

Suppose now that firms can chose the rate of own-innovation I subject to some costs. In particular, assume that the cost function (in terms of labor) of achieving a drift I of a particular product is given by

$$c\left(I;q/Q\right)=\left(\frac{q}{Q_t}\right)^{\sigma-1}\frac{1}{\varphi_I}I^{\zeta}.$$

Hence, the cost of innovation are convex in I (for simplicity we assume the same convexity as for firms' product creation technology). Additionally, the cost of innovation depend on firms' relative efficiency q/Q_t. Allowing for this cost-shifter is required to make the model consistent with balanced growth (see e.g. Atkeson and Burstein (2010)) and Gibrat's law for large firms.

Most results of the baseline model generalize in a straightforward way. In particular, Proposition 2 is exactly the same in this more general framework, except I in the expression for the growth rate is no longer a parameter but a choice variable. The characterization of the value function contained in Proposition 1 is also strikingly similar. The value function is still additive across products and the value of a given product with efficiency q is given by

$$V_t\left(q\right)=\frac{\pi_t\left(q\right)}{\rho+\tau+\left(g^{Q}-\frac{\zeta-1}{\zeta}I\right)\left(\sigma-1\right)}+\frac{\frac{\zeta-1}{\varphi_x}x^{\zeta}w_t}{\rho+\tau}. \tag{A-51}$$

Hence, the only difference to the baseline model is the term $\frac{\zeta-1}{\zeta}$ in front of I in the discount rate. Given $V_t^I\left(q\right)$ the optimal rate of own-innovation is therefore defined by

$$\max_I\left\{I\frac{\partial V_t\left(q\right)}{\partial q}q-\left(\frac{q}{Q_t}\right)^{\sigma-1}\frac{1}{\varphi_I}I^{\zeta}w_t\right\}. \tag{A-52}$$

Using (A-51), the optimal innovation rate associated with (A-52) is given by

$$I=\left(\frac{\left(\sigma-1\right)\left(\mu-1\right)\ell}{\rho+\tau+\left(g^{Q}-\frac{\zeta-1}{\zeta}I\right)\left(\sigma-1\right)}\frac{\varphi_I}{\zeta}\right)^{\frac{1}{\zeta-1}}, \tag{A-53}$$

where again $\ell=L_t^P/N_t$. Hence, the optimally chosen drift is indeed independent of the efficiency q and constant in a BGP. Importantly, because ℓ, τ and g_Q depend on the rate of population growth η, I also changes when population growth declines.

To see how I depends on the rate of population growth η, note that the free entry condition now implies

$$\frac{1}{\varphi_E}=\frac{\left(\mu-1\right)\left(\alpha\lambda^{\sigma-1}+\left(1-\alpha\right)\overline{\omega}^{\sigma-1}\right)\ell}{\rho+\tau+\left(g^{Q}-\frac{\zeta-1}{\zeta}I\right)\left(\sigma-1\right)}+\frac{\frac{\zeta-1}{\varphi_x}\left(\frac{1}{\zeta}\frac{\varphi_x}{\varphi_E}\right)^{\frac{\sigma}{\zeta-1}}}{\rho+\tau}.$$

Hence,

$$I = \varsigma \left(1 - \frac{\left(\frac{\zeta-1}{\zeta}\right)\left(\frac{1}{\zeta}\frac{\varphi_x}{\varphi_E}\right)^{\frac{1}{\zeta-1}}}{\rho+\tau}\right)^{\frac{1}{\zeta-1}}, \tag{A-54}$$

where ς is a collection of structural parameters.[28] Importantly, this expresses the optimal rate of own-innovation I directly as a function of parameters and a single endogenous variable - the rate of creative destruction. In particular, I only depends on the rate of population growth through τ. And because I is increasing the rate of creative destruction, a decline in population growth *reduces* firms' own innovation incentives.

The fact that I is increasing in the rate of creative destruction might at first seem surprising. After all, a higher rate of creative destruction reduces the expected life-span, which should reduce firms' incentives to invest in productivity improvements. To see that this intuition is correct, consider the (A-53): holding market size ℓ and the rate of efficiency growth g_Q constant, an increase in τ indeed *lowers I.* However, once one realizes that all these objects are linked through the free entry condition, the general equilibrium effect of a higher rate of creative destruction becomes positive. Economically: free entry requires the average production value *plus* the innovation value to be equal to the entry costs. A higher rate of creative destruction *lowers* the innovation value. Hence, for the free entry condition to be satisfied, the production value has to *increase.* This increase is achieved through an increase in market size ℓ. And as the returns to own-innovation scale with the production value but not the innovation value, the returns to own-innovation are higher in an environment with higher creative destruction. Conversely, lower population growth lowers own-innovation. This endogenous response of incumbents' own-innovation efforts amplifies the negative consequences of population growth.

A-1.4 Characterization of the Model with Bertrand Competition (Section 3)

In this section we derive the results for the model with Bertrand competition described in Section 3

A-1.4.1 The Value Function

The only difference relative to the baseline case characterized in Section A-1.1.3 is that the static profit function is given by (A-5), i.e.

$$\pi\left(q_i, \Delta_i\right) = \left(1 - \frac{1}{\mu\left(\Delta_i\right)}\right)\mu\left(\Delta_i\right)^{1-\sigma}\left(\frac{q_i}{Q_t}\right)^{\sigma-1}\frac{1}{\left(\mathcal{M}_t\Lambda_t\right)^{\sigma-1}}\frac{Y_t}{N_t}. \tag{A-55}$$

The value function is still additive across products, i.e. $V_t\left(\left[q_i, \Delta_i\right]\right) = \sum_{i=1}^{n} V_t\left(q_i, \Delta_i\right)$. The HJB equation for $V_t\left(q_i, \Delta_i\right)$ is given by

[28] In particular, $\varsigma = \left(\frac{\sigma-1}{\alpha\lambda^{\sigma-1}+(1-\alpha)\omega^{\sigma-1}}\frac{\varphi_I}{\zeta}\right)^{\frac{1}{\zeta-1}}\frac{1}{\varphi_E}$.

$$r_t V_t(q,\Delta) - \dot{V}_t(q,\Delta) = \pi_t(q,\Delta) + I\left\{\frac{\partial V_t(q,\Delta)}{\partial q}q + \frac{\partial V_t(q,\Delta)}{\partial \Delta}\Delta\right\} - \tau V_t(q,\Delta) + \Xi_t, \tag{A-56}$$

where $\Xi_t = \max_x\left\{x\left(\alpha V_t^{CD} + (1-\alpha)V_t^{NV}\right) - \frac{1}{\varphi_x}x^{\zeta}w_t\right\}$ with $V_t^{CD} = \int V_t(\lambda q, 1)\,dF_t(q)$ and $V_t^{NV} = \int V_t\left(\omega Q_t, \frac{\sigma}{\sigma-1}\right)d\Gamma(\omega)$. Note that for notational simplicity we denote the quality gap for the creation of a new variety by $\frac{\sigma}{\sigma-1}$ to indicate that new varieties are able to charge the monopolistic markup.

To characterize the value function, note first that the definition of Ξ_t still implies that

$$0 = \left(\alpha V_t^{CD} + (1-\alpha)V_t^{NV}\right) - \frac{\zeta}{\varphi_x}x^{\zeta-1}w_t.$$

The free entry condition thus still requires that

$$w_t = \varphi_E\left(\alpha V_t^{CD} + (1-\alpha)V_t^{NV}\right).$$

Hence, as in the baseline model, $x = \left(\frac{\varphi_x}{\varphi_E}\frac{1}{\zeta}\right)^{\frac{1}{\zeta-1}}$. Therefore $\Xi_t = \frac{\zeta-1}{\varphi_x}x^{\zeta}w_t \propto w_t$.

To solve for $V_t(q,\Delta)$ in (A-56) along the BGP, conjecture that $V_t(q,\Delta)$ takes the form

$$V_t(q,\Delta) = k(\Delta)\,q^{\sigma-1}U_t + M_t.$$

This implies that

$$\frac{\partial V_t(q,\Delta)}{\partial q}q = (\sigma-1)k(\Delta)\,q^{\sigma-1}U_t \quad \text{and} \quad \frac{\partial V_t(q,\Delta)}{\partial \Delta}\Delta = k'(\Delta)\,\Delta q^{\sigma-1}U_t,$$

so that

$$\frac{\partial V_t(q,\Delta)}{\partial q}q + \frac{\partial V_t(q,\Delta)}{\partial \Delta}\Delta = ((\sigma-1)+\varepsilon_k(\Delta))\,k(\Delta)\,q^{\sigma-1}U_t,$$

where $\varepsilon_k(\Delta) \equiv \frac{k'(\Delta)\Delta}{k(\Delta)}$.

Equation (A-56) thus implies that $k(\Delta)$, U_t and M_t solve the equations

$$(r_t+\tau)M_t - \dot{M}_t = \frac{\zeta-1}{\varphi_x}x^{\zeta}w_t \tag{A-57}$$

and

$$\begin{aligned}(r_t+\tau)k(\Delta)\,q^{\sigma-1}U_t - k(\Delta)\,q^{\sigma-1}\dot{U}_t &= \pi_t(q,\Delta) + I\left((\sigma-1)+\varepsilon_k(\Delta)\right)k(\Delta)\,q^{\sigma-1}U_t \\ &= h(\Delta)\left(\frac{q_i}{Q_t}\right)^{\sigma-1}\frac{1}{(\mathcal{M}_t\Lambda_t)^{\sigma-1}}\frac{Y_t}{N_t} + I\left((\sigma-1)+\varepsilon_k(\Delta)\right)k(\Delta)\,q^{\sigma-1}U_t,\end{aligned}$$

where $h(\Delta) = \left(1 - \frac{1}{\mu(\Delta)}\right) \mu(\Delta)^{1-\sigma}$. Thus

$$(r_t + \tau) k(\Delta) U_t - k(\Delta) \dot{U}_t = h(\Delta) \left(\frac{1}{Q_t}\right)^{\sigma-1} \frac{1}{(\mathcal{M}_t \Lambda_t)^{\sigma-1}} \frac{Y_t}{N_t} + I((\sigma-1) + \varepsilon_k(\Delta)) k(\Delta) U_t.$$

Along the BGP, $\mathcal{M}_t \Lambda_t$ is constant and U_t grows at the same rates as $\frac{Y_t}{N_t} Q_t^{1-\sigma}$. From (A-3) and (A-4) this implies that

$$g_U = \frac{\dot{U}_t}{U_t} = g_w - (\sigma - 1) g_Q.$$

Hence,

$$k(\Delta) U_t = \frac{h(\Delta) \left(\frac{1}{Q_t}\right)^{\sigma-1} \frac{1}{(\mathcal{M}\Lambda)^{\sigma-1}} \frac{Y_t}{N_t}}{r + \tau - g_w + (\sigma-1)(g_Q - I) - I\varepsilon_k(\Delta)}.$$

The solution to the value function is therefore

$$U_t = \left(\frac{1}{Q_t}\right)^{\sigma-1} \frac{1}{(\mathcal{M}\Lambda)^{\sigma-1}} \frac{Y_t}{N_t}$$

and

$$\begin{aligned} k(\Delta) &= \frac{h(\Delta)}{r + \tau - g_w + (\sigma-1)(g_Q - I) - I\varepsilon_k(\Delta)} \\ &= \frac{h(\Delta)}{r + \tau - g_w + (\sigma-1)(g_Q - I) - I\frac{k'(\Delta)\Delta}{k(\Delta)}}. \end{aligned}$$

Along the BGP we have

$$r + \tau - g_w + (\sigma-1)(g_Q - I) = \rho + \left(\frac{\alpha\lambda^{\sigma-1}}{1-\alpha} + \varpi^{\sigma-1} - 1\right)\eta.$$

Hence, the function $k(\Delta)$ solves the differential equation

$$k(\Delta)\mathcal{C} - Ik'(\Delta)\Delta = \frac{\min\left\{\frac{\sigma}{\sigma-1}, \Delta\right\} - 1}{\min\left\{\frac{\sigma}{\sigma-1}, \Delta\right\}^{\sigma}},$$

where $\mathcal{C} = \rho + \left(\frac{\alpha\lambda^{\sigma-1}}{1-\alpha} + \varpi^{\sigma-1} - 1\right)\eta$.

For $\Delta \geq \frac{\sigma}{\sigma-1}$ we have

$$k(\Delta)\mathcal{C} - Ik'(\Delta)\Delta = \frac{1}{\sigma-1}\left(\frac{\sigma-1}{\sigma}\right)^{\sigma}.$$

Hence, $k'(\Delta) = 0$ and

$$k(\Delta) = \frac{\frac{1}{\sigma-1}\left(\frac{\sigma-1}{\sigma}\right)^{\sigma}}{\mathcal{C}} \text{ for } \Delta \geq \frac{\sigma}{\sigma-1}. \tag{A-58}$$

For $\Delta < \frac{\sigma}{\sigma-1}$, we have

$$k(\Delta)\mathcal{C} - Ik'(\Delta)\Delta = \frac{\Delta-1}{\Delta^{\sigma}}. \tag{A-59}$$

We can solve this differential equation together with the terminal condition $k\left(\frac{\sigma}{\sigma-1}\right)$ given in (A-58). Equation (A-57) implies M_t grows at g_w along the BGP. Hence,

$$M_t = \frac{1}{r+\tau-g_w}\frac{\zeta-1}{\varphi_x}x^{\zeta} = \frac{1}{\rho+\tau}\frac{\zeta-1}{\varphi_x}x^{\zeta},$$

because $r = \rho + g_w$. Together with the solution for $k(\Delta)$, the value function along the BGP is given by

$$V_t(q,\Delta) = k(\Delta)\left(\frac{q}{Q_t}\right)^{\sigma-1}\frac{1}{(\mathcal{M}\Lambda)^{\sigma-1}}\frac{Y_t}{N_t} + \frac{1}{\rho+\tau}\frac{\zeta-1}{\varphi_x}x^{\zeta}w_t.$$

Using (A-4) we get that

$$\frac{1}{(\mathcal{M}\Lambda)^{\sigma-1}}\frac{Y_t}{N_t} = \frac{1}{\mathcal{M}^{\sigma-1}\Lambda^{\sigma}}\frac{L_P}{N_t}w_t = \frac{1}{\mathcal{M}^{\sigma-1}\Lambda^{\sigma}}\frac{\ell^P}{\mathcal{N}}w_t,$$

where $\ell^P = L_t^P/L_t$ and $\mathcal{N} = N_t/L_t$ are constant along a BGP. Hence,

$$V_t(q,\Delta) = \left\{k(\Delta)\left(\frac{q}{Q_t}\right)^{\sigma-1}\frac{1}{\mathcal{M}^{\sigma-1}\Lambda^{\sigma}}\frac{\ell_t^P}{\mathcal{N}_t} + \frac{1}{\rho+\tau}\frac{\zeta-1}{\varphi_x}x^{\zeta}\right\}w_t. \tag{A-60}$$

A-1.4.2 The Free Entry Condition

Using (A-60) we can derive the free entry condition. The value of creative destruction is given by

$$V_t^{CD} = \int V_t(\lambda q,\lambda)\,dF_t(q) = \left\{k(\lambda)\lambda^{\sigma-1}\frac{1}{\mathcal{M}^{\sigma-1}\Lambda^{\sigma}}\frac{\ell^P}{\mathcal{N}} + \frac{1}{\rho+\tau}\frac{\zeta-1}{\varphi_x}x^{\zeta}\right\}w_t.$$

The value of variety creation is

$$V_t^{NV} = \int V_t\left(\omega Q_t, \frac{\sigma}{\sigma-1}\right)d\Gamma(\omega) = \left\{k\left(\frac{\sigma}{\sigma-1}\right)\overline{\omega}^{\sigma-1}\frac{1}{\mathcal{M}^{\sigma-1}\Lambda^{\sigma}}\frac{\ell^P}{\mathcal{N}} + \frac{1}{\rho+\tau}\frac{\zeta-1}{\varphi_x}x^{\zeta}\right\}w_t,$$

where $\overline{\omega} = E\left[\omega^{\sigma-1}\right]^{1/(\sigma-1)}$. Hence,

$$V_t^{Entry} = \alpha V_t^{CD} + (1-\alpha) V_t^{NV} = \left\{ \frac{\alpha k(\lambda)\lambda^{\sigma-1} + (1-\alpha) k\left(\frac{\sigma}{\sigma-1}\right)\overline{\omega}^{\sigma-1}}{\mathcal{M}^{\sigma-1}\Lambda^{\sigma}} \frac{\ell^P}{\mathcal{N}} + \frac{1}{\rho+\tau}\frac{\zeta-1}{\varphi_x} x^{\zeta} \right\} w_t.$$

The free entry condition, is thus given by

$$\frac{1}{\varphi_E} = \frac{V_t^{Entry}}{w_t} = \frac{\alpha k(\lambda)\lambda^{\sigma-1} + (1-\alpha) k\left(\frac{\sigma}{\sigma-1}\right)\overline{\omega}^{\sigma-1}}{\mathcal{M}^{\sigma-1}\Lambda^{\sigma}} \frac{\ell^P}{\mathcal{N}} + \frac{1}{\rho+\tau}\frac{\zeta-1}{\varphi_x} x^{\zeta}.$$

Because $\mathcal{M}^{\sigma-1}\Lambda^{\sigma}$ can be calculated along a BGP, the free entry condition determines $\frac{\ell^P}{\mathcal{N}}$ as in the model with constant markups. In particular, x, τ and $k(\Delta)$ can be calculated as functions of parameters and $\mathcal{M}^{\sigma-1}\Lambda^{\sigma}$ is fully determined from the joint distribution $F(\Delta, q)$, which is also only a function of parameters along the BGP.

A-1.4.3 Proposition 2 in the model with Bertrand Competition

To see that Proposition 2 still applies in the model with Bertrand competition, note first that creative destruction and the rate of variety creation are still given by

$$\begin{aligned} \tau &= \alpha(z+x) \\ g_N &= (1-\alpha)(z+x). \end{aligned}$$

Moreover, the optimality condition for incumbent expansion x is still given by (A-12) and the free entry condition still holds. Hence,

$$x = \left(\frac{\varphi_x}{\zeta}\right)^{\frac{1}{\zeta-1}} \left(\alpha \frac{V_t^{CD}}{w_t} + (1-\alpha)\frac{V_t^{NV}}{w_t}\right)^{\frac{1}{\zeta-1}} = \left(\frac{1}{\zeta}\frac{\varphi_x}{\varphi_E}\right)^{\frac{1}{\zeta-1}}.$$

These three equations together with BGP condition $g_N = \eta$ are sufficient to derive Proposition 2.

A-1.4.4 The Joint Distribution of Gaps and Productivity

In the model with Bertrand competition in Section 3, the joint distribution of relative quality $\hat{q}_t = \ln(q_t/Q_t)^{\sigma-1}$ (see (A-32)) and quality gaps Δ, $F_t(\Delta, \hat{q}_t)$ emerges as a key endogenous object. To solve for $F_t(\Delta, \hat{q}_t)$, it is useful to separate the problem by focusing individually on products with competitors (i.e. where creative destructions has happened at some point in the past) and products without competitors (i.e. products which are still owned by the firms that introduced the variety originally). We denote these distributions by $F_t^C(\Delta, \hat{q})$ and $F_t^{NC}(\hat{q})$.[29] They are characterized from the

[29] Note that we do not need to keep track of the quality gap among products without competitors because markups are always given by $\frac{\sigma}{\sigma-1}$.

two differential equations $\frac{\partial F_t^C(\Delta,\hat{q})}{\partial t}$ and $\frac{\partial F_t^{NC}(\hat{q})}{\partial t}$ given in the main text. In this section we derive these expressions. We denote the mass of products with and without competitors respectively by N_t^C and N_t^{NC}. Of course, $N_t = N_t^C + N_t^{NC}$. Also recall that $\hat{q}_t$ has a drift of $g_{\hat{q}} = (\sigma-1)(I - g_{Q,t})$ (see (A-33)).

Let $\bar{F}_t^C(\Delta,\hat{q})$ denote the *mass* of products with a gap less than Δ and relative productivity less than $\hat{q}$, for products with a direct competitor. Similarly, let $\bar{F}_t^{NC}(\hat{q})$ denote the *mass* of the products who have no direct competitor at time t with relative productivity less than $\hat{q}$. Hence, $F_t^{NC} = \bar{F}_t^{NC}/N_t^{NC}$ and $F_t^C = \bar{F}_t^C/N_t^C$. The evolution of the non-competitor mass $\bar{F}_t^{NC}(\hat{q})$ satisfies

$$\bar{F}_t^{NC}(\hat{q}) = \underbrace{\bar{F}_{t-\iota}^{NC}\left(\hat{q} - g_{\hat{q}}\iota\right)\left(1-(\tau_t+\delta)\iota\right)}_{\text{existing mass that survives and improves/falls}} + \underbrace{\left(\frac{1-\alpha}{\alpha}\right)\tau_t N_t \Gamma\left(\frac{\exp(\hat{q})}{\sigma-1}\right)\iota}_{\text{new products that enter}}.$$

where we have used the fact that the new product creation rate is $\frac{1-\alpha}{\alpha}\tau = g_t^N$. Note also that $N_t(1-\alpha)$ will be the equilibrium mass of non-competitive products, which we will call N_t^{NC}. As ι becomes small this leads to the differential equation

$$\frac{\partial \bar{F}_t^{NC}(\hat{q})}{\partial t} = -g_{\hat{q}}\frac{\partial \bar{F}_t^{NC}(\hat{q})}{\partial \hat{q}} - (\tau_t+\delta)\bar{F}_t^{NC}(\hat{q}) + \left(\frac{1-\alpha}{\alpha}\right)\tau_t\frac{N_t^{NC}}{(1-\alpha)}\Gamma\left(\frac{\exp(\hat{q})}{\sigma-1}\right). \tag{A-61}$$

Defining the distribution $F_t^{NC} \equiv \bar{F}_t^{NC}/N_t^{NC}$, A-61 implies

$$\frac{\partial F_t^{NC}(\hat{q})}{\partial t} = -g_{\hat{q}}\frac{\partial F_t^{NC}(\hat{q})}{\partial \hat{q}} - (\tau_t+\delta+\eta)F_t^{NC}(\hat{q}) + \frac{\tau_t}{\alpha}\Gamma\left(\frac{\exp(\hat{q})}{\sigma-1}\right).$$

This is the equation reported in Section 3.

For the mass of products with a competitor, $\bar{F}_t^C(\Delta,\hat{q})$, we not only need to keep track of the relative quality $\hat{q}$ but also of the quality gap Δ. This mass evolves according to

$$\begin{aligned}\bar{F}_t^C(\Delta,\hat{q}) = & \underbrace{\bar{F}_{t-\iota}^C\left(\Delta e^{-I\iota}, \hat{q} - g_{\hat{q}}\iota\right)\left(1-(\tau_t+\delta)\iota\right)}_{\text{existing mass that survives and improves/falls}} \\ & + \underbrace{\lim_{s\to\infty}\iota\tau_t\bar{F}_{t-\iota}^C(s,\hat{q}-\hat{\lambda})}_{\text{Creative destruction of } C \text{ products below } \hat{q}\text{-}\hat{\lambda}} + \underbrace{\tau\iota\bar{F}_{t-\iota}^{NC}(\hat{q}-\hat{\lambda})}_{\text{Creative destruction of } NC \text{ products below } \hat{q}\text{-}\hat{\lambda}},\end{aligned}$$

where again we defined $\hat{\lambda} = \ln \lambda^{\sigma-1}$. As ι becomes small this leads to the differential equation

$$\frac{\partial \bar{F}_t^C(\Delta,\hat{q})}{\partial t} = -\frac{\partial \bar{F}_t^C(\Delta,\hat{q})}{\partial \Delta} I\Delta - g_{\hat{q}} \frac{\partial \bar{F}_t^C(\Delta,\hat{q})}{\partial \hat{q}} - \bar{F}_t^C\left(\Delta,\hat{q}\right)(\tau_t+\delta) + \lim_{s\to\infty} \tau_t \bar{F}_t^C(s,\hat{q}-\hat{\lambda}) + \tau \bar{F}_t^{NC}(\hat{q}-\hat{\lambda}).$$

Defining $\bar{F}_t^C(\Delta,\hat{q}) = N_t^C F_t^C(\Delta,\hat{q})$, we get

$$\frac{\partial F_t^C(\Delta,\hat{q})}{\partial t} = -\Delta I \frac{\partial F_t^C\left(\Delta,\hat{q}\right)}{\partial \Delta} - g_{\hat{q}} \frac{\partial F_t^C\left(\Delta,\hat{q}\right)}{\partial \hat{q}} - (\tau_t+\delta+\eta)\, F_t^C(\Delta,\hat{q}) + \lim_{s\to\infty} \tau_t F_t^C(s,\hat{q}-\hat{\lambda}) + \tau_t \frac{N_t^{NC}}{N_t^C} F_t^{NC}(\hat{q}-\hat{\lambda}).$$

Note that the latter term depends on the relative share of products without a competitor N_t^{NC}/N_t^C.

To derive N_t^{NC}/N_t^C, note that N_t^{NC} and N_t^C evolve according to

$$\dot{N}_t^{NC} = \underbrace{N_t\tau\left(\frac{1-\alpha}{\alpha}\right)}_{\text{New varieties}} - \underbrace{N_t^{NC}(\delta+\tau)}_{\text{Products that turn into C products or exit}},$$

And

$$\dot{N}_t^C = \underbrace{-\delta N_t^C}_{\text{Exiting products}} + \underbrace{N_t^{NC}\tau}_{\text{New inflows}}.$$

The steady state share of NC products is therefore given by

$$\frac{N_t^{NC}}{N_t} = \frac{\tau\left(\frac{1-\alpha}{\alpha}\right)}{\eta+\delta+\tau} = 1-\alpha, \tag{A-62}$$

i.e. the steady-state share of NC products is simply given by its share in the process of product creation.

A-1.4.5 Marginal gap distribution

We now derive the distribution of efficiency gaps given in (32). Let $F_t^C(\Delta)$ denote the cdf of quality gaps among products with a competitor. Let, as before, denote the number of competitor and non-competitor products as N_t^C and N_t^{NC}. The distribution $F_t^C(\Delta)$ the solves the differential equation

$$\frac{\partial F_t^C(\Delta)}{\partial t} + F_t^C(\Delta)\frac{1}{N_t^C}\frac{\partial N_t^C}{\partial t} = \underbrace{-I\Delta\frac{\partial F_t^C(\Delta)}{\partial \Delta}}_{\text{Upward drift of own-innovation}} - \underbrace{\delta F_t^C(\Delta)}_{\text{Exit}} + \underbrace{(1-F_t^C(\Delta))\tau}_{\text{Inflow through CD}} + \underbrace{\frac{N_t^{NC}}{N_t^C}\tau}_{\text{Inflow from } NC \text{products}}.$$

Along a BGP, this distribution is stationary (i.e. $\frac{\partial F_t^C(\Delta)}{\partial t} = 0$), the number of competitive products grows at rate η and $N_t^{NC}/N_t^C = \frac{1-\alpha}{\alpha}$ (see (A-62)). Hence,

$$\begin{aligned} I\Delta\frac{\partial F^C(\Delta)}{\partial \Delta} &= -(\delta+\eta)F^C(\Delta) + (1-F^C(\Delta))\tau + \frac{1-\alpha}{\alpha}\tau \\ &= -(\delta+\eta+\tau)F^C(\Delta) + \frac{1}{\alpha}\tau. \end{aligned}$$

Together with the initial condition $F^C(\lambda) = 0$ and the fact that $\frac{1-\alpha}{\alpha}\tau = \eta + \delta$, it is easy to verify that the solution to this differential equation is

$$F^C(\Delta) = 1 - \left(\frac{\lambda}{\Delta}\right)^{\frac{\delta+\eta+\tau}{I}}.$$

A-2 Quantitative Analysis

A-2.1 Data Description

Our main data is the LBD, which contains information for employment and age for the population of firms in the US. In Table A-1 we report a set of descriptive statistics from this data. The firm size distribution in the US has been changing. Between 1980 and 2010 average firm size increased from 20 employees to about 22 employees. This increase in firm size is mostly due to a change in the concentration of economic activity. As seen in Panel B, the employment share of firms with more than 10,000 employees increased and the employment share of firms with less than 20 employees declined. Finally, an important mechanisms underlying these changes in the size distribution are shifts in the age distribution. As seen in the lowest panel, young firms account for much lower share of aggregate employment then they used to in 1980.

Table A-1: Summary of Data

Aggregate Statistics				
Year	Number of Firms	Employees	Average Employment	
1980	3,606,457	73,753,303	20.04	
1995	4,613,849	99,243,906	21.20	
2010	4,953,425	111,189,088	22.15	
Size Distribution				
	Firms with <20 Employees		Firms with >10,000 Employees	
Year	Firm Share	Employment Share	Firm Share	Employment share
1980	89.38	21.58	0.0002	25.71
1995	88.95	20.74	0.0002	23.84
2010	88.88	18.8	0.0002	27.02
Age Distribution				
	Firms with <5 years		Firms with >5 years	
Year	Firm Share	Employment Share	Firm Share	Employment share
1980	13.84 %	38.50 %	86.16%	61.50 %
1995	13.12 %	35.34 %	86.88 %	64.66 %
2010	9.43%	30.02%	91.57 %	69.98 %

Notes: This Table gives basic summary information about the firms in the LBD through time.

A-2.2 Estimating the Pareto tail of the employment distribution

One of our target moments is the pareto tail of the employment distribution. The distribution of firm employment at time t is - for large firms - given by $P_t\left(l_f > x\right) = \left(\bar{l}/x\right)^{\varsigma}$.Hence,

$$\ln P_t\left(l_f > x\right) = \delta - \varsigma \ln x. \tag{A-63}$$

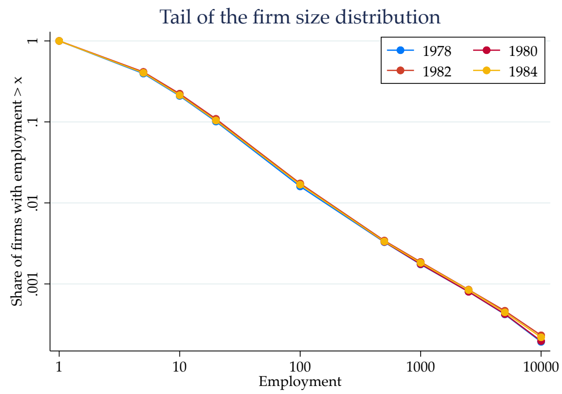
In Figure A-2 we show the empirical relationship between $\ln P_t\left(l_f > x\right)$ and $\ln x$ for different years. As predicted by (A-63), the relationship is almost perfectly linear and stable across years.

In Table A-2 we estimate (A-63) and we include a set of year fixed effects in lieu of δ. The parameter ς is constrained to be common across years. We estimate (A-63) for different samples, i.e. by only including firm above a certain threshold (as the employment distribution only has a pareto tail for large firms). We estimate $\varsigma \approx 1$. If anything ς is slightly smaller than 1. To keep average size bounded, we require $\varsigma > 1$. We therefore opt to calibrate our model to a tail of 1.1.

A-2.3 Computing the sales and markups lifecycle

In this section we derive the details of our characterization of the firms' lifecycle of markup and sales that we use to calibrate the model (see Section 4.2). In particular, we show that relative sales by age

Figure A-2: Estimating the tail of the firm size distribution



Notes: The figure plots $\ln P_t\left(l_f > x\right)$ against $\ln x$ for different years. The data is taken from the BDS.

	Full Sample	Empl >= 5	Empl >= 20	Empl >= 100
log employment	-0.963***	-1.000***	-0.981***	-0.935***
	(0.010)	(0.004)	(0.009)	(0.006)
Year FE	Yes	Yes	Yes	Yes
Observations	80	72	56	48
R^2	0.995	0.998	0.997	0.999

Standard errors in parentheses
$^{*}\ p < 0.10$, $^{**}\ p < 0.05$, $^{***}\ p < 0.01$

Table A-2: Estimates of the pareto tail ς

of the product is given by

$$s_P(a_P) \equiv E\left[\frac{p_i y_i}{Y}\middle| a_p\right] = \mu(a_p)^{1-\sigma} e^{(\sigma-1)(I-g^Q)a_p}\left(\alpha\lambda^{\sigma-1} + (1-\alpha)\bar{\omega}^{\sigma-1}\right). \tag{A-64}$$

Moreover we derive the distribution of product age a_P as a function of firm age a_f and the number of products N. Given this distribution we can then easily evaluate $s_f(a_f)$ and $\mu_f(a_f)$ computationally.

A-2.3.1 Derivation of $s_P(a_P)$ in (A-64)

Consider a BGP where $\mathcal{M}_t$ and Λ_t are constant. (A-5) then implies that sales of product i relative to average sales are

$$s_P(a_P) \equiv E\left[\frac{p_i y_i}{Y_t/N_t}\middle| a_p\right] = E\left[\mu_i^{1-\sigma}\left(\frac{q_i}{Q_t}\right)^{\sigma-1}\middle| a_p\right]\left(\frac{1}{\mathcal{M}_t\Lambda_t}\right)^{\sigma-1}.$$

Note first that markups are a deterministic function of Δ and Δ is a deterministic function of the age of the product. In particular,

$$\mu_i = \mu(a_P) = \min\left\{\lambda e^{Ia_P}, \frac{\sigma-1}{\sigma}\right\}.$$

Similarly, Q_t is given by $Q_t = e^{g_Q a_p} Q_{t-a_p}$.

Now consider the distribution of q_i conditional on a_P. This distribution is given by

$$P(q_i \leq q|a_P) = P(q_i \leq q|a_P, CD)\,\alpha + P(q_i \leq q|a_P, NV)(1-\alpha),$$

where $P(q_i \leq q|a_P, CD)$ and $P(q_i \leq q|a_P, NV)$ denotes the conditional probability, conditional on the firm having acquired product i through creative destruction or new variety creation respectively. Then

$$P(q_i \leq q|a_P, CD) = F_{t-a_P}\left(\frac{1}{\lambda}qe^{-Ia_P}\right),$$

where $F_{t-a_P}(q)$ denotes the productivity distribution at time $t-a_P$. Similarly,

$$P(q_i \leq q|a_P, NV) = \Gamma\left(qe^{-Ia_P}\frac{1}{Q_{t-a_P}}\right).$$

Hence.

$$\begin{aligned} E\left[q_i^{\sigma-1}\middle| a_P\right] &= \alpha\int q^{\sigma-1} dF_{t-a_P}\left(\frac{1}{\lambda}qe^{-Ia_P}\right) + (1-\alpha)\int q^{\sigma-1} d\Gamma\left(qe^{-Ia_P}\frac{1}{Q_{t-a_P}}\right) \\ &= e^{(\sigma-1)Ia_P} Q_{t-a_P}^{\sigma-1}\left(\alpha\lambda^{\sigma-1} + (1-\alpha)\,\overline{\omega}^{\sigma-1}\right), \end{aligned}$$

so that

$$s_P\left(a_P\right)=\mu\left(a_P\right)^{1-\sigma}e^{(\sigma-1)\left(I-g_Q\right)a_P}\left(\alpha\lambda^{\sigma-1}+\left(1-\alpha\right)\overline{\omega}^{\sigma-1}\right)\left(\frac{1}{\mathcal{M}_t\Lambda_t}\right)^{\sigma-1},$$

which is the expression in (A-64).

A-2.3.2 Life-Cycle Dynamics

Relative sales and markups at the product level as a function of the state variables Δ and q are given by

$$\begin{aligned}\mu_i &= \mu\left(\Delta_i\right)=\min\left\{\frac{\sigma}{\sigma-1},\Delta_i\right\}\\ \frac{s_i}{Y_t/N_t} &= s_P\left(\Delta_i,q_i\right)=\left(\frac{1}{\mathcal{M}_t\Lambda_t}\right)^{\sigma-1}\mu\left(\Delta_i\right)^{1-\sigma}\left(\frac{q_i}{Q_t}\right)^{\sigma-1}.\end{aligned}$$

Relative sales and average markups of firm f level as a function of the random vector $\left[\Delta_i,q_i\right]_{i=1}^{N_f}$ are then given by

$$\frac{s_{ft}}{Y_t/N_t}=\sum_{n=1}^{N_f}s_P\left(\Delta_i,q_i\right)\quad\text{and}\quad\mu_f=\frac{1}{N_f}\sum_{i=1}^{N_f}\mu\left(\Delta_n\right).$$

Expected relative sales as a function of firm age a_f are given by

$$\begin{aligned}E\left[\frac{s_{ft}}{Y_t/N_t}\middle|a_f\right] &= E\left[E\left[\frac{s_{ft}}{Y_t/N_t}\middle|a_f,a_P,N_f\right]\middle|a_f\right]\\ &= E\left[\sum_{n=1}^{N_f}E\left[s_P\left(\Delta_i,q_i\right)|\,a_f,a_P,N_f\right]\middle|a_f\right]\\ &= E\left[\sum_{n=1}^{N_f}s_P\left(a_P\right)\middle|a_f\right],\end{aligned}$$

where $s_P\left(a_P\right)$ is given in (A-64). The last equality exploits the fact that conditional on product age a_P, product level sales are independent of firm age a_f and the number of products N_f. Letting $f_{a_P|A_f,N}\left(a_P|a,n\right)$ denote the conditional distribution of product age a_P conditional on firm age a_f and the number of products n and $\overline{p}_n\left(a_f\right)$ the probability a firm of age a_f having n products (conditional on survival). Then

$$E\left[\frac{s_{ft}}{Y_t/N_t}\middle|a_f\right]=\sum_{n=1}^{\infty}n\left(\int_{a_P}s_P\left(a_P\right)f_{a_P|A_f,N}\left(a_P|a_f,n\right)da_P\right)\overline{p}_n\left(a_f\right).$$

Using the expression for $\overline{p}_n\left(a_f\right)$ in (A-27) yields

$$E\left[\frac{s_{ft}}{Y_t/N_t}\middle|a_f\right]=\left(1-\gamma\left(a_f\right)\right)\sum_{n=1}^{\infty}n\left(\int_{a_P}s_P\left(a_P\right)f_{a_P|A_f,N}\left(a_P|a_f,n\right)da_P\right)\gamma\left(a_f\right)^{n-1},$$

where $\gamma(a)$ is given in (A-24). Using the same logic, the average markup as a function of firm age a_f is given by

$$E\left[\mu_f \middle| a_f\right] = \sum_{n=1}^{\infty} \left(\int_{a_P} \mu(a_P) f_{a_P|A_f,N}(a_P|a_f,n)\, da_P \right) (1-\gamma(a_f))\, \gamma(a_f)^{n-1}.$$

Given the density $f_{a_P|A_f,N}(a_P|a_f,n)$, these expressions can be directly evaluated. We now show how to compute this density.

A-2.3.3 Calculating the conditional density $f_{a_P|A_f,N}(a_P|a_f,n)$

We now derive the conditional density of product age a_P, $f_{a_P|A_f,N}(a_P|a_f,n)$.

For illustration, we first derive the expected age of the products in a firm's portfolio as it ages. To do so, consider the mass of firms with n products at age A. We are going to derive the law of motion for the *total number of years* the products that this mass of firms owns have been alive (think of products accumulating years for every instant they have been alive). Call this object $\Psi_A(n)$, where

$$\Psi_A(n) = \underbrace{\Lambda_A(n)n}_{\text{Total number of products by firms of age } A} \underbrace{\mathbb{E}_A[a|n]}_{\text{Average age of products of firms of age } A \text{ and } n \text{ products}}.$$

The pool of total years $\Psi_A(n)$ is equal to the number of firms of age A with n products, denoted $\Lambda_A(n)$, times the number of products they own n, times the average age of all those products $\mathbb{E}_A[a|n]$.

We are going to consider how this object evolves through a discrete time approximation. For a small time interval ι,

$$\begin{aligned}\mathbb{E}_A[a|n]\Lambda_A(n)n = &\underbrace{(\mathbb{E}_{A-\iota}[a|n]+\iota)\Lambda_{A-\iota}(n)n(1-(\tau+\delta+x)n\iota)}_{\text{drift from existing mass}} \\ &+\underbrace{\iota x(n-1)\Lambda_{A-\iota}(n-1)\Big((n-1)\mathbb{E}_{A-\iota}[a|n-1]\Big)}_{\text{flow in from n-1 firms}} \\ &+\underbrace{\iota(\tau+\delta)(n+1)\Lambda_{A-\iota}(n+1)\Big(n\mathbb{E}_{A-\iota}[a|n+1]\Big)}_{\text{flow in from n+1 firms}}\end{aligned}$$

The first term in this expression is the drift in total years from an increment of time ι, multiplied by the fraction of firms who don't drop or gain a product in this increment. Intuitively, these products age with a unit drift. The second term is the flow of total years into the pool $\Psi_A(n)$ from the mass of firms with $n-1$ products who are each gaining a product. Importantly, while they bring n products each into the year pool, only $n-1$ have a positive age, and their average age is $\mathbb{E}_{A-\iota}[a|n-1]$. Lastly, the third term is the flow from the mass of firms with $n+1$ products who are losing a product. They bring n products with average age $\mathbb{E}_A[a|n+1]$ with them.

Rewrite this as

$$\begin{aligned}\frac{\Psi_A(n)-\Psi_{A-\iota}(n)}{\iota} &= \Lambda_A(n)n-(\tau+\delta+x)n\mathbb{E}_{A-\iota}[a|n]\Lambda_A(n)n \\ &+ x(n-1)\Lambda_A(n-1)\Big((n-1)\mathbb{E}_{A-\iota}[a|n-1])\Big) \\ &+ (\tau+\delta)(n+1)\Lambda_A(n+1)\Big(n\mathbb{E}_{A-\iota}[a|n+1]\Big)\end{aligned}$$

so that

$$\begin{aligned}\frac{\Psi_A(n)}{dA} &= \Lambda_A(n)n-(\tau+\delta+x)n\mathbb{E}_A[a|n]\Lambda_A(n)n \\ &+ x(n-1)\Lambda_A(n-1)\Big((n-1)\mathbb{E}_A[a|n-1])\Big) \\ &+ (\tau+\delta)(n+1)\Lambda_A(n+1)\Big(n\mathbb{E}_A[a|n+1]\Big)\end{aligned} \tag{A-65}$$

This gives us a set of equations for the evolution of $\Psi_A(n)$ for all $n>1$ that can be solved computationally given initial conditions. We also need one for $n=1$, which comes from

$$\begin{aligned}\frac{d\mathbb{E}_A[a|1]\Lambda_A(1)1}{dA} &= \Lambda_A(1)-(\tau+\delta+x)\mathbb{E}_{A-\iota}[a|1]\Lambda_A(1) \\ &+ (\tau+\delta)(2)\Lambda_A(2)\Big(\mathbb{E}_A[a|2]\Big)\end{aligned}$$

The initial condition is that

$$\mathbb{E}_0[a|n]\Lambda_0(n)n = \Psi_0(n) = 0$$

for all n. The equations we solve computationally are

$$\begin{aligned}\frac{\Psi_A(n)}{dA} &= \Lambda_A(n)n-(\tau+\delta+x)n\Psi_A(n) \\ &+ x(n-1)\Psi_A(n-1) \\ &+ (\tau+\delta)n\Psi_A(n+1)\end{aligned} \tag{A-66}$$

Lastly, to recover $\mathbb{E}_A[a|n]$ after computing $\Psi_A(n)$, note that

$$\Lambda_A(n) = F_0 p_A(n)$$

where F_0 is the initial number of firms, and $p_A(n)$ as above is the probability that a firm of age A will have n products , for which we have closed form expressions. Then

$$\mathbb{E}_A[a|n] = \frac{\Psi_A(n)}{\Lambda_A(n)n}$$

Finally, to compute the expected age of products for surviving firms of age A, we have

$$E_A[a] = \sum_{n=1}^{\infty} \mathbb{E}_A[a|n] \frac{p_A(n)}{1 - p_A(0)}$$

We use this object in computing markups and sales by firm age, since product markup is a deterministic function of product age.

A-2.3.4 Full Product Age Distribution

Consider the object $X_{A,n}(a) = \Lambda_A(n)n\Phi_{A,n}(a)$, the total number of products with age less than a by firms of age A with n products. Recall that $\Lambda_A(n)$ is the total number of firms of age A with n products. Define $\Phi_{A,n}(a)$ as the probability that a product of a firm of age A with $n$products is less than or equal to a. This evolves as

$$\begin{aligned}
X_{A,n}(a) = {} & \Lambda_{A-\iota}(n)\Phi_{A-\iota,n}(a-\iota)n(1-(\tau+\delta+x)n\iota) \\
& + \iota x(n-1)\Lambda_{A-\iota}(n-1)\Big((n-1)\Phi_{A-\iota,n-1}(a)+1\Big) \\
& + \iota(\tau+\delta)(n+1)\Lambda_{A-\iota}(n+1)\Big(n\Phi_{A-\iota,n+1}(a)\Big)
\end{aligned}$$

Note the difference on the second line now, because the new product has age $0 < a$.

Write this as

$$\begin{aligned}
\frac{X_{A,n}(a) - X_{A-\iota,n}(a-\iota)}{\iota} = & -(\tau+\delta+x)nX_{A-\iota,n}(a-\iota) \\
& + x(n-1)\Lambda_{A-\iota}(n-1) + x(n-1)X_{A-\iota,n-1}(a) \\
& + (\tau+\delta)nX_{A-\iota,n+1}(a)
\end{aligned} \tag{A-67}$$

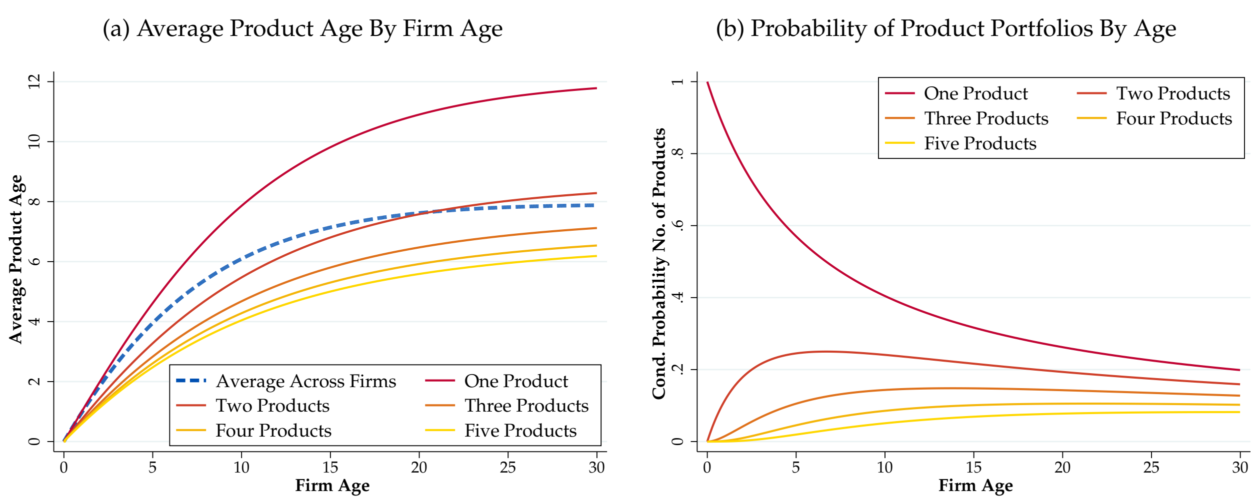
$$\begin{aligned}
\frac{X_{A,n}(a) - X_{A-\iota,n}(a) + X_{A-\iota,n}(a) - X_{A-\iota,n}(a-\iota)}{\iota} = & -(\tau+\delta+x)nX_{A-\iota,n}(a-\iota) \\
& + x(n-1)\Lambda_{A-\iota}(n-1) + x(n-1)X_{A,n-1}(a) \\
& + (\tau+\delta)nX_{A,n+1}(a)
\end{aligned}$$

which goes to

$$\frac{\partial X_{A,n}(a)}{\partial A} + \frac{\partial X_{A,n}(a)}{\partial a} = -(\tau + \delta + x) n X_{A,n}(a) + x(n-1)\Lambda_A(n-1) + x(n-1)X_{A,n-1}(a) + (\tau+\delta) n X_{A,n+1}(a) \tag{A-68}$$

In Figure A-3 we depict the average product by firm age (left panel) and the probability of having n products as a function of age. The left panel shows the effect of selection on the average product age of multi-product firms. Conditional on the age of the firm, the average product is declining in the number of products because newly added products are - by construction - younger. In the right panel we show five "slices" of the joint distribution of age and the number of products. One-product firms are mostly young firms as all firms enter with a single product. Old firms only rarely have a single product as they either grew or exited already. The remaining lines show that older firms are more and more likely to have many products.

Figure A-3: Product Age and Firm Age



Notes: Panel (a) of this figure plots the average product age for a firm of n products for $n = 1, ..., 5$ as the firm ages in the calibrated model. These objects are computed using the productivity distribution $X_{A,n}(a)$ (see (A-68)). Panel (b) plots the conditional probability of a portfolio of n products for $n = 1, ..., 5$ in the calibrated model for surviving firms by firm age.

A-2.4 Computing the the cross-sectional size and age distribution (Section 4.3)

In Section 4.3 we reported the model-implied distribution of firm size and firm age. We now show how to derive these objects.

A-2.4.1 The cross-sectional size distribution

We have expressions for the probability distribution of number of products by age p_n, as well as the age distribution of firms. So the final piece is the distribution of employment conditional on firm age and number of products. To do so, define

$$l_{A,n} = \mu_i^{-\sigma}\left(\frac{q_i}{Q_t}\right)^{\sigma-1}\frac{L_t}{N_t}\Lambda_t^{-\sigma}\mathcal{M}_t^{1-\sigma}$$

as the random variable of employment at the product level, conditional on the firm having age of A and n products. Conditional on the age of the firm and the number of products, the distribution of $l_{A,n}$ is independent across products, so we derive the distribution of the sum of these objects through a convolution. For the first product

$$Prob(l_{A,n} \leq y) \equiv D^1_{A,n}(y)$$

$$= Prob(\log\left((\frac{q_i}{Q_t})^{\sigma-1}\right) \leq \log(y) + \log(\mu_i^{\sigma}) - log(\frac{L_t}{N_t}\Lambda_t^{-\sigma}\mathcal{M}_t^{1-\sigma}))$$

Now note that $\mu = \Delta = \lambda e^{Ia}$ for a below the critical threshold. Define the joint density of (log) productivity and gaps $f^C(\tilde{q}, \Delta)$from

$$F^C(\hat{q}, \Delta) = \int_{-\infty}^{\hat{q}}\int_{\lambda}^{\Delta} f(x, y)dxdy$$

with associated conditional density $f^C_{\hat{q}|\Delta}(\tilde{q}|\Delta)$ and conditional distribution function $F^C_{\hat{q}|\Delta}(\hat{q}|\Delta)$. Lastly, denote the distribution of productivity for the non-competitor mass as $F^{NC}(\hat{q}|a)$. Given that incumbent innovation is constant for non-competitive products, the law of motion for the mass of products $\bar{F}^{NC}_a(\hat{q})$ at age a is given by

$$\frac{\partial \bar{F}^{NC}_a(\hat{q})}{\partial a} = \frac{\partial F^{NC}(\hat{q}|a)}{\partial \hat{q}}(\sigma-1)((I(\bar{\Delta}) - \gamma) - \bar{F}^{NC}_a(\hat{q})(\tau_t + \delta)$$

with initial condition

$$\bar{F}^{NC}_0(\hat{q}) = \Gamma\left(\frac{\exp(\hat{q})}{\sigma-1}\right)$$

From this we can compute the conditional distribution of productivity $F^{NC}(\hat{q}|a)$.With these pieces we can compute the distribution of employment as

$$
\begin{aligned}
D^1_{A,n}(y) = & \int_0^{\bar{a}} d\Phi_{A,n}(a) F^C(log(y) + \log(\mu^\sigma) - c|\Delta(a)) \\
& + \int_{\bar{a}}^{A} d\Phi_{A,n}(a)) F^C(log(y) + \log(\bar{\mu}^\sigma) - c|\Delta(a)) \\
& + \int_0^{A} d\Phi_{A,n}(a)) F^{NC}(log(y) + \log(\bar{\mu}^\sigma) - c|a)
\end{aligned}
$$

where $c \equiv log(\frac{L_t}{N_t}\Lambda_t^{-\sigma}\mathcal{M}_t^{1-\sigma})$. Now we have the distribution of this object, we can define recursively the distribution of the sums of employment across products from a convolution. Define

$$
Z^j_{A,n} = \sum_{i=1}^{j} l^i_{A,n}
$$

and then

$$
P(Z^j_{A,n} \leq y) = D^j_{A,n}(y) = \int_0^y \int_{-\infty}^{\infty} \frac{dD^{j-1}_{A,n}(x)}{dx} \frac{dD^1(z-x)}{dx} dxdz
$$

for $j \geq 2$. Then, for each age of the firm we can define the conditional employment distribution as

$$
Prob(E_f \leq y | a_f = A) = \sum_{n=1}^{\infty} \frac{p_n(A)}{1 - p_o(A)} D^n_{A,n}(y).
$$

A-2.4.2 The cross-sectional age distribution

Let Y_{ta} be the number of firms who are a years old at time t. The total number of firms at time t is then given by $Y_t = \int_{a=0}^{\infty} Y_{ta} da$. Let E_τ denote the number of entrants at time τ. Then

$$
Y_{ta} = \underbrace{E_{t-a}}_{\text{Entrants}} \underbrace{S(a)}_{\text{Survival}}.
$$

Note also that the number of entrants is given by $E_\tau = zN_\tau$. And as N_τ grows at rate η, we have $E_\tau = zN_0 e^{\eta\tau}$. Hence

$$
Y_{ta} = zN_0 e^{\eta(t-a)} S(a).
$$

The density of firms which are a years old is therefore given by

$$
\omega_t^F(a) = \frac{Y_{ta}}{Y_t} = \frac{zN_0 e^{\eta(t-a)} S(a)}{\int_{a=0}^{\infty} zN_0 e^{\eta(t-a)} S(a)\, da} = \frac{\frac{\psi e^{-(\psi+\eta)a}}{\psi + x(1-e^{-\psi a})}}{\int_{a=0}^{\infty} \frac{\psi e^{-(\psi+\eta)a'}}{\psi + x\left(1-e^{-\psi a'}\right)} da'},
$$

where the last line uses (A-26).

A-2.5 Exit rates by size

To compute exit rates by size (show in Figure 8) we can compute an exit rate per product. Then, the probability of having a number of products n by age A of the firm, conditional on being a certain size y (between l and $u$employees) is

$$Prob(A,n|l \le y \le u) = \frac{Prob(l \le y \le u|A,n) \times P(A,n)}{\sum_{A'} \sum_{n'} Prob(l \le y \le u|A',n') \times P(A',n')}$$

The joint probability of the age bins and number of products is

$$P(A,n) = p_n(A)\omega_t^F(A)$$

Then we can construct the probability

$$Prob(n|l \le y \le u) = \sum_A \frac{Prob(l \le y \le u|A,n) \times P(A,n)}{\sum_{A'} \sum_{n'} Prob(l \le y \le u|A',n') \times P(A',n')}$$

where we are using discrete A bins to compute this object. Once we have the conditional probabilities of numbers of products by size bins, we can compute exit rates by size bins, since exit only depends on the number of products. The exit probability for each number of products n can be calculated as follows.

The probability of losing k products in an interval Δ if you lose each product at rate $\tau + \delta$ is

$$p_n(k,\tau) = e^{-\tau n\Delta} \frac{((\tau+\delta)n\Delta)^k}{k!}$$

The probability of winning m products in an interval Δ if you expand at rate x

$$g_n(m,x) = e^{-xn\Delta} \frac{(xn\Delta)^m}{m!}$$

Hence, the probability of exit when a firm has n products is

$$\begin{aligned}
Prob(k-m>=n) &= E_m\left[Prob(k>=n+m)\right] \\
&= E_m\left[\sum_{k=n+m+1}^{\infty} p_n(k,(\tau+\delta))\right] \\
&= \sum_{m=0}^{\infty} e^{-xn\Delta}\frac{(xn\Delta)^m}{m!} \sum_{k=n+m}^{\infty} e^{-\tau n\Delta}\frac{((\tau+\delta)n\Delta)^k}{k!}.
\end{aligned}$$

A-2.6 Firm Heterogeneity: Young Firm Rockets

Very young firms tend to grow fast even conditional on survival (Dunne et al., 1989). Luttmer (2011) discusses how a violation of Gibrat's law is needed to deliver the relatively low age of very large firms: to match the data, there must be a mechanism whereby young firms can grow quickly for a time. A similar reasoning is also discussed in Pugsley et al. (2019). In our baseline model, such a mechanism is absent. Young firms do indeed violate Gibrat's law in the model, but this is only because of survival bias. Here we discuss the implications for the effects of a population growth slowdown of introducing a subset of young firms that act as "rockets", growing and innovating quickly for a time, before their growth rate slows to look like other ordinary firms (see also Acemoglu et al. (2012))

Suppose that when a firm is born, it can be a rocket (R) or slow (S) type. The only difference between the two is the speed with which a firm can invent new products, such that $x = \{x^R, x^S\}$ To highlight the central differences with the main model, we take these rates to be exogenous. In addition, assume that a rocket firm transitions into being a slow firm at rate ξ. For exposition, assume labor is perfectly substitutable between research and production, and the rate of own product improvement I is fixed. The value of such a firm can be written

$$\begin{aligned}
r_t V_t^R\left(\left[\Delta_i, q_i\right]\right) \quad = \quad & \sum_{i=1}^{n} \pi_t\left(\left[\Delta_i, q_i\right]\right) + \dot{V}_t^R\left(\left[\Delta_i, q_i\right]\right) + \sum_{i=1}^{n}(\tau+\delta)\left[V_t^R\left(\left[\Delta_j, q_j\right]_{j\neq i}\right) - V^R\left(\left[\Delta_i, q_i\right]\right)\right] \\
& + \sum_{j=1}^{n} I \frac{\partial V_t^R\left(\left[\Delta_i, q_i\right]\right)}{dq_j} q_j \\
& + n \max_x \left\{ x\left[\alpha \int_q V_t^R\left(\left[\Delta_i, q_i\right], 1, \lambda q\right) dF_t(q) + (1-\alpha)\int_\omega \int_\Delta V_t^R\left(\left[\Delta_i, q_i\right], \Delta, \omega Q_t\right) dG(\Delta) d\Gamma(\omega) \right.\right. \\
& \left.\left. - V_t^R\left(\left[\Delta_i, q_i\right]\right)\right] - \frac{1}{\varphi_x^R} x^\zeta w_t \right\} \\
& \xi\left(V_t^S\left(\left[\Delta_i, q_i\right]\right) - V_t^R\left(\left[\Delta_i, q_i\right]\right)\right).
\end{aligned}$$

with an analogous equation holding for V_t^S. Suppose lastly that entrants cannot choose whether they are going to be a rocket or slow, but become a rocket at entry with fixed probability κ. It can be shown that under these assumptions, the solution to the value functions are

$$V_t^R\left(\left[\Delta_i, q_i\right]\right) = \sum_{i=1}^{n} U_t\left(\Delta_i, q_i\right) + n\phi^R w_t$$

$$V_t^S\left(\left[\Delta_i, q_i\right]\right) = \sum_{i=1}^{n} U_t\left(\Delta_i, q_i\right) + n\phi^S w_t$$

Where

$$U_t\left(\Delta_i, q_i\right) = \frac{u\left(\Delta_i\right)}{g(\sigma-1)+\rho+\tau+\delta-\eta} \frac{q_i^{\sigma-1} Y_t}{\left(\mathcal{M}_t \Lambda_t\right)^{\sigma-1} N_t Q_t^{\sigma-1}}$$

and ϕ^R and ϕ^S are the solutions to

$$(\rho+\tau+\delta)\phi^R = x^R\left[\frac{1}{\varphi_E}+(1-\kappa)(\phi^R-\phi^S)\right]-\frac{1}{\varphi_x^R}(x^R)^\zeta-\xi(\phi^R-\phi^S)$$

$$(\rho+\tau+\delta)\phi^S = x^S\left[\frac{1}{\varphi_E}-\kappa(\phi^R-\phi^S)\right]-\frac{1}{\varphi_x^S}(x^S)^\zeta$$

The share of rocket firms v^R in the population depends on the entry rate, and so changes in population growth will affect the average rate of incumbent expansion. To see this, note that the share of rockets firm $Y^R_{a,t}$ of age a at time $t$denoted is given by

$$Y^R_{a,t} = \kappa e^{-\xi a}$$

where $Y^R_{a,t}$and so the share of rockets in the economy is given by integrating this object over the age distribution. The age distribution is defined by the following two pieces. First, for fast firms the fraction of firms with n products evolves with age a as

$$\dot{p}^R_n(a) = (n-1)\,x^R p^R_{n-1}(a)+(n+1)\,(\tau+\delta)\,p^R_{n+1}(a)-n\left(x^R+\tau+\delta\right)p^R_n(a)-\xi p^R_n(a)\,. \quad \text{(A-69)}$$

Because exit is an absorbing state, $\dot{p^R}_0(a) = (\tau+\delta)\,p^R_1(a)$. The fraction of firms that have survived by a is $S^R(a) = \frac{1-p^R_0(a)}{\sum_{n=0}^{\infty} p^R_n(a)}$. Similarly for slow firms, we have

$$\dot{p}^S_n(a) = (n-1)\,x^S p^S_{n-1}(a)+(n+1)\,(\tau+\delta)\,p^S_{n+1}(a)-n\left(x^S+\tau+\delta\right)p^S_n(a)+\xi p^R_n(a)\,.$$

with $\dot{p^S}_0(a) = (\tau+\delta)\,p^S_1(a)$ and $S^S(a) = \frac{1-p^S_0(a)}{\sum_{n=0}^{\infty} p^S_n(a)}$. The total fraction of surviving firms is then given by

$$S(a) = 1-\kappa S^R(a)-(1-\kappa)S^S(a)$$

m

The age distribution can be obtained from calculating the density of firms by age using

$$\omega_t(a) = \frac{(1-\alpha)\,zN_0 e^{\eta(t-a)}S(a)}{\int_{a=0}^{\infty}(1-\alpha)\,zN_0 e^{\eta(t-a)}S(a)\,da}$$

$$= \frac{e^{-\eta a}S(a)}{\int_{a=0}^{\infty} e^{-\eta a}S(a)\,da}$$

Then the share of rockets in the overall population of is

$$v^R = \int_0^\infty \kappa e^{-\xi a}\omega(a)\,da$$

The share of products that are owned by rockets however, is not quite the same thing. This is given by

$$\hat{\upsilon}^R = \frac{F_t^R \int_0^\infty \sum_{n=1}^\infty \frac{p_n^R(a)n}{1-p_0^R(a)} \omega^R(a)da}{F_t^S \int_0^\infty \sum_{n=1}^\infty \frac{p_n^S(a)n}{1-p_0^S(a)} \omega^S(a)da + F_t^S \int_0^\infty \sum_{n=1}^\infty \frac{p_n^R(a)n}{1-p_0^R(a)} \omega^R(a)da}$$
$$= \frac{\upsilon^R \int_0^\infty \sum_{n=1}^\infty \frac{p_n^R(a)n}{1-p_0^R(a)} \omega^R(a)da}{(1-\upsilon^R) \int_0^\infty \sum_{n=1}^\infty \frac{p_n^S(a)n}{1-p_0^S(a)} \omega^S(a)da + \upsilon^R \int_0^\infty \sum_{n=1}^\infty \frac{p_n^R(a)n}{1-p_0^R(a)} \omega^R(a)da}$$

where the numerator is the number of rocket firms times the average products of a rocket firm.

Creative destruction in this economy is given by

$$\tau = \alpha \left(z + \hat{\upsilon}^R x^R + (1-\hat{\upsilon}^R) x^S \right)$$

However, it is still the case that

$$\tau = \frac{\alpha}{1-\alpha}(\eta + \delta)$$

So while changes in the age distribution will affect the share of rockets in the population, the overall effect on growth is the same as the baseline model. To characterize the equilibrium with rockets, we close the model with the labor market clearing condition

$$L_t = L_t^P + L_t^R = L_t^P + N_t \left(\frac{1}{\varphi_E} z_t + \frac{\hat{\upsilon}^R}{\varphi_x^R} (x^R)^\zeta + \frac{(1-\hat{\upsilon}^R)}{\varphi_x^S} (x^S)^\zeta \right).$$

Which we can characterise in terms of the production share ℓ^P and variety intensity $\mathcal{N}$ on the BGP as

$$\left(\frac{1-\ell_t^P}{\mathcal{N}_t} \right) = \frac{1}{\varphi_E} z + \frac{\hat{\upsilon}^R}{\varphi_x^R} (x^R)^\zeta + \frac{(1-\hat{\upsilon}^R)}{\varphi_x^S} (x^S)^\zeta$$

There are xx things we need to calculate the equilibrium.

Free entry requires that

$$\frac{1}{\phi_E} w_t = \kappa\alpha \int V_t^R(\lambda, q)\, dF_t(q) + \kappa(1-\alpha) \int V_t^R(\bar{\Delta}, Q_t q)\, dG(q)$$
$$+ (1-\kappa)\alpha \int V_t^S(\lambda, q)\, dF_t(q) + (1-\kappa)(1-\alpha) \int V_t^S(\bar{\Delta}, Q_t q)\, dG(q)$$

$$= \frac{\alpha u(\lambda)\lambda^{\sigma-1} + (1-\alpha)u(\bar{\omega})\bar{\omega}^{\sigma-1}}{g(\sigma-1) + \rho + \tau + \delta - \eta} \frac{L_t^P / N_t}{(\mathcal{M}_t)^{\sigma-1} \Lambda_t^\sigma} w_t + \kappa\phi_R + (1-\kappa)\phi_S$$

where in the exogenous I case, we can solve for $u(\Delta)$ from the differential equation

$$u(\Delta) = h(\Delta) + \frac{(\sigma-1)u(\Delta) + u'(\Delta)\Delta}{g(\sigma-1) + \rho + \tau + \delta - \eta} I$$

A-2.6.1 Pareto Tail in the Product Distribution

Now we the following law for the evolution of the rocket distribution. Consider $n \geq 2$. Then, the number of rocket firms with each number of products n evolves according to

$$\dot{\omega}_t^R(n) = \underbrace{\omega_t^R(n-1)(n-1)x}_{\text{From } n-1 \text{ to} n \text{ products}} + \underbrace{\omega_t^R(n+1)(n+1)(\tau+\delta)}_{\text{From } n+1 \text{ to} n \text{ products}} - \underbrace{\omega_t^R(n)n(\tau+x+\delta)}_{\text{From } n \text{ to } n-1 \text{ or } n+1 \text{ products}} - \underbrace{\xi\omega_t^R(n)}_{\text{Transition to slow}}.$$

For $n = 1$ we have

$$\dot{\omega}_t^R(1) = \kappa Z_t + \omega_t^R(2)\,2(\tau+\delta) - \omega_t^R(1)(\tau+x+\delta).$$

Along the BGP the mass of firms grows at rate η. . Hence, the mass of firms is increasing at rate η. Hence, along the BGP we have

$$\dot{\omega^R}_t(n) = \eta\omega_t^R(n).$$

Along the BGP, $\{\nu(n)\}_{n=1}^{\infty}$ is determined by

$$\nu^R(2) = \frac{\nu^R(1)\left(\tau + x^R + \delta + \eta + \xi\right) - \frac{\kappa z}{\bar{\nu}^R}}{2(\tau+\delta)} \tag{A-70}$$

and

$$\nu^R(n+1) = \frac{\nu^R(n)\,n\left(\tau + x^R + \delta\right) + \nu^R(n)(\eta+\xi) - \nu^R(n-1)(n-1)x^R}{(n+1)(\tau+\delta)} \quad \text{for } n \geq 2 \tag{A-71}$$

Again we can apply the result from Luttmer (2011) and the Pareto tail is given by

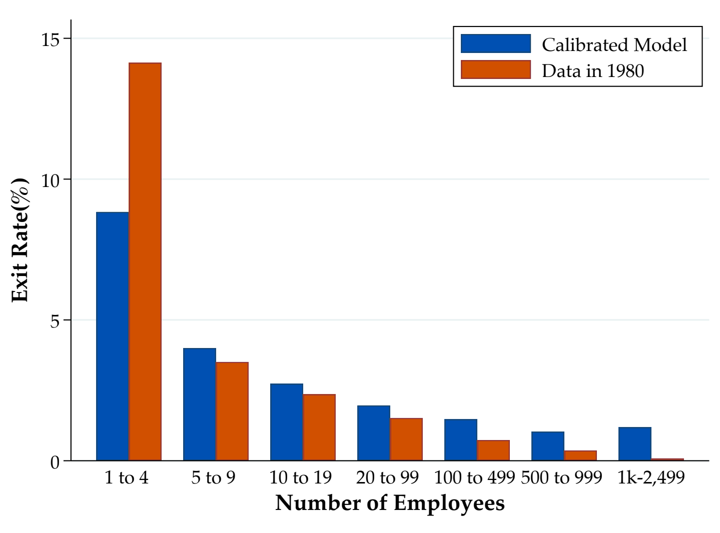
$$\begin{aligned}\zeta_n &= \frac{\eta+\xi}{x^R - \tau - \delta}\\ &= \frac{\eta+\xi}{x^R - \frac{\alpha}{1-\alpha}(\eta+\delta) - \delta}\end{aligned}$$

and we again have the result that lower population growth lowers this tail coefficient. Note also that a smaller transition rate ξ reduces the Pareto tail, that is concentration in the top rises as rockets transition into the slow types at a slower pace.

A-2.6.2 Exit Rates by Size with Firm Heterogeneity

The introduction of type heterogeneity substantially improves the fit of the model against the data on exit rates by size. Simply put, this heterogeneity allows some firms to grow large by adding more products, an outcome which is relatively rare in the baseline model. Because of diversification across products, this lowers the exit rate for large firms (whereas in the baseline model, the overwhelming majority of large firms are so because they have a single, high q product). Figure A-4 shows an illustrative calibration of the model with rockets, demonstrating a declining exit rate with size.

Figure A-4: Exit Rate By Size with Rocket Firms



Notes: This figure shows the exit rate by size in an illustrative calibration of the model with rocket firms. The transition rate is set to $\xi = 0.25$, and the share of rockets at 0.1. The innovation rate of rockets x^R is chosen to match average sales growth by age 10 from the LBD, as in the main quantitative section, while the rate of slow firms is set to 0.1.

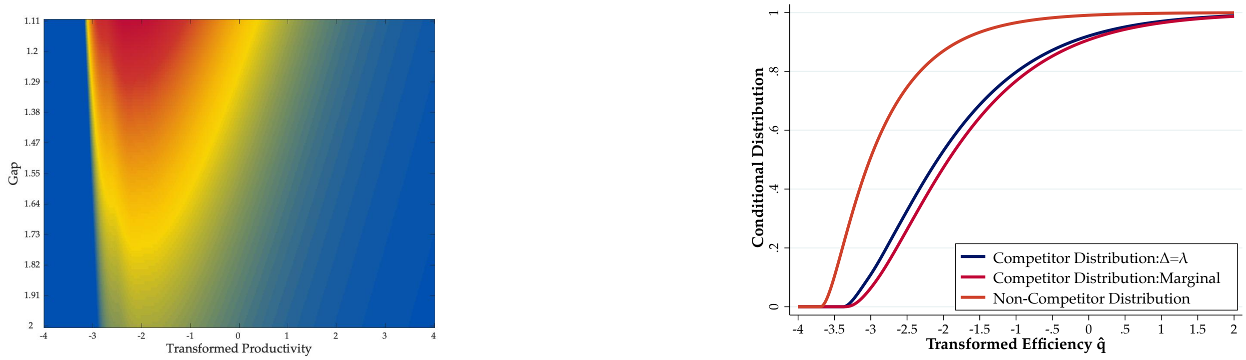
A-2.7 The Joint Distribution of Efficiency and Markups

All aggregate allocations in our model depend on the misallocation wedge $\mathcal{M}$ and the labor share Λ. These aggregate wedges in turn depend on the joint distribution of relative efficiency q/Q and efficiency gaps Δ. In the left panel of Figure A-5 we display this distribution. Multiple forces shape this distribution. On the one hand, firms increase their efficiency q over their life-cycle. This tends to generate a positive correlation between relative efficiency and efficiency gaps. On the other hand, successful creative destruction events also increase relative efficiency but reduce efficiency gaps and hence markups. Moreover, new products have - in our calibration - low efficiency (because $\overline{\omega} < 1$) and high efficiency gaps. In the right panel we look at the efficiency distributions of the different type of products more directly. We depict the overall cross-sectional distribution of competitive products in red and compare it to the efficiency of products conditional on having a quality gap of λ (blue) and to the products that just entered and are still without a competitor (orange). The overall distribution dominates the distribution of new products in a first-order stochastic dominance sense because new products have on average lower qualities. The conditional efficiency distribution, conditional on having a quality gap of Δ, is also lower because some of these products are non-competitive products that just experienced their first creative destruction event.

Figure A-5: The Distributions of Efficiency q and Gaps Δ

(a) Joint Distribution of Efficiency and Gaps

(b) Conditional Productivity Distributions



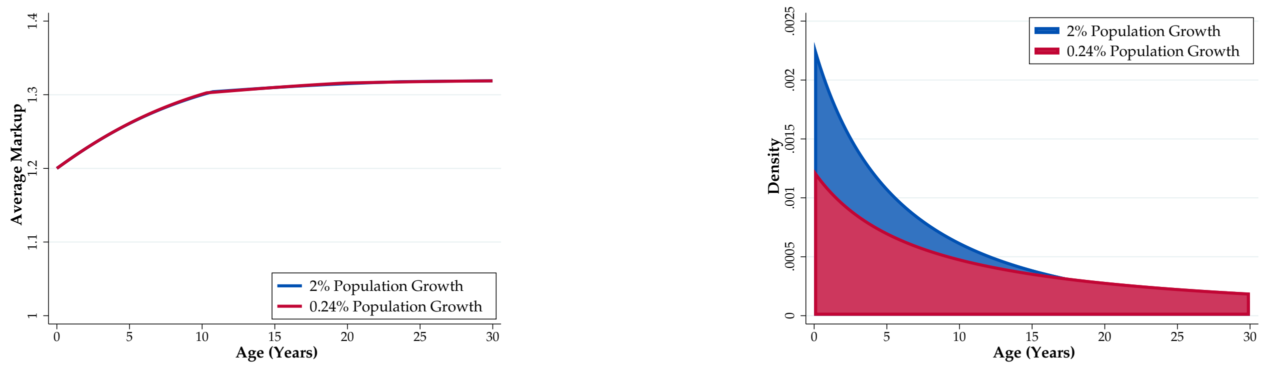
Notes: The left panel shows the joint density of $\hat{q}$ (relative efficiency) and Δ (the gap between the leading product and the next best product) in the calibrated BGP. The right panel shows the productivity distributions in the calibrated model for three types of products: non-competitive products (orange), products which have just seen a creative destruction event and have a gap of $\Delta = \lambda$ (red) and all competitive products (blue)

A-2.8 Decomposing the Impact of Falling Population Growth

Our analysis in Section 5 showed that the experienced and projected decline in population growth increased firm size substantially. In addition, the average markup also increased. In principle this rise in firm size and markups can be due to both changes in the age distribution and changes in firms' size conditional on age. In Figures A-6 and A-7 we show that the lion share of these changes is due to changes in the age distribution. Consider first Figure A-6, where we show the exit rate by age (left panel) and the sales life-cycle (right panel) both in the original BGP (blue) and the new BGP when population growth is 0.24% (red line). While both the age conditional exit rate and the life-cycle do change, the changes are qualitatively small. In Figure A-7 we report the life-cycle of markups (left panel) and the age distribution (right panel) again for both BGPs. The life-cycle of markups is essentially unchanged. By contrast, the age distribution shifts substantially: declining population growth causes firms to become older. And because older firms are larger, charge higher markups and exit at a lower rate, such shifts in the age distribution explain most of the observed change in concentration in our model. This result is consistent with the findings of Hopenhayn et al. (2018) and Karahan et al. (2016), who document empirically that the age-conditional allocations have been relatively constant since the 1980s.

Figure A-7: Markups Within and Across Firms

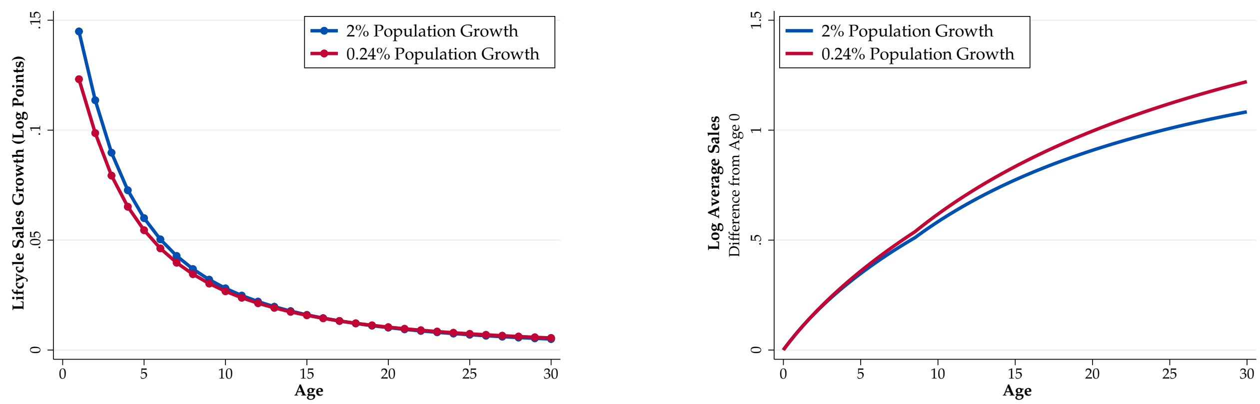
(a) *Lifecycle Firm-level Markups* (b) *Age Distribution*



Note: Panel A shows the average firm-level markup in the model at the calibrated baseline (blue) and and the counterfactual of a 1.74% decline in population growth (red). Panel B shows the density of the age distribution for the same cases.

Figure A-6: Model Counterfactual Lifecycle Moments

(a) Exit Rates (b) Sales Growth



Notes: This left panel shows the model prediction for firm exit rates by age when population growth is 2% (blue) and 0.24% (red). The right panel shows the same for sales growth.

A-2.9 Computing the Transitional Dynamics

In this section we characterize the transitional dynamics of our model. In Section A-2.9.1 we solve for the value function without imposing the economy to be on the BGP. In Section A-2.9.2 we characterize the value of entry during the transitional dynamics. In Section A-2.9.3 we use the free entry condition to characterize the differential equation for the free entry equilibrium during the transition. In Section A-2.9.4 we derive the characterization of the system of equations that fully characterize the transitional dynamics. In Section (A-2.9.5) we derive the differential equation for the joint distribution of quality q and quality gaps Δ, $F_t(q, \Delta)$, that we need to compute the evolution of markups along the transition.

A-2.9.1 The value function

As shown in Section A-1.4, the value function is additive across products and the value of a single product with quality q and quality gap Δ is described by the HJB equation

$$r_t V_t(q,\Delta) - \dot{V}_t(q,\Delta) = \pi_t(q_i,\Delta_i) + \left(\frac{\partial V_t(q,\Delta)}{\partial q} + \frac{\partial V_t(q,\Delta)}{\partial \Delta}\frac{\partial \Delta}{\partial q}\right)\dot{q} - (\tau_t+\delta) V_t(q,\Delta) + \Xi_t,$$

where

$$\pi_t(q_i,\Delta_i) = \left(1-\frac{1}{\mu(\Delta_i)}\right)\mu(\Delta_i)^{1-\sigma}\left(\frac{q_i}{Q_t}\right)^{\sigma-1}\frac{1}{\mathcal{M}_t^{\sigma-1}\Lambda_t^{\sigma}}\frac{L_P}{N_t}w_t.$$

Note that $\frac{\partial \Delta}{\partial q} = \frac{1}{q}\Delta$. Also note that the free entry condition still implies that $\Xi_t = \frac{\zeta-1}{\varphi_x}x^{\zeta}w_t$. Hence, the HJB equations reduces to

$$r_t V_t(q,\Delta) - \dot{V}_t(q,\Delta) = \pi_t(q_i,\Delta_i) + \left(\frac{\partial V_t(q,\Delta)}{\partial q}q + \frac{\partial V_t(q,\Delta)}{\partial \Delta}\Delta\right) I - (\tau_t+\delta) V_t(q,\Delta) + \frac{\zeta-1}{\varphi_x}x^{\zeta}w_t,$$

where $\frac{\dot{q}}{q} = I$.

Now conjecture that the value function takes the form

$$V_t(q,\Delta) = \left(\frac{q}{Q_t}\right)^{\sigma-1} U_t(\Delta) + M_t. \tag{A-72}$$

This implies that

$$\frac{\partial V_t(q,\Delta)}{\partial q}q + \frac{\partial V_t(q,\Delta)}{\partial \Delta}\Delta = ((\sigma-1)+\varepsilon_k(\Delta))\left(\frac{q}{Q_t}\right)^{\sigma-1} U_t(\Delta),$$

where

$$\varepsilon_t(\Delta) \equiv \frac{U_t'(\Delta)\Delta}{U_t(\Delta)}. \tag{A-73}$$

Using the conjecture in (A-72), the HJB simplifies to the following two equations:

1. The function M_t in (A-72) solves the differential equation

$$(r_t+\tau_t+\delta) M_t - \dot{M}_t = \frac{\zeta-1}{\varphi_x}x^{\zeta}w_t$$

2. The function $U_t(\Delta)$ in (A-72) solves the differential equation

$$\left(r_t+\tau_t+\delta+(\sigma-1)\left(g_Q-I\right)-I\varepsilon_t\left(\Delta\right)\right)U_t\left(\Delta\right)-\dot{U}_t\left(\Delta\right)=h\left(\Delta\right)\frac{1}{\mathcal{M}_t^{\sigma-1}\Lambda_t^{\sigma}}\frac{L_P}{N_t}w_t$$

where

$$h\left(\Delta\right) \quad = \quad \left(1-\frac{1}{\mu\left(\Delta_i\right)}\right)\mu\left(\Delta_i\right)^{1-\sigma}=\left(\frac{\mu\left(\Delta_i\right)-1}{\mu\left(\Delta_i\right)^{\sigma}}\right)=\frac{\min\left\{\Delta,\frac{\sigma}{\sigma-1}\right\}-1}{\left(\min\left\{\Delta,\frac{\sigma}{\sigma-1}\right\}\right)^{\sigma}}$$

and $\varepsilon_t\left(\Delta\right)$ is given in (A-73).

A-2.9.2 The value of entry

The value of entry is given by

$$V_t^{Entry} \quad = \quad \underbrace{\alpha\int V_t\left(\lambda q,\lambda\right)dF_t\left(q\right)}_{\text{CD with gap }\lambda\text{ and quality }\lambda q}+\underbrace{\left(1-\alpha\right)\int V_t\left(\omega Q_t,\frac{\sigma}{\sigma-1}\right)d\Gamma\left(\omega\right)}_{\text{New variety with gap }\frac{\sigma}{\sigma-1}\text{ and quality }\omega Q_t}$$

Using the conjecture in (A-72), V_t^{Entry} can be written as

$$V_t^{Entry} \quad = \quad \alpha\lambda^{\sigma-1}U_t\left(\lambda\right)+\left(1-\alpha\right)\overline{\omega}^{\sigma-1}U_t\left(\frac{\sigma}{\sigma-1}\right)+M_t.$$

Upon defining $v_t^{Entry}=\frac{V_t^{Entry}}{w_t}$, $m_t=\frac{M_t}{w_t}$ and $\underline{u}_t\left(\Delta\right)=\frac{U_t(\Delta)}{w_t}$, we get

$$v_t^{Entry}=\alpha\lambda^{\sigma-1}\underline{u}_t\left(\lambda\right)+\left(1-\alpha\right)\overline{\omega}^{\sigma-1}\underline{u}_t\left(\frac{\sigma}{\sigma-1}\right)+m_t \tag{A-74}$$

where m_t solves

$$\left(r_t+\tau_t+\delta-g_w\right)m_t-\dot{m}_t=\frac{\zeta-1}{\varphi_x}x^{\zeta} \tag{A-75}$$

and $\underline{u}_t\left(\Delta\right)$ solves

$$\begin{aligned}\left(r_t+\tau_t+\delta-g_w+(\sigma-1)\left(g_Q-I\right)-I\varepsilon_t\left(\Delta\right)\right)\underline{u}_t\left(\Delta\right)-\dot{\underline{u}}_t\left(\Delta\right) \quad &= \quad h\left(\Delta\right)\frac{1}{\mathcal{M}_t^{\sigma-1}\Lambda_t^{\sigma}}\frac{L_P}{N_t} \qquad \text{(A-76)}\\ &= \quad h\left(\Delta\right)\frac{1}{\mathcal{M}_t^{\sigma-1}\Lambda_t^{\sigma}}\ell_t s_t^P,\end{aligned}$$

where $\ell_t=\frac{L_t}{N_t}$ and $s_t^P=\frac{L_{Pt}}{L_t}$.

A-2.9.3 Free Entry

Free entry requires $v_t^{Entry} = 1/\varphi_E$ so that

$$\frac{1}{\varphi_E} = \alpha\lambda^{\sigma-1}\underline{u}_t\left(\lambda\right) + \left(1-\alpha\right)\overline{\omega}^{\sigma-1}\underline{u}_t\left(\frac{\sigma}{\sigma-1}\right) + m_t. \tag{A-77}$$

This also implies $\dot{v}_t^{Entry} = 0$. From (A-74) this means $\underline{\dot{u}}_t$ and $\dot{m}_t$ satisfy the restriction.

$$0 = \alpha\lambda^{\sigma-1}\underline{\dot{u}}_t\left(\lambda\right) + \left(1-\alpha\right)\overline{\omega}^{\sigma-1}\underline{\dot{u}}_t\left(\frac{\sigma}{\sigma-1}\right) + \dot{m}_t.$$

Together with (A-75), we can use this restriction to solve for m_t in terms of $\underline{u}_t$ as

$$m_t = \frac{\frac{\zeta-1}{\varphi_x}x^{\zeta} - \left(\alpha\lambda^{\sigma-1}\underline{\dot{u}}_t\left(\lambda\right) + \left(1-\alpha\right)\overline{\omega}^{\sigma-1}\underline{\dot{u}}_t\left(\frac{\sigma}{\sigma-1}\right)\right)}{r_t+\tau_t+\delta-g_w}.$$

Substituting this in (A-77) yields

$$\frac{1}{\varphi_E} = \alpha\lambda^{\sigma-1}\left(\underline{u}_t\left(\lambda\right) - \frac{\underline{\dot{u}}_t\left(\lambda\right)}{r_t+\tau_t+\delta-g_w}\right) + \left(1-\alpha\right)\overline{\omega}^{\sigma-1}\left(\underline{u}_t\left(\frac{\sigma}{\sigma-1}\right) - \frac{\underline{\dot{u}}_t\left(\frac{\sigma}{\sigma-1}\right)}{r_t+\tau_t+\delta-g_w}\right) + \frac{\zeta-1}{\varphi_x}\frac{x^{\zeta}}{r_t+\tau_t+\delta-} \tag{A-78}$$

where u_t solves the differential equation (A-76).

Note that two discount rates appear in these equations.

1. First we have $r_t+\tau_t+\delta-g_w+(\sigma-1)\left(g_Q-I\right)$ in (A-76). This can be written as

   $$r_t+\tau_t+\delta-g_w+(\sigma-1)\left(g_Q-I\right) = \rho+g_{L^P}-g_{\Lambda}-\eta+\left(\overline{\omega}^{\sigma-1}+\lambda^{\sigma-1}\frac{\alpha}{1-\alpha}\right)\delta+\left(\overline{\omega}^{\sigma-1}-1+\lambda^{\sigma-1}\frac{\alpha}{1-\alpha}\right)g_t$$

   where $g_{\Lambda} = \dot{\Lambda}_t/\Lambda_t$ and $g_{L^P} = \dot{L}_t^P/L_t^P$. Using $s_t^P = L_t^P/L_t$ we have$g_{s_t^P} = g_{L^P}-\eta$. Hence,

   $$r_t+\tau_t+\delta-g_w+(\sigma-1)\left(g_Q-I\right) = \rho+g_{s_t^P}-g_{\Lambda}+\left(\overline{\omega}^{\sigma-1}+\lambda^{\sigma-1}\frac{\alpha}{1-\alpha}\right)\delta+\left(\overline{\omega}^{\sigma-1}-1+\lambda^{\sigma-1}\frac{\alpha}{1-\alpha}\right)g_t^N,$$

2. Second we have the expression $r_t+\tau_t+\delta-g_w$ in (A-78). This can be written as

   $$r_t+\tau_t+\delta-g_w = \rho+g_{s_t^P}-g_{\Lambda}+\frac{\alpha}{1-\alpha}g_N+\frac{1}{1-\alpha}\delta$$

A-2.9.4 Final Dynamic system

We now derive the final characterization equations characterizing the transitional dynamics. Note first that labor market requires

$$\ell_t\left(1-s_t^P\right)=\frac{1}{\varphi_E}\left(\frac{g_t^N+\delta}{1-\alpha}-\frac{\zeta-1}{\zeta}x\right), \tag{A-79}$$

where $\ell_t=\frac{L_t}{N_t}$ and $s_t^P=\frac{L_{Pt}}{L_t}$.

Using the discount rates defined in Section A-2.9.3, equation A-78 reads

$$\begin{aligned}\frac{1}{\varphi_E} &= \alpha\lambda^{\sigma-1}\left(\underline{u}_t(\lambda)-\frac{\dot{\underline{u}}_t(\lambda)}{\rho+g_{s_t^P}-g_\Lambda+\frac{\alpha}{1-\alpha}g_t^N+\frac{1}{1-\alpha}\delta}\right)+(1-\alpha)\,\overline{\omega}^{\sigma-1}\left(\underline{u}_t\left(\frac{\sigma}{\sigma-1}\right)-\frac{\dot{\underline{u}}_t\left(\frac{\sigma}{\sigma-1}\right)}{\rho+g_{s_t^P}-g_\Lambda+\frac{\alpha}{1-\alpha}g_t^N+\cdots}\right. \\ &\quad+\frac{\zeta-1}{\varphi_x}\frac{x^\zeta}{\rho+g_{s_t^P}-g_\Lambda+\frac{\alpha}{1-\alpha}g_t^N+\frac{1}{1-\alpha}\delta}\end{aligned}$$

where $\underline{u}_t(\Delta)$ solves

$$\left(\rho+g_{s_t^P}-g_\Lambda+\left(\overline{\omega}^{\sigma-1}+\lambda^{\sigma-1}\frac{\alpha}{1-\alpha}\right)\delta+\left(\overline{\omega}^{\sigma-1}-1+\lambda^{\sigma-1}\frac{\alpha}{1-\alpha}\right)g_t^N\right)\underline{u}_t(\Delta)-I\frac{\partial u_t(\Delta)}{\partial\Delta}\Delta-\dot{\underline{u}}_t(\Delta)\;=\;h(\Delta)\cdots$$

This is a differential equation in Δ and t. We have two terminal conditions. For $\Delta\geq\frac{\sigma}{\sigma-1}$ we have

$$h\left(\frac{\sigma}{\sigma-1}\right)=\frac{\frac{\sigma}{\sigma-1}-1}{\left(\frac{\sigma}{\sigma-1}\right)^\sigma}=\left(\frac{1}{\sigma-1}\right)^{1-\sigma}\frac{1}{\sigma^\sigma}=\frac{(\sigma-1)^{\sigma-1}}{\sigma^\sigma}$$

and $\frac{\partial u_t(\Delta)}{\partial\Delta}=0$. Furthermore, we have that $\frac{1}{\mathcal{M}_t^{\sigma-1}\Lambda_t^\sigma}\ell_t s_t^P$ is constant in the steady state so that $\dot{\underline{u}}_t(\Delta)\to 0$.

The transitional dynamics of the system is a path of $\left\{\ell_t,s_t^P\right\}_t$ that solves the equations above. Note that given $\left\{\ell_t,s_t^P\right\}_t$, we can calculate g_t^N from A-79. Given g_t^N we can calculate τ_t and $g_{Q,t}$ as

$$\tau_t = \frac{\alpha}{1-\alpha}\left(g_t^N+\delta\right) \tag{A-82}$$

$$(g_Q-I)(\sigma-1) = \left(\left(\lambda^{\sigma-1}-1\right)\frac{\alpha}{1-\alpha}+\overline{\omega}^{\sigma-1}-1\right)\left(g_t^N+\delta\right). \tag{A-83}$$

As we show in Section , this is also sufficient to compute $\left\{\mathcal{M}_t^{\sigma-1}\Lambda_t^\sigma\right\}_t$ and g_Λ.

A-2.9.5 The Evolution of $F_t(\Delta,q)$: Calculating $\{\mathcal{M}_t,\Lambda_t\}_t$

To calculate $\mathcal{M}_t$ and Λ_t we require the joint distribution of productivity q and quality gaps Δ, $F_t(\Delta,q)$. Note that $\mathcal{M}_t$ and Λ_t only depend on q via $(q/Q)^{\sigma-1}$. Hence, it is useful to characterize the distribution of $F_t(\Delta,\hat{q})$, where $\hat{q}_t=\ln(q_t/Q_t)^{\sigma-1}$. Let $F_t^C(\Delta,\hat{q})$ denote the distribution among products that have a competitor and $F_t^{NC}(\hat{q})$ denote the distribution for products without a competitor.[30] Let N_t^C and N_t^{NC} denote the mass of these products. Let $\hat{F}_t^C(\Delta,\hat{q})=F_t^C(\Delta,\hat{q})\frac{N_t^C}{N_t}$ and $\hat{F}_t^{NC}(\hat{q})=F_t^{NC}(\hat{q})\frac{N_t^C}{N_t}$.

[30] Recall that we do not need Δ for the non-competitor products as they all have a markup of $\frac{\sigma-1}{\sigma}$.

If we have the full evolution of $\left\{N_t, \hat{F}_t^C(\Delta,\hat{q}), \hat{F}_t^{NC}(\hat{q})\right\}_t$ we can calculate Λ_t and $\mathcal{M}_t$ as

$$\begin{aligned}
\Lambda_t &= \frac{N_t \int \mu(\Delta)^{-\sigma} e^{\hat{q}} d\hat{F}_t^C(\Delta,\hat{q}) + N_t \left(\frac{\sigma}{\sigma-1}\right)^{-\sigma} \int e^{\hat{q}} d\hat{F}_t^{NC}(\hat{q})}{N_t \int \mu(\Delta)^{1-\sigma} e^{\hat{q}} d\hat{F}_t^C(\Delta,\hat{q}) + N_t \left(\frac{\sigma}{\sigma-1}\right)^{1-\sigma} \int e^{\hat{q}} d\hat{F}_t^{NC}(\hat{q})} \\
\mathcal{M}_t &= \frac{\left(N_t \int \mu(\Delta)^{1-\sigma} e^{\hat{q}} d\hat{F}_t^C(\Delta,\hat{q}) + N_t \left(\frac{\sigma}{\sigma-1}\right)^{1-\sigma} \int e^{\hat{q}} d\hat{F}_t^{NC}(\hat{q})\right)^{\frac{\sigma}{\sigma-1}}}{N_t \int \mu(\Delta)^{-\sigma} e^{\hat{q}} d\hat{F}_t^C(\Delta,\hat{q}) + N_t \left(\frac{\sigma}{\sigma-1}\right)^{-\sigma} \int e^{\hat{q}} d\hat{F}_t^{NC}(\hat{q})},
\end{aligned}$$

where $e^{\hat{q}} = (q_t/Q_t)^{\sigma-1}$ as $\hat{q}_t = \ln(q_t/Q_t)^{\sigma-1}$ and $\mu(\Delta)$ is the markup function $\mu(\Delta) = \min\left\{\frac{\sigma}{\sigma-1}, \Delta\right\}$.

We now derive expressions to calculate the evolution of $\left\{N_t, \hat{F}_t^C(\Delta,\hat{q}), \hat{F}_t^{NC}(\hat{q})\right\}_t$. Let $\left(N_0^C, N_0^{NC}, \hat{F}_0^C(\Delta,\hat{q}), \hat{F}_0^{NC}(\hat{q})\right)$ be given. In practice these objects are determined in the initial BGP. This in particular implies that $N_0^C = \alpha N_t$ and $N_0^{NC} = (1-\alpha) N_t$. Suppose a path for $\{g_t^N\}$ is given. Then we can calculate and $\{\tau_t\}$ and $\{g_{Qt}\}$ from (A-82) and (A-83)

1. Given $\{\tau_t\}$ and $\{g_{Qt}\}$, we can calculate $\left\{\hat{F}_t^C(\Delta,\hat{q}), \hat{F}_t^{NC}(\hat{q})\right\}_t$ as follows:

   (a) The evolution of $\hat{F}_t^{NC}(\hat{q})$ is given by

   $$\frac{\partial \hat{F}_t^{NC}(\hat{q})}{\partial t} = -g_{\hat{q}} \frac{\partial \hat{F}_t^{NC}(\hat{q})}{\partial \hat{q}} - \left(\tau_t + \delta + g_t^N\right) \hat{F}_t^{NC}(\hat{q}) + \left(\frac{1-\alpha}{\alpha}\right) \tau_t \Gamma\left(\exp\left(\frac{\hat{q}}{\sigma-1}\right)\right),$$

   where $g_{\hat{q}} = (\sigma-1)(I - g_{Qt})$ is given in (A-83) and $\Gamma\left(\exp\left(\frac{\hat{q}}{\sigma-1}\right)\right)$ is the exogenous source distribution.

   (b) Given $\left\{\hat{F}_t^{NC}(\hat{q})\right\}_t$ we can solve for $\left\{\hat{F}_t^C(\Delta,\hat{q})\right\}_t$. In particular, $\left\{\hat{F}_t^C(\Delta,\hat{q})\right\}_t$ then solves the differential equation

   $$\frac{\partial \hat{F}_t^C(\Delta,\hat{q})}{\partial t} = -\Delta I \frac{\partial \hat{F}_t^C\left(\Delta,\hat{q}\right)}{\partial \Delta} - g_{\hat{q}} \frac{\partial \hat{F}_t^C\left(\Delta,\hat{q}\right)}{\partial \hat{q}} - \left(\tau + \delta + g_t^N\right) \hat{F}_t^C(\Delta,\hat{q}) + \lim_{s\to\infty} \tau \hat{F}_t^C(s,\hat{q}-\hat{\lambda}) + \tau \hat{F}_t^{NC}(\hat{q}-\hat{\lambda}$$

   where $\hat{\lambda} = \ln \lambda^{\sigma-1}$. Given that we solved for $\hat{F}_t^{NC}(\hat{q}-\hat{\lambda})$ already, this determines $\left\{\hat{F}_t^C(\Delta,\hat{q})\right\}_t$ given an initial condition $\hat{F}_0^C(\Delta,\hat{q})$

A-2.9.6 Firm-level moments along the transition

Given the equilibrium path $\{g_{N,t}, z_t\}$ we can compute the time series of the entry rate and average firm size. To do so, let $\omega_t(n)$ be the mass of firms with n products at time t. Consider $n \geq 2$. Then

$$\dot{\omega}_t(n) = \underbrace{\omega_t(n-1)(n-1)x}_{\text{From } n-1 \text{ to} n \text{ products}} + \underbrace{\omega_t(n+1)(n+1)(\tau+\delta)}_{\text{From } n+1 \text{ to} n \text{ products}} - \underbrace{\omega_t(n)\, n(\tau+x+\delta)}_{\text{From } n \text{ to } n-1 \text{ or } n+1 \text{ products}}.$$

For $n = 1$ we have

$$\dot{\omega}_t(1) = Z_t + \omega_t(2)\, 2(\tau+\delta) - \omega_t(1)(\tau+x+\delta).$$

Let $\nu_t(n) = \frac{\omega_t(n)}{N_t}$, which is stationary along the BGP. Then

$$\begin{aligned}
\frac{\dot{\omega}_t(n)}{N_t} &= \nu_t(n-1)(n-1)x + \nu_t(n+1)(n+1)(\tau+\delta) - \nu_t(n)n(\tau+x+\delta) \\
\frac{\dot{\omega}_t(1)}{N_t} &= z_t + \nu_t(2)2(\tau+\delta) - \nu_t(1)(\tau+x+\delta).
\end{aligned}$$

Now

$$\dot{\omega}_t(n) = \dot{\nu}_t(n) N_t + \nu_t(n) \dot{N}_t$$

so that

$$\frac{\dot{\omega}_t(n)}{N_t} = \dot{\nu}_t(n) + \nu_t(n) g_{N,t}$$

Hence,

$$\begin{aligned}
\dot{\nu}_t(n) &= \nu_t(n-1)(n-1)x + \nu_t(n+1)(n+1)(\tau+\delta) - \nu_t(n)n(\tau+x+\delta) - \nu_t(n) g_{N,t} \\
\dot{\nu}_t(1) &= z_t + \nu_t(2)2(\tau+\delta) - \nu_t(1)(\tau+x+\delta) - \nu_t(1) g_{N,t}.
\end{aligned}$$

Given an initial condition$\{\nu_0(n)\}_n$we can calculate the evolution of $\{\nu_t(n)\}_n$ for given $\{g_{N,t}, z_t\}$. Given $\{\nu_t(n)\}_n$ we can calculate some objects:

1. The number of firms at time t:

$$F_t = \sum_{n=1}^{\infty} \omega_t(n) = N_t \sum_{n=1}^{\infty} \nu_t(n)$$

   and hence average firm size L_t/F_t

2. The entry rate

$$Entry - rate_t = \frac{z_t N_t}{F_r} = \frac{z_t}{\sum_{n=1}^{\infty} \nu_t(n)}.$$

3. The exit rate

$$Exit - rate_t = \frac{\tau_t N_t \nu_t(1)}{F_t}.$$

A-2.10 The Impact of Falling Population Growth: The Model with Endogenous Own-Innovation

We calibrate the model under this specification, and find that quantitatively the slowdown in growth is significantly amplified by this endogenous response of own innovation. The aggregate consequences appear in Table A-3.