Behavioral Influence*

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January 22, 2019

Abstract

In the context of stochastic choice, we introduce and behaviorally characterize a choice theoretic model which admits a notion of interactive influence among individuals. The model presumes that individual choice is not only determined by idiosynratic evaluations of the alternatives but also by the influence from other individuals. We establish that the model is uniquely identified; hence, the degree of influence can be inferred from the observable choice behaviors. We also show that the behavior produced by our model constitutes a stable equilibrium when embedded in a dynamic environment.

Keywords: Social influence, peer effects, stochastic choice, conformity.

JEL classification numbers: D01; D91.

*We thank Miguel Ballester, Yoram Halevy, Paola Manzini, Marco Mariotti, Irina Merkurieva and John Quah for their valuable comments.

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1 Introduction

It is well-known that individual choices are directly influenced by the choices of others. Behavioral evidence on whether social interactions alter individual behavior is conclusive and indisputable. Examples abound. Peer behavior has a significant influence not only on a student's school achievement [Calvo-Armengol et al., 2009], but also on social behavior such as consumption of recreational activities, drinking, smoking, etc. [Sacerdote, 2011]. High productivity co-workers are found to increase one's own productivity [Mas and Moretti, 2009]. Involvement in crime [Glaeser et al., 1996], job search [Topa, 2001], adolescent pregnancy [Case and Katz, 1991], college major choice [De Giorgi et al., 2010] are other prominent examples in which social interactions are shown to be crucial constituents of individual behavior. An abundance of empirical evidence corroborates the role of social influence on individual behavior. We know that *it happens*. What is less clear is *how it happens*. How exactly does influence from others alter one's behavior? More importantly, viewing behavior as resulting from an unobservable cognitive process, how can we identify the extent to which one's behavior is attributed to influence as opposed to one's own preferences?

In this paper, we answer these two questions via a microfoundational approach. We propose and characterize a simple decision making procedure for interacting individuals. Our main contribution is to provide an intuitive and tractable choice model which affords a meaningful, and measurable, definition of "influence" as derived from choice behavior alone. Hence, our work provides the first meaningful behavioral language for discussing influence in an abstract framework.

Before introducing the setting, we lay out the basic principles of social influence on which we base our model. First of all, we presume that influence alters the way individuals evaluate the alternatives. Individuals possess idiosynratic "preferences" as usual. However, influence from others directly distorts their perception of these alternatives instead of modifying the choice set.[1] This is in line with many findings from the social psychology or experimental

[1] The traditional assumption of exogenous and fixed individual preferences has also been frequently challenged by economic theorists over the last couple of decades; see, for instance, Bowles [1998], Bisin and Verdier [2001], Bar-Gill and Fershtman [2005], Doepke and Zilibotti [2017]. Closer to our perspective is Fehr and Hoff [2011], who argue that individual preferences are susceptible to social effects via cognitive channels such as framing, anchoring and identity effects.

economics literatures. For instance, Kremer and Levy [2008] show that alcohol consumption by one roommate is more likely to influence the alcohol consumption of another roommate via a preference change rather than a modification of the choice set. Kenrick and Gutierres [1980] show that individual evaluations of physical attractiveness of random people are directly altered by evaluations from peers. According to the notion of (mis)identification in social psychology, when some alternatives become identified with certain identities, they become more likely to be preferred by aspiring individuals, whereas despising individuals avoid them in order not to be misindentified [Berger, 2016].

Second, not everybody is equally influential toward a given individual. Individuals have different levels of susceptibility to influence from different agents: Aral and Walker [2012] investigate this heterogeneity over social media networks, Frey and Meier [2004] for prosocial behavior of university students and Glaeser et al. [1996] for criminal behavior.[2]

Lastly, we treat influence as a mutual notion in line with the entire literature on social interactions.[3] Not only is one influenced by her peers, but she also possesses the potential to influence them.

Our model lives in a stochastic set up and, for brevity, takes the classical Luce model [Luce, 1959] of stochastic choice as a benchmark. In Luce's model, each alternative is parameterized by a "weight" reflecting its strength of choice for the decision maker. Choice probabilities from any given budget are always proportional to the strength of choice. Our model likewise relies on a weight, reflecting its strength of choice and choice probabilities. However, the strength of choice in our model not only accommodates an idiosyncratic component but also a component incorporating influence. The idiosyncratic strength of choice is directly modified by a weighted version of the other individuals' observable choice probabilities from the same budget. Thus there are two parameters in our model. One parameter is each individual's idiosyncratic strength of choice (or Luce weights). The other parameter is each individual's measure of the degree of influence. The higher is this measure, the more an

[2]The majority of the literature on social interactions refers to one's reference group as the main source of influence. Although we do not model reference groups explicitly, our framework is entirely in line with this view. Moreover, our identification strategy in the multi-individual setting of Section 3 enables the revelation of reference groups from observable choice behavior.

[3]Although the mutuality of influence is acknowledged, for some social interaction models tractability requires that the individual ignores her own effect on the society. See, for instance, Brock and Durlauf [2003].

individual's behavior conforms with others' behavior. Absent influence, our model reverts to Luce's. On the other hand, with influence our model diverges from Luce's model.

The main result of our paper is a characterization of the model in terms of observed stochastic choice alone. The power of our representation theorem lies in its implications for an outside observer. First, testable properties enable an outside observer to detect interacting individuals from observable choice frequencies alone. Second and more importantly, owing to the unique identification granted by the representation, the outside observer can fully disentangle underlying idiosyncratic motives (preferences) from social motives (influence). Unique identification of the "hidden" motives is considered crucial for policy and welfare purposes.[4]

A secondary result imagines a dynamic adjustment procedure. When two individuals first interact, in general we have no particular reason to suppose that their behavior conforms to our model. We show under reasonably general conditions that, through time, as each individual responds to the other's choices via the linear aggregation procedure, the predictions of our model will be borne out. In other words, if we believe that each individual aggregates behaviorally according to our procedure, we should expect their behavior to conform to our model in the long-run. The result also illustrates that our model is *stable*. If one individual mistakenly chooses, or one of them misobserves the other's choices, still their behavior will revert to the predictions of our model.

Let us provide an example demonstrating the basic idea of our model.

An example: Consider two colleagues, Dan and Bob, who potentially influence each other the way that they receive their news. There are two different online news sources: BBC (B) and Daily Mail (D). Their browser histories suggest that Dan uses B approximately 71% of the time whereas this frequency is 78% for Bob, as summarized on the left panel of the table below. Assume that these browsing frequencies constitute the only information available to an outside observer, who aims to learn about the underlying preferences as well

[4]The economics literature on identification of social interactions has developed many econometric tools and techniques to detect the direct effect of peers on one's choices, and more importantly to differentiate the direct behavioral influence from other related effects, such as correlation in tastes (see Blume et al. [2011] for a comprehensive review). Our paper contributes to this literature by focusing on microfoundations. Studying *how* interacting individuals choose allows us to deduce the unobservables from the observables.

as peer influence over online news sources by means of revealed preference analysis. Absent any further information, one might be tempted to disregard peer influence and think that these choice frequencies directly capture the underlying preferences, as in the classical Luce model. Hence, it is natural to infer that both Dan and Bob prefer B to D. However, when a new online news source, the Conversation (C) is launched, these frequencies change, as presented on the right panel:

	Dan	Bob	Dan	Bob
BBC	0.71	0.78	0.60	0.70
Conversation	-	-	0.14	0.11
Daily Mail	0.29	0.22	0.26	0.19

This pair of behaviors are consistent with our model, hence we can reveal the underlying preferences and the interaction parameters uniquely. Interestingly, our identification implies that although Dan and Bob's choice frequencies have the same ranking over the news sources, their idiosyncratic preferences are not aligned. For Bob, indeed the weight of B is the highest and D is the lowest, whereas for Dan, the exact opposite holds. However Bob's behavior has great influence on Dan. To be precise, the weights of B, C, D for Dan and Bob are $0.1, 0.3, 0.6$ and $0.8, 0.08, 0.12$, with interaction parameters 5 and 1, respectively. This means for Dan, Bob's behavior is five times more important then his own subjective weights, whereas for Bob they are equivalent. Thus strong conformity motives have resulted in the observed behavior. □

The example suggests that the presence of other individuals allows us to infer different information about preferences than we would if we supposed the individual to be alone. A natural question is whether *any* individual choice behavior could be rationalized by the presence of some unobserved individual's influence. While this question is certainly interesting, it is beside the point of what we are doing. Our model postulates a given, observable set of individuals, and tests *whether* these individuals' behavior is in line with our predictions. This is much in the same spirit as the theory of consumer choice. Afriat [1967] characterizes the empirical content of such choice, but Varian [1988] shows that, in principle, if some commodities are unobservable, then any behavior can be rationalized. The tradition in this literature is to test, for a fixed set of assets, whether data can be rationalized. In general, the more data one observes, the more restrictive are the predictions on the model.

The special case of our model in which there is no influence coincides with the Luce model. In the Luce model, the weighting function is invariant to rescaling; its units have no empirical meaning. In our model, on the other hand, we interpret the weight as the probability that a given alternative would have been chosen in isolation *even though* we never actually observe choice in isolation.

Our baseline model involves two individuals with conformity motives, as in the example above. An action's choice probability increases as the action is chosen more frequently by one's peer. However our model easily adapts to more individuals. We present a simple extension, incorporating multi-individual interaction, where an individual have different degrees of dependence on the behaviors of her peers. Despite it simplicity, our model is versatile enough to capture a wide range of social phenomena involving interactions. Let us provide a couple of applications to exemplify this.

Homophily: Homophily refers to the tendency to create social ties with people that are similar to one's self [McPherson et al., 2001, Blackwell and Lichter, 2004, Currarini et al., 2009]. Since both homophily and peer-influence result in behavioral resemblance among peers, an identification problem arises. For instance, consider a group of high school students with a tendency towards delinquent behavior. The interpretation is twofold: It might be the case that all of these kids have high aspirations towards criminal behavior (and that is why they hang out together). Or it might as well be the case that conformity motives with one (or some) influential members have resulted in this group behavior. Diagnosing the correct interaction dynamic is imperative for effective treatment of the issue. Since our model allows for unique and full identification of the underlying parameters, it enables the differentiation of homophily (similar underlying preferences) from peer-influence (high conformity parameters), as long as the observed behavior satisfies the characterizing properties of our model.

Social norms: One of the most fundamental concepts in the study of social influence is that of social norms. Thanks to the versatility of our model, we can accommodate different attitudes towards social norms in our framework. We can treat the behavior induced by social norms as the behavior of an exogenous hypothetical individual. In this case, different levels of compliance with social norms can easily be captured by different individual interaction parameters. Alternatively, the formation of social norms can also be modelled in our framework, in the dynamic setting of Subsection 2.1. In that case, we can treat social norms as a hypothetical individual, but this time with a high level of dependence on the behavior of the others. The equilibrium then pictures a society with established and stable social norms and yet different levels of compliance with them.

Economics research on social interactions has mainly utilized econometrics tools and techniques both for theoretical and empirical works. Most of these studies employ *linear social interaction models* [Manski, 1993, Blume et al., 2011, Jackson, 2011, Blume et al., 2015], where individual utility of an action is defined as a linear additive function with two components: an individual private utility and a social utility. Social utility depends on the (expected) behaviors of one's peers. Linear social interaction models are defined for continuous choice variables. An alternative to this is developed by incorporating the linear additive utility function with interaction effects into a discrete choice setting [Blume, 1993, Brock and Durlauf, 2001, 2003]. Binary or multinomial discrete choice models with social interactions make use of random fields models to study the equilibrium. Three critical assumptions ensure tractability of the model. First, the assumption of constant strategic complementarity: the cross-partial of social utility is a positive constant that is the same for all individuals. Second, rational expectations: the expected average behavior is simply the objective average behavior. Finally, the error terms follow a relevant extreme value distribution. This assumptions are sufficient to produce individual choice outcomes that are consistent with logistic choice with multiple equilibria. The majority of these papers assume large populations in order to justify the assumption that each individual ignores the effect of their own choice on the average choice of the society. An exception to this is Soetevent and Kooreman [2007], where they consider interaction in small groups in which choices of other individuals is fully observable. Thus, the choice of an individual directly depends on the observed behavior of the others. Our model also uses this intuition. It is interesting to note that under certain assumptions the behavior produced by a multinomial discrete choice model with social interactions coincides with the behavior produced by our model. We clarify this connection in Subsection 2.2.

The use of micro-foundational tools to study social interactions is quite recent. As far as we know the first choice-theoretic work investigating influence across individuals is Cuhadaroglu [2017]. This work introduces a deterministic model of two-stage optimization where the first stage involves maximization of own preferences (transitive but not necessarily complete), and the second stage accommodates social influence to further refine first stage outcomes. Recently, Borah and Kops [2018] propose a choice procedure in a group setting that makes use of 'a consideration set' approach. According to their model, individuals only consider those alternatives that are chosen sufficiently enough by the members of their reference group. Then, in a second stage, they choose their personal best out of those considered. The main difference of our work from this model is about the channel through which others' behavior influence the individual. Our model presumes that social influence alters one's preferences, whereas Borah and Kops' model assume a limitation of the choice set due to social influence.

Fershtman and Segal [2018] also consider a social interaction set up where individual behavior not only depends on one's own preferences but also on the behavior of other agents. Each individual possesses a private vNM utility and a perfectly observable vNM utility. A social influence function converts the private utility of the agent and the observable utilities of everyone else to an observable utility for the agent. They study certain properties of social influence functions and their implications for the equilibrium without proposing an explicit behavioral model.

Finally, our work is related to the literature discussing the revealed preference implications of solution concepts in games; for example, Sprumont [2000], Lee [2012]. One interpretation of the mathematics of our model is as formalizing, for each choice set, a game and a solution concept. Thus, our model provides observable predictions of our concept as strategy sets vary. The aforementioned papers also study the predictions of game theory as strategy sets vary.

The organization of the paper is as follows. The next section introduces the baseline model with two individuals. In subsection 2.1 we discuss the stability properties of the model in a dynamic set up. Subsection 2.2 is devoted to the link between our model and the discrete choice models developed in the empirical social interactions literature, whereas subsection 2.3 discusses the connection to Quantal Response Equilibrium. Section 3 presents an extension of the baseline model to multi-agent settings. All proofs are left to an appendix.

2 The Baseline Model

Let X be a finite set of *alternatives*. There are two individuals, 1 and 2. A *stochastic choice rule* is a map $p : 2^X \setminus \{\emptyset\} \to \bigcup_{E \in X} \Delta_{++}(E)$ such that for all $E \subseteq X$, $p(E) \in \Delta_{++}(E)$.[5]

We propose a simple model of influence. Each individual is influenced by the choices of the other individual. We first consider the extreme case where there is no influence between individuals. For such cases our model boils down to the classical Luce model [Luce, 1959].

Let us revisit the Luce model. In this model, each alternative has a (subjective) decision weight $w(x)$, which measures the strength of preferences associated with the alternative x.[6] Then the probability of choosing x is written as the relative weight of preference strength compared to other alternatives in the choice set. Formally,

Definition. A stochastic choice rule p has a **Luce** representation if there exists a weight function $w : X \to (0, 1)$ with $\sum_{x \in X} w(x) = 1$ such that

$$p(x, S) = \frac{w(x)}{\sum_{y \in S} w(y)}$$

for all $x \in S, S \in 2^X \setminus \emptyset$.

The Luce model has a very simple graphical representation. Consider three alternatives x, y, and z. The simplex in Figure 1 illustrates a stochastic choice rule represented by a Luce model. Each vertex of the simplex represents one of three alternatives. To be more precise, the vertex x ($(1, 0, 0)$) represents the degenerate distribution on $\{x, y, z\}$ where x is chosen with probability one from the choice set $\{x, y, z\}$. The points closer to any vertex represent a relative preference for the corresponding alternative. The solid dot in the interior of the simplex represents the choice distribution from $\{x, y, z\}$, which is a 3-dimensional

[5] The notation Δ_{++} refers to the set of probability distributions with full support.

[6] Bradley and Terry [1952] introduced the same model for binary choices.

vector: $(p(x|\{x,y,z\}), p(y|\{x,y,z\}), p(z|\{x,y,z\}))$. We abuse the notation and denote this 3-dimensional vector by $\mathbf{p}(xyz)$. The little squares on the left, right and bottom sides of the triangle represent the binary probabilities $\mathbf{p}(xz)$, $\mathbf{p}(yz)$, and $\mathbf{p}(xy)$, respectively. In Figure 1, $\mathbf{p}(xyz)$ lines up with $\mathbf{p}(xy)$, $\mathbf{p}(yz)$, and $\mathbf{p}(xz)$ perfectly. These imply a strong relationship in this model:

$$\frac{p(x,S)}{p(y,S)} = \frac{p(x,T)}{p(y,T)}$$

for all $x, y \in S \cap T$. That is, the ratio of the probability of choosing one alternative to the probability of choosing another should be constant regardless of the context.

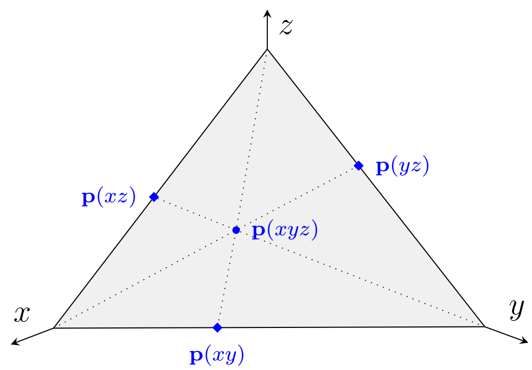


Figure 1: Graphical illustration of a Luce Model (No Influence)

Let us now introduce our baseline model for two individuals who are potentially influenced by each other. The primitive of the model is a pair of stochastic choice rules (p_1, p_2) where p_i stands for individual i's choices. We use the notation $i, j \in \{1, 2\}$ with $i \neq j$ for the individuals in general. Throughout the paper, we focus our attention to the pair of stochastic choice rules with some variation in the overall behavior, i.e., $p_1 \neq p_2$.[7]

We postulate that the choice behavior of individual j regarding an alternative $x \in S$ directly influences individual i's evaluation of that alternative for the same choice set. Specifically individual i now assigns $w_i(x) + \alpha_i p_j(x, S)$ as the subjective weight of x where α_i is the degree of influence of j on i. We assume that $\alpha_i \geq 0$, hence α_i acts as a conformity parameter. The higher the probability that j chooses x from S, the higher is i's evaluation

[7]This is because one cannot learn much from the data when $p_1 = p_2$. Having exactly the same behavior in any choice set might be due to identical preferences of 1 and 2, i.e, $w_1 = w_2$; or it might be because one of the individuals only cares about imitating the other individual. It is not possible to distinguish between these cases without any additional information such as their choice behavior in isolation.

of x in S and hence i chooses x with a higher probability as well. Let us now define our model formally.

Definition. (p_1, p_2) has a **dual interaction** representation if there exist two functions $w_1, w_2 : X \to (0,1)$, $w_2 : X \to (0,1)$ with $\sum_{x \in X} w_1(x) = \sum_{x \in X} w_2(x) = 1$ and $\alpha_1, \alpha_2 \in \Re^+$ such that

$$p_i(x, S) = \frac{w_i(x) + \alpha_i p_j(x, S)}{\sum_{y \in S} [w_i(y) + \alpha_i p_j(y, S)]}$$

for all $x \in S, S \in 2^X \setminus \emptyset$ and $i, j \in \{1, 2\}$ with $j \neq i$.

When (p_1, p_2) has a dual interaction representation with parameters $(w_1, w_2, \alpha_1, \alpha_2)$, we say that $(w_1, w_2, \alpha_1, \alpha_2)$ *represent* (p_1, p_2).

Given parameters of the individual's preferences $\{(w_i, \alpha_i)\}_{i=1,2}$, each p_i is defined implicitly by the procedure above. Note that p_1 is not explicitly defined: p_2 needs to be known in order to determine p_1 and vice versa. However we can obtain an explicit representation by solving the system of simultaneous equations, to arrive at:

$$p_i(x, S) \equiv \lambda_i(S) \frac{w_i(x)}{\sum_{x \in S} w_i(x)} + (1 - \lambda_i(S)) \frac{w_j(x)}{\sum_{x \in S} w_j(x)}$$

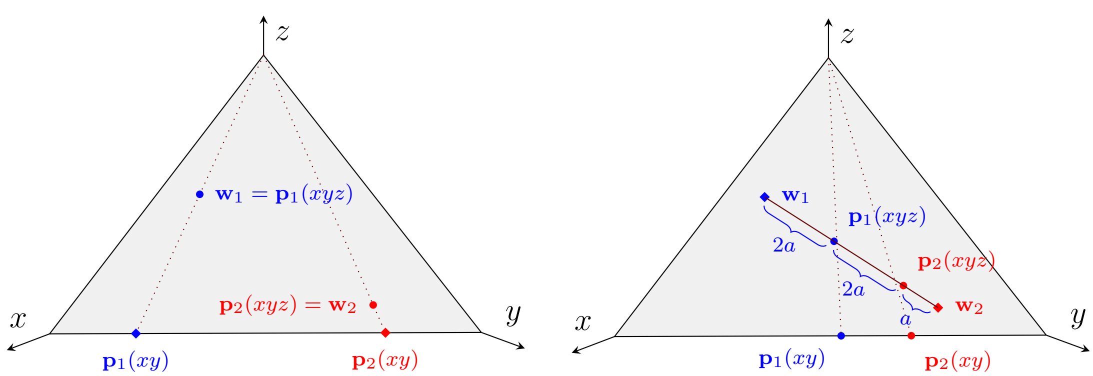
for $\lambda_i(S) \in (0, 1)$, defined explicitly below. Hence each p_i can be expressed as a linear combination of their Luce ratios. This is "as if" each individual knows exactly not only her own Luce's weights but also those of the other individual, which are not necessarily observable. Notice that in our original formulation, each individual utilizes each others' observable choice probabilities rather than their unobservable Luce's weights. We believe influence based on observed behaviour is more plausible. Nevertheless, this explicit formulation provides more insight about the model. Here, the weight attached to each individual's Luce ratio depends on the budget set. That is,

$$\lambda_i(S) = \frac{w_i(S)[w_j(S) + \alpha_j]}{w_i(S)w_j(S) + \alpha_i w_j(S) + \alpha_j w_i(S)}$$

where $w_i(S) = \sum_{x \in S} w_i(x)$. $\lambda_i(S)$ is decreasing in α_i and increasing in α_j. In other words, the more influenced by the other person the more weight attached to other individual's Luce ratio. In the extreme case, when $\alpha_i = 0$, $\lambda_i(S)$ is equal to 1, independent of the budget set.

Notice also that this formulation marks out the uniqueness of the behaviour produced by the model. In other words for a given $(w_1, w_2, \alpha_1, \alpha_2)$, there is a unique pair (p_1, p_2) consistent with the dual interaction model.

We illustrate our model in Figure 2. The first panel refers to the case that there is no interaction between 1 and 2. Hence we are back to the classical Luce model. The little squares in the interior of the simplex in the second panel correspond to Luce weights for the individuals, w_1 and w_2. We denote these vectors by $\mathbf{w}_1$ and $\mathbf{w}_2$, respectively. Observe that all $\mathbf{w}_1$, $\mathbf{w}_2$, $\mathbf{p}_1(xyz)$ and $\mathbf{p}_2(xyz)$ are on the same line. This is due to the linear structure of our model. Since $\mathbf{p}_1(xyz)$ is between $\mathbf{w}_1$ and $\mathbf{p}_2(xyz)$, and $\mathbf{p}_2(xyz)$ is between $\mathbf{w}_2$ and $\mathbf{p}_1(xyz)$, individual 1 is positively influenced by individual 2 and vice versa. Another observation is that $\mathbf{p}_1(xyz)$ is the mid-point of $\mathbf{w}_1$ and $\mathbf{p}_2(xyz)$. This implies that individual 1 treats his own weights and the choices of individual 2 equally. Hence, individual 1's imitation parameter is 1 ($\alpha_1 = 1$). On the other hand, $\mathbf{p}_2(xyz)$ is closer to $\mathbf{w}_2$ than $\mathbf{p}_1(xyz)$, which indicates that individual 2 puts less weight on individual i's choices. Since the distance between $\mathbf{p}_1(xyz)$ and $\mathbf{p}_2(xyz)$ twice as much as the distance between $\mathbf{p}_2(xyz)$ and $\mathbf{w}_2$, the imitation parameter of individual 2 is 0.5. Unlike the Luce model, $\mathbf{p}_i(xyz)$ does not line up with $\mathbf{p}_i(xy)$ for all i.



$\alpha_1 = \alpha_2 = 0$ $\qquad$ $\alpha_1 = 1$ and $\alpha_2 = 0.5$

Figure 2: Graphical illustration of Dual Interaction Model

The components w_i and α_i are not directly observable. We will illustrate how to reveal each component from the observed behavior. We first try to understand the empirical implications of this model. To do this, define, for each $i = 1, 2$, for any pair (x, S) with $x \in S$,

$d_i : (x, S) \mapsto \Re$, by

$$d_i(x, S) := p_i(x, S) - p_i(x, X).$$

The quantity $d_i(x, S)$ is simply the change in the probability of i's choosing x as the set X shrinks to S. In the Luce model, i.e. when each $\alpha_i = 0$, this change is always nonnegative. In a larger set, there are more alternatives from which to choose. In the dual interaction model, this change instead is governed by two separate effects. First, there is a direct effect, corresponding to the effect in the Luce model. A smaller set includes less alternatives, rendering any given alternative relatively more attractive. In addition, there is also an "indirect" effect imposed by the direct effect on the other individual's choice probability. Since $\alpha_i > 0$, as the set shrinks, the indirect effect contributes to the gain in choice probability of any given alternative.

Our interaction model enjoys a linear structure. The characterizing axioms highlight this linearity. All three properties impose conditions on two variables $\beta_1(x, y, S)$ and $\beta_2(x, y, S)$ that are driven from the choice behaviours, p_1, p_2 as follows:

Let $i \neq j$. For any $S \neq X$, and any $x, y \in S$ for which $x \neq y$, define

$$\beta_i(x, y, S) \equiv \frac{\frac{d_i(x,S)}{p_i(x,S)} - \frac{d_i(y,S)}{p_i(y,S)}}{\frac{d_j(x,S)}{p_i(x,S)} - \frac{d_j(y,S)}{p_i(y,S)}}.$$

Observe that, for any x, S, $\dfrac{d_i(x, S)}{p_i(x, S)}$ is the percentage change in agent i's choice probability of x in expanding S to X. So, $\dfrac{d_i(x, S)}{p_i(x, S)} - \dfrac{d_i(y, S)}{p_i(y, S)}$ is a differential in percentage changes. On the other hand, $\dfrac{d_j(x, S)}{p_i(x, S)}$ is a bit more subtle. It reflects a differential change in choice probability of x by agent j, normalized by the choice probabilities of i. Recall that we are trying to capture a direct effect of i's choice behavior on j's behavior. To this end, this seems to be a relevant quantity, if we believe that individual i's choice probability enters linearly into j's behavior. Thus, in very rough terms, $\beta_i(x, y, S)$ is a measure of differential cross-elasticity of choice probabilities in expanding the set S to X.

Three independent properties on $\beta_i(x, y, S)$ characterize the dual interaction model. Let $i \in \{1, 2\}$:

Axiom 1 (Independence) *$\beta_i(x, y, S)$ is independent of S, x, and y.*

Axiom 2 (Uniform Boundedness) *$\beta_i(x, y, S) < \min_{z \in X} \left\{ \dfrac{p_i(z, X)}{p_j(z, X)} \right\}$ for all $S \neq X$, and $x, y \in S$ with $x \neq y$.*

Axiom 3 (Non-negativeness) *$\beta_i(x, y, S) \geq 0$ for all $S \neq X$, and $x, y \in S$ with $x \neq y$.*

Independence is the property that restores the additive linear influence structure among individuals. Uniform Boundedness guarantees that the idiosynratic evaluations of alternatives, w_i are positive. And finally, Non-negativeness restricts the interaction among individuals to conformity behaviour rather than diversification. Now, let us state the representation theorem for our baseline model:

Theorem 1 *Suppose that $p_1 \neq p_2$. Then (p_1, p_2) has a **dual interaction** representation with nonnegative $\alpha_i \geq 0$ and $w_i \gg 0$ for each i if and only if it satisfies Independence, Uniform Boundedness, and Non-negativeness. If in addition $\{(w_i, \alpha_i)\}_{i=1,2}$ and $\{(w'_i, \alpha'_i)\}_{i=1,2}$ each represent (p_1, p_2), then $\alpha_i = \alpha'_i$ and $w_i = w'_i$ for all x and i.*

Theorem 1 is a strong result. For a pair of stochastic choice behaviours, three properties are not only necessary and sufficient for consistency with an underlying dual interaction model but also the unobservable parameters, preferences and levels of influence, are identified uniquely. The proof constructs the model thanks to the structure granted by Independence and by the help of restrictions imposed by the remaining two axioms. We take $\alpha_i(x, y, S) := \alpha_i = \dfrac{\beta_i}{1 - \beta_i}$ (well-defined by Axioms 1 and 2 and non-negative by Axioms 2 and 3) and $w_i(x) := p_i(x, X) + \alpha_i(p_i(x, X) - p_j(x, X))$ (positive by Axiom 2). We then show that for any S and $x, y \in S$, Independence builds up to

$$\frac{p_i(x, S)}{p_i(y, S)} = \frac{w_i(x) + \alpha_i p_j(x, S)}{w_i(y) + \alpha_i p_j(y, S)}.$$

The fact that this holds for each pair of alternatives immediately gives us the dual interaction model. Uniqueness of $\{(w_i, \alpha_i)\}_{i=1,2}$, on the other hand, follows from the rank of system of equations.

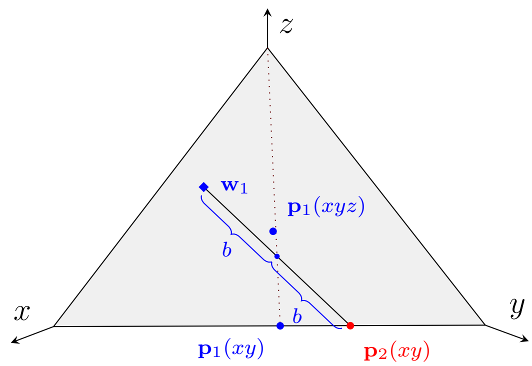


Figure 3: The dynamic adjustment procedure

2.1 Stability

The dual interaction model involves a dynamic adjustment procedure where an individual's evaluation of an alternative is adjusted by each other's behaviour as well as the level of susceptibility to influence. Figure 3 illustrates this dynamic procedure. The point $\mathbf{w_1}$ reflects individual 1's idiosyncratic weights, as usual. Now, suppose individual 1 observes individual 2's choice behavior from the set $\{x, y\}$, $\mathbf{p_2}(xy)$. Suppose that individual 1 cares about 2's behavior as much as her own tastes, e.i., $\alpha_1 = 1$. Then the geometric interpretation of $\mathbf{p_1}(xy)$ is as follows: on the line segment connecting $\mathbf{w_1}$ and $\mathbf{p_2}(xy)$, find the midpoint (owing to the fact that $\alpha_1 = 1$; a different weight would be reflected in a different proportion on the segment). The point $\mathbf{p_1}(xy)$ is then the projection from the z-vertex of this midpoint. In the figure, $\mathbf{p_1}(xyz)$ is also illustrated, to emphasize the point that *it* need not project onto $\mathbf{p_1}(xy)$.

We now embed this adjustment procedure in a dynamic setting, where individuals start interaction from possibly unrelated behaviors. Specifically let (p_1^t, p_2^t) denote the behaviors of 1 and 2 at period $t > 0$ and assume that their initial behaviors (p_1^1, p_2^1) are given. One can think of fresh roommates or teenagers just enrolled in a new school as examples. Below we show that although these individuals start interacting from possibly unrelated behaviors, as long as they adjust accordingly, eventually they converge to (p_1^*, p_2^*), the unique pair of behaviors that the model yields for the given set of parameters. In other words, the behavior produced by the dual interaction model constitutes a stable equilibrium when embedded in a dynamic environment.

Theorem 2 *Take* $w_i \gg 0$, $\alpha_i \geq 0$, $p_i^*(S) \in \Delta_{++}(S)$ *for all* $S \in 2^X \setminus \{\varnothing\}$ *and for each* $i \in \{1,2\}$ *and let* $(w_1, w_2, \alpha_1, \alpha_2)$ *represent* (p_1^*, p_2^*)*. Further, let* $(p_1^1, p_2^1) \in \Delta(S) \times \Delta(S)$*. Define for each* $i \in \{1,2\}$ *and* $t \geq 2$, $p_i^t(\cdot, S) \in \Delta(S)$ *via*

$$p_i^t(x,S) \equiv \frac{w_i(x) + \alpha_i p_j^{t-1}(x,S)}{\sum_{y \in S} w_i(y) + \alpha_i p_j^{t-1}(y,S)}.$$

Then for each $i \in \{1,2\}$, $\lim_{t \to \infty} p_i^t = p_i^*$.

The proof is an application of Contraction Mapping.

2.2 Connection to Empirical Models

As surveyed in the introduction, the standard econometric tools to study social interactions include discrete choice models [Blume, 1993, Brock and Durlauf, 2001, 2003]. In this subsection we show that our model can indeed be reproduced in a discrete choice setting with peer effects. Specifically, under certain assumptions the behavior produced by a multinomial discrete choice model applied to social interactions coincides with the behavior described by our model. This is because both models take logistic choice as the basis. To see this, consider a specific budget set, i.e., a choice problem and let the deterministic part of individual utility constitute two components: the idiosyncratic weight (Luce weight) and the social influence as defined in our model. Now assume a multiplicative form for individual utility as follows:

$$U_i(x) = V_i(x)\varepsilon_i(x) \quad \text{where } V_i(x) = w_i(x) + \alpha_i p_j(x)$$

Under the assumption that the disturbances are i.i.d. with a Log-logistic distribution (i.e., $\log \varepsilon_i$ follows a Type 1 extreme value distribution) with $f(\log \varepsilon_i) = e^{-\log \varepsilon_i} e^{-e^{-\log \varepsilon_i}}$, we have the following:

$$\begin{aligned}
\log U_i(x) &= \log V_i(x) + \log \varepsilon_i(x) \\
p_i(x) &= Prob\left(\log V_i(x) + \log(\varepsilon_i(x)) > \log V_i(y) + \log(\varepsilon_i(y)),\ \forall y \neq x\right) \\
&= Prob\left(\log \varepsilon_i(y) < \log\left(\frac{V_i(x)\varepsilon_i(x)}{V_i(y)}\right),\ \forall y \neq x\right)
\end{aligned}$$

Then for a given $\varepsilon_i(x)$, using $F(\log \varepsilon_i)$:

$$Prob\,(x|\varepsilon_i(x)) = \prod_{y \neq x} \exp\Big\{ - e^{-\log\left(\frac{V_i(x)\varepsilon_i(x)}{V_i(y)}\right)} \Big\}$$

which leads to:

$$p_i(x) = \int_{-\infty}^{+\infty} \left(\prod_{y \neq x} \exp\Big\{ - e^{-\log\left(\frac{V_i(x)\varepsilon_i(x)}{V_i(y)}\right)} \Big\} \right) e^{-\log \varepsilon_i} \exp\{-e^{-\log \varepsilon_i}\} d\log(\varepsilon_i)$$
$$p_i(x) = \int_{-\infty}^{+\infty} \left(\prod_{y} \exp\Big\{ - e^{-\log\left(\frac{V_i(x)\varepsilon_i(x)}{V_i(y)}\right)} \Big\} \right) e^{-\log \varepsilon_i} d\log(\varepsilon_i)$$

The second line above is observed by collecting terms in the exponent of e given that $\dfrac{V_i(x)}{V_i(x)} = 1$.

$$\begin{aligned} p_i(x) &= \int_{-\infty}^{+\infty} \exp\Big\{ - \sum_y e^{-\log\left(\frac{V_i(x)\varepsilon_i(x)}{V_i(y)}\right)} \Big\} e^{-\log \varepsilon_i} d\log(\varepsilon_i) \\ &= \int_{-\infty}^{+\infty} \exp\Big\{ - e^{-\log \varepsilon_i} \sum_y e^{-\log\left(\frac{V_i(x)}{V_i(y)}\right)} \Big\} e^{-\log \varepsilon_i} d\log(\varepsilon_i) \end{aligned}$$

Apply a transformation of variables as $t = e^{-\log(\varepsilon_i(x))}$ such that $dt = -e^{-\log(\varepsilon_i(x))} d\log(\varepsilon_i)$. Note that as $\log(\varepsilon_i)$ approaches infinity, t approaches zero, and as $\log(\varepsilon_i)$ approaches negative infinity, t becomes infinitely large.

$$\begin{aligned} p_i(x) &= \int_{\infty}^{0} - \exp\Big\{ - t \sum_y e^{-\log\left(\frac{V_i(x)}{V_i(y)}\right)} \Big\} dt \\ &= \int_{\infty}^{0} - \exp\Big\{ - t \sum_y \frac{V_i(y)}{V_i(x)} \Big\} dt \\ &= \left. \frac{e^{-t \frac{\sum V_i(y)}{V_i(x)}}}{\frac{\sum V_i(y)}{V_i(x)}} \right|_{\infty}^{0} = \frac{V_i(x)}{\sum_y V_i(y)} = \frac{w_i(x) + \alpha_i p_j(x)}{\sum_y (w_i(y) + \alpha_i p_j(y))} \end{aligned}$$

Thus, two specific assumptions lead to the behavior granted by our model: a logarithmic transformation of the deterministic individual utility and a relevant extreme value distribution for the error terms. Clarifying its connection to the widespread econometric models of social interactions, this observation provides another justification for our choice-theoretic model of influence.

2.3 Connection to Quantal Response Equilibrium

In a similar fashion to the preceding subsection, our model appears conceptually related to Quantal Response Equilibrium (QRE), which is a solution concept for normal form games [McKelvey and Palfrey, 1995]. And indeed, it is possible to reproduce the behavior granted by our model as a logit QRE. But two caveats must be mentioned: first, QRE is a prediction for a single game, whereas the testable implications of our model derive their power from the ability to observe behavior *across* choice sets. Indeed, QRE affords basically no predictions on a single-game (much like classical choice theory generates no predictions from a single budget). See for example, Haile et al. [2008]. Thus, a suitable extension of the notion of QRE *across* game forms must be described.[8] Second, just as in the preceding subsection, our model results from a very specific choice of error distribution (one of the parameters of the QRE model) and a very specific choice of utility (the other main parameter). Put differently, the behavior produced by our model may be viewed as being *rationalized* by a particular choice of game forms and the logit QRE solution concept, suitably extended to across games. Details are available upon request.

3 Multi-agent Interaction

One of the strenghts of our model is that it is easily generalizable to multi individual settings with more intricate forms of social interactions. We can easily capture the heterogeneities that drive different behavioral outcomes in a social context. Not only individuals have different preferences but they also have different levels of susceptibility to influence. Or

[8] In particular one must take care to ensure the error distributions across game forms coincide in a natural way.

similarly, different people might influence an individual in different ways. The generalization of our model to multi individual settings allow for these variations.

It is crucial to note that, thanks to our identification strategy, we do not need assume exogenous network structure. In other words, for identification purposes it is not required to know the underlying network structure. On the contrary, our representation theorem reveals the unknown network of social interactions in addition to individual preferences and influence patterns. Specifically, given the behavior of a group of individuals that is consistent with our characterizing properties, we can *uniquely* identify the underlying preferences, represented by w_i, and the interaction patterns, represented by α_{ij}, capturing how individual i is influenced by the behavior of individual j for all pairs of individuals i and j. Note that the interaction between i and j might be asymmetric, i.e., α_{ij} need not be equal to α_{ji}.

Let us now formally introduce the generalized model. Let N denote a set of $n < +\infty$ individuals interacting. As before, for each choice problem, $S \in 2^X \setminus \emptyset$, we observe agent i's stochastic choice, $p_i(x, S)$. Let $p_{-i}(x, S) \in R^{n-1}$ denote the vector of $p_j(x, S)$ and $d_{-i}(x, S) \in \Re^{n-1}$ the vector of $d_j(x, S)$ for all $j \neq i$.

Definition. $(p_1, p_2, ..., p_n)$ has a **social interaction** representation if for each $i \in N$ there exist $w_i : X \to (0, 1)$ with $\sum\limits_{x \in X} w_i(x) = 1$ and $\bar{\alpha}_i \in R_+^{n-1}$ such that

$$p_i(x, S) = \frac{w_i(x) + \bar{\alpha}_i \cdot p_{-i}(x, S)}{\sum_{y \in S}[w_i(y) + \bar{\alpha}_i \cdot p_{-i}(y, S)]}$$

for all $x \in S$ and for all S.

The parameter $\bar{\alpha}_i$ captures different levels of susceptibility to influence from different individuals, i.e., agent i can be influenced differently by different j's. Let α_{ij} denote the entry of $\bar{\alpha}_i$ relating to the influence of individual j on i. If $\alpha_{ij} = 0$ for all $j \neq i$, once again i's choice behavior reduces down to Luce.

The characterizing properties listed below are immediate generalizations of our baseline properties to multi-individual environenements. First let us define our new $\bar{\beta}_i$. Let $i \in N$. For any $S \neq X$, and any $x, y \in S$ for which $x \neq y$, define $\bar{\beta}_i(x, y, S) \in R^{n-1}$ such that

$$\bar{\beta}_i(x,y,S) \cdot \left(\frac{d_{-i}(x,S)}{p_i(x,S)} - \frac{d_{-i}(y,S)}{p_i(y,S)} \right) = \frac{d_i(x,S)}{p_i(x,S)} - \frac{d_i(y,S)}{p_i(y,S)}.$$

Axiom 4 (N-Conditional Independence) *For all $i \in N$, $\bar{\beta}_i(x,y,S)$ is independent of S, x, and y.*

Axiom 5 (N-Uniform Boundedness) *For all $z \in X$, $p_i(z,X) > \bar{\beta}_i(x,y,S) \cdot p_{-i}(z,X)$ for all $S \neq X$, $x,y \in S$ with $x \neq y$ and for all $i \in N$.*

Axiom 6 (N-Nonnegativeness) *$\bar{\beta}_i(x,y,S) \in R_+^{n-1}$ for all $S \neq X$, $x,y \in S$ with $x \neq y$ and for all $i \in N$.*

For identification purposes, we focus our attention to linearly independent stochastic choice behaviors. Specifically, we assume that there does not exist any p_i such that for all x and S, $p_i(x,S)$ can be expressed as a convex combination of $\{p_j(x,S)\}_{j \neq i}$.

Theorem 3 *Let $\{p_i\}_{i \in N}$ as defined. Then, $\{p_i\}_{i \in N}$ has a* ***social interaction*** *representation if and only if N-Conditional Independence, N-Uniform Boundedness and N-Nonnegativeness hold. Moreover, $\{w_i, \bar{\alpha}_i\}_{i \in N}$ are uniquely identified.*

The proof of Theorem 3 closely follows that of Theorem 1. In this case, we take $\alpha_{ij} = \dfrac{\bar{\beta}_{ij}}{1 - \sum_{j \neq i} \bar{\beta}_{ij}}$, well-defined and non-negative as guaranteed by the axioms.

4 Concluding Remarks

The identification of social interactions out of observable behaviour is a challenging, yet relevant question for economists. We believe that the use of microfoundational tools to study social interactions brings about a new perspective that proves itself useful for identification of unobservable underlying interaction structures and parameters. We suggest the dual interaction and social interaction models as simple tools to this purpose. Yet there is much room for possible extensions and applications.

One such avenue is the study of negative interactions. Most of the theoretical tools developed to study social interactions are restricted by strategic complementarity or conformity type assumptions. However in certain contexts, where individuals especially do not want to behave similarly, negative interactions are in play.[9] In our setting the inclusion of negative interactions seems rather straightforward, via a possibly negative interaction parameter α_i. Indeed we can show that simple alterations of our axioms would suffice to characterize both of our models preserving the unique identification of the underlying parameters. These results are available upon request. However, in this case we do have an existence problem. Specifically, when we allow for negative αs, not every combination of $\{(w_i, \alpha_i)\}$ yields a representation. Let us exemplify this for the dual interaction model. The left panel of Figure 4 represents a dual interaction model with $\alpha_1 = -.5$ and $\alpha_2 = 1$. Since individual 2 is negatively influenced by individual 1, $\mathbf{p}_1$ is no longer between $\mathbf{w}_1$ and $\mathbf{p}_2$, but instead further away from $\mathbf{p}_2$, yet still on the same line, as expected. On the right panel of Figure 4, however, the interaction parameters are chosen such that the resulting behavior cannot be expressed as a stochastic choice function. One can find the restrictions on the admissible combinations of parameters that ensure representation. Indeed we have this result available on request. However these restrictions remain to be unintuitive and technical, pointing out the need for further study to investigate more convoluted forms of social interactions.

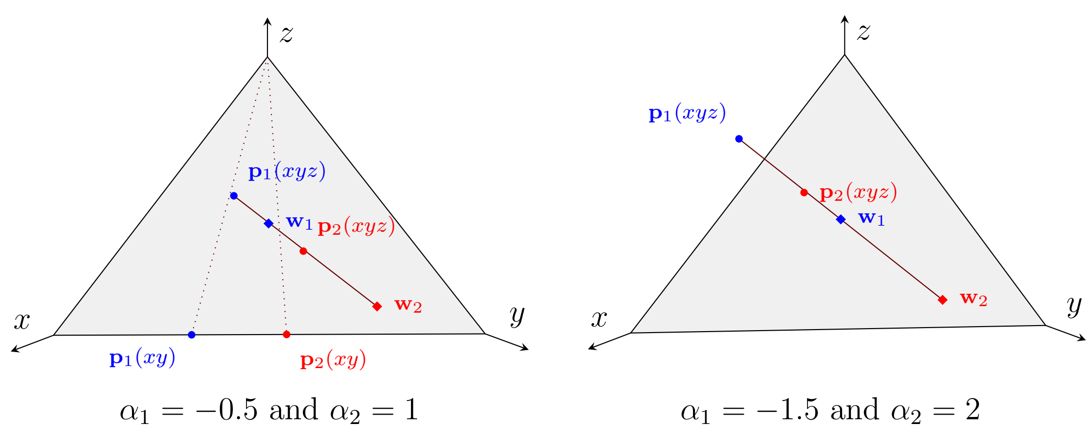


Figure 4: Dual interaction model with negative interactions

[9] Examples include fashions and fads, where the the choice of a fashion product not only signals which social group you would like to identify with but also signals who you would like to differentiate from [Pesendorfer, 1995]; deidentification among siblings, the choice of different paths by the siblings for the sake of differentiating from each other, especially performed by the second-born in order to avoid sibling competition [Schachter et al., 1976, Sulloway, 2010]; or among criminals due to competition for resources [Glaeser et al., 1996].

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5 Appendix

Proof of Theorem 1. ($\Rightarrow$) That the axioms are satisfied is straightforward; observe that if the representation is satisfied, then

$$\begin{aligned}
\beta_i(x,y,S) &= \frac{\frac{d_i(x,S)}{p_i(x,S)} - \frac{d_i(y,S)}{p_i(y,S)}}{\frac{d_j(x,S)}{p_i(x,S)} - \frac{d_j(y,S)}{p_i(y,S)}} = \frac{\frac{p_i(y,X)}{p_i(y,S)} - \frac{p_i(x,X)}{p_i(x,S)}}{\frac{p_j(x,S)}{p_i(x,S)} - \frac{p_j(y,S)}{p_i(y,S)} + \frac{p_j(y,X)}{p_i(y,S)} - \frac{p_j(x,X)}{p_i(x,S)}} \\
&= \frac{\alpha_i\left(\frac{p_i(y,X)}{p_i(y,S)} - \frac{p_i(x,X)}{p_i(x,S)}\right)}{\frac{w_i(x)+\alpha_i p_j(x,S)}{p_i(x,S)} - \frac{w_i(y)+\alpha_i p_j(y,S)}{p_i(y,S)} + \frac{w_i(y)+\alpha_i p_j(y,X)}{p_i(y,S)} - \frac{w_i(x)+\alpha_i p_j(x,X)}{p_i(x,S)}} \\
&= \frac{\alpha_i\left(\frac{p_i(y,X)}{p_i(y,S)} - \frac{p_i(x,X)}{p_i(x,S)}\right)}{\frac{p_i(x,S)[w_i(S)+\alpha_i]}{p_i(x,S)} - \frac{p_i(y,S)[w_i(S)+\alpha_i]}{p_i(y,S)} + \frac{p_i(y,X)[1+\alpha_i]}{p_i(y,S)} - \frac{p_i(x,X)[1+\alpha_i]}{p_i(x,S)}} \\
&= \frac{\alpha_i\left(\frac{p_i(y,X)}{p_i(y,S)} - \frac{p_i(x,X)}{p_i(x,S)}\right)}{[1+\alpha_i]\left(\frac{p_i(y,X)}{p_i(y,S)} - \frac{p_i(x,X)}{p_i(x,S)}\right)} \\
&= \frac{\alpha_i}{1+\alpha_i}
\end{aligned}$$

From this the axioms 1 and 3 follow directly. Axiom 2 follows from $w_i(x) > 0$ for all x since $w_i(x) = (1+\alpha_i)p_i(x,X) - \alpha_i p_j(x,X)$. Then we have $\dfrac{p_i(x,X)}{p_j(x,X)} > \beta_i$.

($\Leftarrow$) We first establish that if the axioms are satisfied, then the representation holds.

Suppose that $p_1 \neq p_2$. Define $\beta_i \equiv \beta_i(x, y, S)$, which is well-defined by Axiom 1. We first show that for each $i \in \{1, 2\}$, $\beta_i \neq 1$. Assume by means of contradiction that $\beta_i = 1$. By Axiom 2, $1 < \dfrac{p_i(x, X)}{p_j(x, X)}$ for all $x \in X$. Observe then that for all $x \in X$, $p_i(x, X) > p_j(x, X)$, from which it follows that $1 = \sum_{x \in X} p_i(x, X) > \sum_{x \in X} p_j(x, X) = 1$, a contradiction.

Now, define, for $S \neq X$, and $x, y \in S$ for which $x \neq y$,

$$\alpha_i(x, y, S) := \frac{\frac{p_i(x,X)}{p_i(x,S)} - \frac{p_i(y,X)}{p_i(y,S)}}{\frac{d_i(x,S)-d_j(x,S)}{p_i(x,S)} - \frac{d_i(y,S)-d_j(y,S)}{p_i(y,S)}} = \frac{\beta_i}{1 - \beta_i}$$

By Axiom 1, α_i does not depend on $x, y \in S$ and $S \neq X$, so, let $\alpha_i(x, y, S) \equiv \alpha_i$. We claim that $\alpha_i \geq 0$ for each $i \in \{1, 2\}$. Observe that by Axiom 2, $\beta_i < 1$. Joint with Axiom 3, this means $\beta_i \in [0, 1)$. Hence it follows that $\alpha_i = \dfrac{\beta_i}{1 - \beta_i} \geq 0$. Next, define

$$w_i(x) \equiv p_i(x, X) + \alpha_i(p_i(x, X) - p_j(x, X)).$$

Observe that $\sum_{x \in X} w_i(x) = 1$.

Then:

$$\frac{p_i(x, X)}{p_i(x, S)} - \frac{p_i(y, X)}{p_i(y, S)} = \alpha_i \left[\frac{d_i(x, S) - d_j(x, S)}{p_i(x, S)} - \frac{d_i(y, S) - d_j(y, S)}{p_i(y, S)} \right]$$

or

$$\frac{p_i(x, X) + \alpha_i d_j(x, S) - \alpha_i d_i(x, S)}{p_i(x, S)} = \frac{p_i(y, X) + \alpha_i d_j(y, S) - \alpha_i d_i(y, S)}{p_i(y, S)}.$$

Adding α_i to both sides of the equality and organizing

$$\begin{aligned}\frac{p_i(x,S)}{p_i(y,S)} =&\frac{p_i(x,X)+\alpha_i d_j(x,S)-\alpha_i d_i(x,S)+\alpha_i p_i(x,S)}{p_j(y,X)+\alpha_i d_j(y,S)-\alpha_i d_i(y,S)+\alpha_i p_i(y,S)}\\ =&\frac{p_i(x,X)+\alpha_i(p_i(x,X)-p_j(x,X))+\alpha_i p_j(x,S)}{p_i(x,X)+\alpha_i(p_i(x,X)-p_j(x,X))+\alpha_i p_j(x,S)}\\ =&\frac{w_i(x)+\alpha_i p_j(x,S)}{w_i(y)+\alpha_i p_j(y,S)}.\end{aligned}$$

Observe in particular that this equality holds even in the case $x = y$.

Now, for any $x, y \in S$, we have

$$p_i(y,S) = p_i(x,S)\frac{w_i(y)+\alpha_i p_j(y,S)}{w_i(x)+\alpha_i p_j(x,S)}$$

so that

$$\sum_{y\in S} p_i(y,S) = \sum_{y\in S} p_i(x,S)\frac{w_i(y)+\alpha_i p_j(y,S)}{w_i(x)+\alpha_i p_j(x,S)}.$$

Conclude

$$1 = p_i(x,S)\frac{\sum_{y\in S}(w_i(y)+\alpha_i p_j(y,S))}{w_i(x)+\alpha_i p_j(x,S)}.$$

Consequently,

$$p_i(x,S) = \frac{w_i(x)+\alpha_i p_j(x,S)}{\sum_{y\in S}(w_i(y)+\alpha_i p_j(y,S))}.$$

We now show that $w_i(x) > 0$ for all $x \in X$. For all $x \in X$, $\frac{p_i(x,X)}{p_j(x,X)} > \beta_i = \frac{\alpha_i}{1+\alpha_i}$. Here, we obtain $(\alpha_i+1)p_i(x,X) > \alpha_i p_j(x,X)$ for all x. Consequently, $w_i(x) = p_i(x,X) + \alpha_i[p_i(x,X) - p_j(x,X)] > 0$ for all x.

We conclude the proof of sufficiency by establishing uniqueness of the representation. The following system defines α_i uniquely given $p_1 \neq p_2$:

$$p_i(x,S) = \frac{w_i(x)+\alpha_i p_j(x,S)}{w_i(S)+\alpha_i} \tag{1}$$

$$p_i(x,X) = \frac{w_i(x)+\alpha_i p_j(x,X)}{1+\alpha_i} \tag{2}$$

since they imply

$$
\begin{aligned}
w_i(x) &= p_i(x, X) + \alpha_i(p_i(x, X) - p_j(x, X)) \\
w_i(x) &= p_i(x, S) \sum_{x \in S} w_i(x) + \alpha_i(p_i(x, S) - p_j(x, S))
\end{aligned} \tag{3}
$$

Unique identification of w_1, w_2 is immediate. ■

Proof of Theorem 2. The proof is via contraction mapping. Let us metrize the set $\Delta(S) \times \Delta(S)$ with the function defined by $d((p, q), (p', q')) \equiv \|p - p'\| + \|q - q'\|$, where $\|p\|$ references the standard Euclidean norm.[10] Observe that this metric generates the standard Euclidean topology on $\Delta(S) \times \Delta(S)$.

Let us use the notation $w_i|_S$ for the restriction of w_i to S. We will establish that the map $f : \Delta(S) \times \Delta(S) \to \Delta(S) \times \Delta(S)$ defined by

$$
f(p, q) \equiv \left(\frac{w_1|_S + \alpha_1 q}{w_1(S) + \alpha_1}, \frac{w_2|_S + \alpha_2 p}{w_2(S) + \alpha_2} \right)
$$

is a contraction. It is straightforward to establish that f is contraction in our metrization of $\Delta(S) \times \Delta(S)$ if and only if the maps $f_i : \Delta(S) \to \Delta(S)$ given by

$$
f_i(p) \equiv \frac{w_i|_S + \alpha_i p}{w_i(S) + \alpha_i}
$$

are contractions on $\Delta(S)$ with the standard Euclidean topology. So this is what we prove.

Observe that

$$
f_i(p) - f_i(p') = \frac{w_i|_S + \alpha_i p}{w_i(S) + \alpha_i} - \frac{w_i|_S + \alpha_i p'}{w_i(S) + \alpha_i} = \left(\frac{\alpha_i}{w_i(S) + \alpha_i} \right) (p - p').
$$

Hence the mapping f_i is a contraction with modulus $\dfrac{\alpha_i}{w_i(S) + \alpha_i} < 1$; consequently, so is the mapping f with the above metric on $\Delta(S) \times \Delta(S)$, with modulus $\max \left\{ \dfrac{\alpha_1}{w_1(S) + \alpha_1}, \dfrac{\alpha_2}{w_2(S) + \alpha_2} \right\}$.

[10] That is, $\|p - p'\| = \sqrt{\sum_{y \in S} (p(y) - p'(y))^2}$.

Now, thanks to Banach Fixed Point Theorem, we can conclude that f has a unique fixed point, $(p_1^*(S), p_2^*(S))$, establishing the first claim of the theorem, and the sequence $(p_1^t(S), p_2^t(S))$ converges to $(p_1^*(S), p_2^*(S))$, establishing the latter. ■

Proof of Theorem 3. ($\Rightarrow$) We skip the proof of necessity since it closely follows that of Theorem 1.

($\Leftarrow$) Let $\{p_i\}_{i\in N}$. Take any $i \in N$, x, y and S and define $\bar{\beta}_i := \bar{\beta}_i(x, y, S)$, by Axiom 3. Further, define $\bar{\alpha}_i \in R^{n-1}$ such that $\alpha_{ij} = \dfrac{\bar{\beta}_{ij}}{1 - \sum_{j\neq i} \bar{\beta}_{ij}}$. We first show that $\bar{\alpha}_i$ is well-defined and nonnegative since $\displaystyle\sum_{j\neq i} \beta_{ij} < 1$. This is because by Axiom 5 $p_i(x, X) > \bar{\beta}_i p_{-i}(x, X)$ for all x, we have $1 = \displaystyle\sum_{x\in X} p_i(x, X) > \sum_{x\in X} \bar{\beta}_i p_{-i}(x, X) = \sum_{j\neq i} \bar{\beta}_{ij}$. Hence, $\bar{\alpha}_i \in R_+^{n-1}$ is well-defined for all $\bar{\beta}_i$ as claimed.

Notice we then have $\dfrac{\bar{\alpha}_i}{1 + \sum_{j\neq i} \alpha_{ij}} = \bar{\beta}_i$.

Now define

$$w_i(x) := p_i(x, X) + \bar{\alpha}_i \cdot [p_i(x, X)\bar{1} - p_{-i}(x, X)]$$

where $\bar{1} \in R^{n-1}$ is a vector of ones and observe that

$$\begin{aligned}
\sum_{x\in X} w_i(x) &= \sum_{x\in X} p_i(x, X) + \bar{\alpha}_i \cdot [p_i(x, X)\bar{1} - p_{-i}(x, X)] \\
&= 1 + \bar{\alpha}_i \cdot [\sum_{x\in X} p_i(x, X)\bar{1} - \sum_{x\in X} p_{-i}(x, X)] \\
&= 1 + \bar{\alpha}_i(\bar{1} - \bar{1}) \\
&= 1.
\end{aligned}$$

By Axiom 3,

$$\frac{\bar{\alpha}_i}{1 + \sum_{j\neq i} \alpha_{ij}} \cdot \left(\frac{d_{-i}(x, S)}{p_i(x, S)} - \frac{d_{-i}(y, S)}{p_i(y, S)}\right) = \frac{p_i(y, X)}{p_i(y, S)} - \frac{p_i(x, X)}{p_i(x, S)}$$

$$\frac{(1 + \sum_{j\neq i} \alpha_{ij})p_i(x, X) + \bar{\alpha}_i \cdot p_{-i}(x, S) - \bar{\alpha}_i \cdot p_{-i}(x, X)}{p_i(x, S)} = \frac{(1 + \sum_{j\neq i} \alpha_{ij})p_i(y, X) + \bar{\alpha}_i \cdot p_{-i}(y, S)}{p_i(y, S)} - \frac{\bar{\alpha}_i \cdot p_{-i}(y, X)}{p_i(y, S)}.$$

Hence

$$\begin{aligned}\frac{p_i(x,S)}{p_i(y,S)} =& \frac{p_i(x,X)+\bar{\alpha}_i\cdot[p_i(x,X)\bar{1}-p_{-i}(x,X)]+\bar{\alpha}_i\cdot p_{-i}(x,S)}{p_i(y,X)+\bar{\alpha}_i\cdot[p_i(y,X)\bar{1}-p_{-i}(y,X)]+\bar{\alpha}_i\cdot p_{-i}(y,S)}\\ =& \frac{w_i(x)+\bar{\alpha}_i\cdot p_{-i}(x,S)}{w_i(y)+\bar{\alpha}_i\cdot p_{-i}(y,S)}.\end{aligned}$$

But then, since this claim holds for all $y \in S$:

$$\begin{aligned}p_i(y,S) &= p_i(x,S)\frac{w_i(y)+\bar{\alpha}_i\cdot p_{-i}(y,S)}{w_i(x)+\bar{\alpha}_i\cdot p_{-i}(x,S)}\\ \sum_{y\in S} p_i(y,S) &= \sum_{y\in S} p_i(x,S)\frac{w_i(y)+\bar{\alpha}_i\cdot p_{-i}(y,S)}{w_i(x)+\bar{\alpha}_i\cdot p_{-i}(x,S)}\\ 1 &= p_i(x,S)\frac{\sum_{y\in S}[(w_i(y)+\bar{\alpha}_i\cdot p_{-i}(y,S)]}{w_i(x)+\bar{\alpha}_i\cdot p_{-i}(x,S)}\\ p_i(x,S) &= \frac{w_i(x)+\bar{\alpha}_i\cdot p_{-i}(x,S)}{\sum_{y\in S}[w_i(y)+\bar{\alpha}_i\cdot p_{-i}(y,S)]}.\end{aligned}$$

We finally show that $w_i(x) > 0$ for all $x \in X$. This is established by Axiom 5. Since $p_i(x,X) > \bar{\beta}_i p_{-i}(x,X)$ and $1+\sum_{j\neq i}\alpha_{ij} > 0$, then, $(1+\sum_{j\neq i}\alpha_{ij})p_i(x,X) > \bar{\alpha}_i p_{-i}(x,X) \Rightarrow w_i(x) > 0$.

We conclude the proof of sufficiency by establishing uniqueness of the representation. The following system defines $\bar{\alpha}_i$ uniquely given $\{p_i\}_{i\in N}$:

$$\begin{aligned}p_i(x,S) &= \frac{w_i(x)+\bar{\alpha}_i\cdot p_{-i}(x,S)}{\sum_{y\in S}[w_i(y)+\bar{\alpha}_i\cdot p_{-i}(y,S)]}\\ p_i(x,X) &= \frac{w_i(x)+\bar{\alpha}_i\cdot p_{-i}(x,X)}{1+\sum_{j\neq i}\alpha_{ij}}\end{aligned}$$

since they imply

$$\begin{aligned}w_i(x) &= p_i(x,X)+\bar{\alpha}_i\cdot[p_i(x,X)\bar{1}-p_{-i}(x,X)]\\ w_i(x) &= w_i(S)p_i(x,S)+\bar{\alpha}_i\cdot[p_i(x,X)\bar{1}-p_{-i}(x,S)]\end{aligned}$$

Unique identification of w_i is immediate. ■